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A NOTE ON POLYNOMIALLY BARRELED LOCALLY CONVEX SPACES

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Résumé : Le principal objectif de l'article est de démontrer que tout espace localement convexe séquentiellement complet polynômialement infratonnelé est polynômialement tonnelé. Ce résultat est une version polynômiale d'un résultat classique de la théorie linéaire des espaces localement convexes.

Summary : The main goal of the article is to prove that sequentially complete polynomially infrabarreled locally convex spaces are polynomially barreled locally convex spaces, a polynomial version of a classical result of the linear theory of locally convex spaces.

In the linear theory of locally convex spaces, it is known that every Hausdorff, sequentially complete, infrabarreled locally convex space is barreled. The purpose of this note is to prove the corresponding result in the polynomial context ; that is, we shall prove the following.

THEOREM. *Every Hausdorff, sequentially complete, polynomially infrabarreled locally convex space E is polynomially barreled.*

Partially supported by CNPq.

We must observe that this theorem is known in the particular case in which E is polynomially bornological and quasi-complete ([8], theorem 3.35).

We shall adopt the notation and terminology of [6], [7] and [8], and the following conventions. All topological vector spaces will be complex. If $E_1,\dots,E_m$ and F are topological vector spaces, $\mathcal{L}(E_1,\dots,E_m ; F)$ will denote the vector space of all continuous m-linear mappings from $E_1 \times \dots \times E_m$ into F. If $E_1 = \dots = E_m = E$, we write $\mathcal{L}(E_1,\dots,E_m ; F) = \mathcal{L}(^mE ; F)$. If E and F are topological vector spaces and $m \in \mathbb{N}$, $\mathcal{L}_s(^mE ; F)$ will denote the vector subspace of $\mathcal{L}(^mE ; F)$ of all continuous symmetric m-linear mappings from E^m into F ; and $\mathcal{P}(^mE ; F)$ the vector space of all continuous m-homogeneous polynomials from E into F.

In order to prove the theorem, we shall need the following material.

DEFINITION 1. *Let* $E_1,\dots,E_m$ *be topological vector spaces and* F *a locally convex space. Let* $\theta_1,\dots,\theta_m$ *be sets formed by bounded subsets of* $E_1,\dots,E_m$ *, respectively. The* $(\theta_1,\dots,\theta_m)$*-topology in* $\mathcal{L}(E_1,\dots,E_m ; F)$ *is the locally convex topology defined by the family of seminorms*

$$A \in \mathcal{L}(E_1,\dots,E_m ; F) \mapsto \sup_{x_1 \in B_1,\dots,x_m \in B_m} \beta(A(x_1,\dots,x_m)) \in \mathbb{R}_+ ,$$

where β *varies in the set of all continuous seminorms on* F, B_1 *varies in* $\theta_1,\dots,B_m$ *varies in* θ_m. *If* $\theta_1,\dots,\theta_m$ *are the sets of all finite (resp. compact) subsets of* $E_1,\dots,E_m$ *, respectively, we shall denote* $(\theta_1,\dots,\theta_m)$ *by* τ_s *(resp.* τ_0*). In the same way, if* E *is a topological vector space,* F *is a locally convex,* θ *is a set formed by bounded subsets of* E *and* $m \in \mathbb{N}$, *we define the* θ*-topology in* $\mathcal{P}(^mE ; F)$. *We shall use the symbols* τ_s *and* τ_0 *with the same meaning as in the multilinear case. Obviously,* $\tau_s \leqslant \tau_0$ *in both cases.*

PROPOSITION 1. *Let* E *be a topological vector space,* F *a locally convex space,* $m \in \mathbb{N}^*$ *and* θ *a set formed by bounded subsets of* E *such that* $\lambda_1\theta + \dots + \lambda_m\theta \subset \theta$, *for every* $\lambda_1,\dots,\lambda_m \in \mathbb{C}$. *Under these assumptions, the vector spaces isomorphism*

$$\phi : A \in \mathcal{L}_s(^mE ; F) \mapsto \hat{A} \in \mathcal{P}(^mE ; F)$$

is a locally convex spaces isomorphism, if $\mathcal{L}_s(^mE ; F)$ *is endowed with the locally convex topology induced by* $(\mathcal{L}(^mE ; F), (\theta,\dots,\theta))$ *and* $\mathcal{P}(^mE ; F)$ *is endowed with the* θ*-topology.*

Proof. Since ϕ is obviously continuous, it remains to show that ϕ^{-1} is continuous. Fix $B_1,\dots,B_m \in \theta$ and a continuous seminorm β on F. For every $A \in \mathcal{L}_s(^mE ; F)$,

$$\sup_{x_1\in B_1,\dots,x_m\in B_m} \beta(A(x_1,\dots,x_m)) \leqslant$$

$$\frac{1}{m!2^m} \sum_{\substack{\epsilon_i=\pm 1\\ 1\leqslant i\leqslant m}} \sup_{x_1\in B_1,\dots,x_m\in B_m} \beta(\hat{A}(\epsilon_1 x_1+\dots+\epsilon_m x_m)), \qquad (*)$$

by the Polarization Formula.

For each choice of $\epsilon_i = \pm 1$ $(1 \leqslant i \leqslant m)$ the bounded set $\{\epsilon_1 x_1 + \dots + \epsilon_m x_m \,;\, x_1 \in B_1,\dots,x_m \in B_m\}$ belongs to θ, by hypothesis. Hence, the function

$$P \in \mathscr{P}(^mE\,;F) \mapsto \sup_{x_1\in B_1,\dots,x_m\in B_m} \beta(P(\epsilon_1 x_1+\dots+\epsilon_m x_m))$$

is a continuous seminorm on $(\mathscr{P}(^mE\,;F),\theta)$, and inequality (*) guarantees the continuity of ϕ^{-1}. QED

Remark 1. τ_s and τ_0 satisfy the condition of Proposition 1 imposed on θ.

PROPOSITION 2. *Let* $E_1,\dots,E_m$ $(m \geqslant 1)$ *and* F *be Hausdorff locally convex spaces,* $E_1,\dots,E_m$ *being sequentially complete. Let* $\theta_1,\dots,\theta_m$ *be coverings of* $E_1,\dots,E_m$, *formed by bounded subsets of* $E_1,\dots,E_m$, *respectively. Then, every subset of* $\mathcal{L}(E_1,\dots,E_m\,;F)$ *which is bounded for* τ_s *is bounded for the* $(\theta_1,\dots,\theta_m)$*-topology.*

Proof. For each $u \in \mathcal{L}(E_1,\dots,E_m\,;F)$, let $g(u) \in \mathcal{L}_a(E_1\,;\mathcal{L}(E_2,\dots,E_m\,;F))$ be defined by :

$$g(u)(x_1)(x_2,\dots,x_m) = u(x_1,\dots,x_m), \quad \text{if } x_1 \in E_1,\dots,x_m \in E_m .$$

Since the continuity of u implies the hypocontinuity of u ([3] , chap. III, § 4), it is easy to see that

$$g(u) \in \mathcal{L}(E_1\,;(\mathcal{L}(E_2,\dots,E_m\,;F),(\theta_2,\dots,\theta_m))).$$

We claim that the injective linear mapping

$$g : u \in \mathcal{L}(E_1,\dots,E_m\,;F) \mapsto g(u) \in \mathcal{L}(E_1\,;(\mathcal{L}(E_2,\dots,E_m\,;F),(\theta_2,\dots,\theta_m)))$$

is an isomorphism of $((E_1,\dots,E_m\,;F),(\theta_1,\dots,\theta_m))$ onto $S = g(\mathcal{L}(E_1,\dots,E_m\,;F))$, S being endowed with the topology induced by $(\mathcal{L}(E_1\,;(\mathcal{L}(E_2,\dots,E_m\,;F),(\theta_2,\dots,\theta_m))),\theta_1)$. In fact, fix $B_1 \in \theta_1,\dots,B_m \in \theta_m$, $\epsilon > 0$, and a continuous seminorm β on F. Then

$$U = \Big\{u \in \mathcal{L}(E_1,\dots,E_m\,;F)\,; \sup_{x_1\in B_1,\dots,x_m\in B_m} \beta(u(x_1,\dots,x_m)) \leqslant \epsilon\Big\}$$

is a sub-basic neighborhood of zero in $(\mathcal{L}(E_1,...,E_m\ ;F),(\theta_1,...,\theta_m))$, and

$$g(U)=\{g(u)\in S\ ;\ \sup_{x_1\in B_1}\ (\sup_{x_2\in B_2,...,x_m\in B_m}\ \beta(g(u)(x_1)(x_2,...,x_m)))\leqslant\epsilon\}$$

is a sub-basic neighborhood of zero in S. Hence, our claim is verified.

To finish the proof we will argue by induction, the linear case being well known ([5], p. 135, (3)). Let $\mathscr{X}\subset\mathcal{L}(E_1,...,E_m\ ;F)$ be bounded for τ_s. By induction, the image $g(\mathscr{X})$ of $\mathscr{X}$ by g is bounded in $(\mathcal{L}(E_1\ ;(\mathcal{L}(E_2,...,E_m\ ;F),(\theta_2,...,\theta_m))),\tau_s)$. Again, by the linear case, $g(\mathscr{X})$ is bounded in S, and, hence, $\mathscr{X}$ is bounded in $(\mathcal{L}(E_1,...,E_m\ ;F),(\theta_1,...,\theta_m))$. Q.E.D.

COROLLARY 1. *Let* E *and* F *be Hausdorff locally convex spaces,* E *being sequentially complete, and let θ be a covering of* E *by bounded subsets of* E *such that $\lambda_1\theta+...+\lambda_m\theta\subset\theta$, for every $\lambda_1,...,\lambda_m\in C$. If $\mathscr{X}\subset\mathscr{P}(^mE\ ;F)$ is bounded for τ_s, then $\mathscr{X}$ is bounded for the θ-topology.*

Proof. The corollary is obvious for m = 0, and is known for m = 1, without any restriction on θ ([5], p. 135, (3)). Let us suppose $m\geqslant 2$. By Proposition 1, the corresponding $\tilde{\mathscr{X}}=\{A\in\mathcal{L}_s(^mE\ ;F)\ ;\hat{A}\in\mathscr{X}\}$ is bounded in $(\mathcal{L}_s(^mE\ ;F),\tau_s)$, and, by Proposition 2, $\tilde{\mathscr{X}}$ is bounded in $(\mathcal{L}_s(^mE\ ;F),(\theta,...,\theta))$. A new application of Proposition 1 completes the proof. QED

Before we prove the theorem, we recall the polynomial notions involved in it (see [1] and [8]).

DEFINITION 2. *A locally convex space* E *is said to be polynomially barreled (resp. polynomially infrabarreled) if, for every locally convex space* F *and for every* m ∈ IN, *a subset $\mathscr{X}\subset\mathscr{P}(^mE\ ;F)$ is equicontinuous if $\mathscr{X}$ is bounded for τ_s (resp. τ_0).*

Proof of the theorem. Let F be a locally convex space, and m ∈ IN. If $\mathscr{X}\subset\mathscr{P}(^mE\ ;F)$ is bounded for τ_s, then $\mathscr{X}$ is bounded for τ_0, by Corollary 1. By using the fact that E is polynomially infrabarreled, we get that $\mathscr{X}$ is equicontinuous, and, hence, E is polynomially barreled. QED

We refer the reader to [1] and [8] for the study of the polynomial theory and to [2] and [7] for the study of the holomorphic theory. Finally, we mention some examples of locally convex spaces which have the polynomial concepts considered in this note (see [1] and [8] for the details).

Example 1. a) Every metrizable locally convex space is polynomially infrabarreled ;

b) Every Baire locally convex space is polynomially barreled ;

c) Every barreled (resp. infrabarreled) DF ([4], Definition 1) locally convex space is polynomially barreled (resp. polynomially infrabarreled).

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