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## Dynamical connections and non-autonomous Lagrangian systems<sup>(1)</sup>

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**RÉSUMÉ.** — On montre que si  $\xi$  est une équation différentielle du deuxième ordre (semigerbe) sur le fibré des jets  $J^1(\mathbf{R}, M)$  telle que les courbes intégrales sont des solutions de l'équation de Lagrange non-autonome alors il existe une connexion  $\Gamma$  sur  $J^1(\mathbf{R}, M)$  dont les courbes intégrales sont aussi des solutions de la même équation. En plus,  $\Gamma$  est une connexion ayant comme semigerbe  $\xi$ . L'étude est une extension à la dynamique Lagrangienne non-autonome de quelques résultats de Grifone pour le cas autonome.

**ABSTRACT.** — We show that if  $\xi$  is a second-order differential equation (semispray) on the jet bundle  $J^1(\mathbf{R}, M)$  whose paths are solutions of the non-autonomous Lagrange equations then there is a connection  $\Gamma$  on  $J^1(\mathbf{R}, M)$  whose paths are also solutions of the same equations. Moreover,  $\Gamma$  is a connection whose associated semispray is precisely  $\xi$ . This is an extension to non-autonomous Lagrangian dynamics of a previous result due to Grifone for autonomous Lagrangians.

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### 1. Introduction

The geometrical description of autonomous Lagrangian systems, started with GALLISOT [G], was elucidated by KLEIN [K1], [K2] (see also GODBILLON [GB]). He showed that the differential geometry of Lagrangian dynamics is intrinsically related to a (1.1) tensor field  $J$ , called *almost tangent structure*, defined on the tangent bundle of a manifold.

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Since the early works of KLEIN some articles have been published showing that almost tangent geometry provides a natural framework in which interesting generalizations of autonomous Lagrangian systems may be developed (see for instance, CRAMPIN [C], CRAMPIN et al. [CSC], de LEON and RODRIGUES [DLR1], [DLA2], GOTAY and NESTER [GN], SARLET et al. [SCC]). In particular, an extensive study about the theory of connections on tangent bundles in terms of the almost tangent geometry, including some aspects of autonomous Lagrangians, was proposed by GRIFONE [GR] around 1972.

As far as we know the non-autonomous case has been practically unknown in the literature (an exception, for example, is the recent paper of CRAMPIN and co-workers [CPT]). It is the purpose of this paper to establish some intrinsic properties about almost tangent theory of connections and its relation with non-autonomous Lagrangian dynamics. We will see that the theory of connections on the jet manifold  $J^1(\mathbf{R}, M)$  is surprisingly more simpler than the theory of connections on  $TM$ , where  $M$  is a given manifold (the reader is invited to compare our results with GRIFONE).

## 2. Preliminaries

Throughout the text we shall keep in mind all results, definitions and notations previously introduced in [DLR 1] (see also [DLR 2]). All structures, functions, etc, are assumed to be smooth ( $C^\infty$ ). Let  $M$  be a manifold of dimension  $m$  (called *configuration* manifold) and  $\Gamma$  a connection on the tangent bundle  $TM$  of  $M$ . We recall here that a connection  $\Gamma$  on  $TM$  generates two projectors  $h : T(TM) \rightarrow Hor(TM)$ ,  $v : T(TM) \rightarrow Ver(TM)$  such that  $T(TM) = Hor(TM) \oplus Ver(TM)$ , where  $Hor(TM)$  (resp.  $Ver(TM)$ ) is the horizontal (resp. vertical) bundle over  $TM$ . If  $\bar{\xi}$  is an arbitrary semispray (second-order differential equation) on  $TM$  then  $\xi = h(\bar{\xi})$  is a semispray on  $TM$  which does not depend on the choice of  $\bar{\xi}$ . We call  $\xi$  the *associated* semispray of  $\Gamma$ . A connection  $\Gamma$  and its associated semispray have same paths.

If  $\xi$  is a semispray on  $TM$  then it can be shown that  $\Gamma = -\mathcal{L}_\xi J$  is a connection on  $TM$  (here  $\mathcal{L}_\xi$  is the Lie derivative and  $\mathcal{L}_\xi J$  is defined by

$$(\mathcal{L}_\xi J)(Y) = ([\xi, JY] - J[\xi, Y]).$$

When  $\xi$  is a spray (homogeneous second-order differential equation) then  $\Gamma = -\mathcal{L}_\xi J$  is a connection on  $M$  such that its associated semispray is precisely  $\xi$ . For a semispray  $\xi$  there is a family of connections  $\Gamma = -\mathcal{L}_\xi J + T$ ,

where  $T$  is a semibasic tensor field of type (1.1) on  $TM$  in equilibrium with  $\xi$  (in fact  $T$  is the strong torsion of  $\Gamma$ ) (see [GR]). In the non-autonomous situation the relation between connections and semisprays becomes much more simpler, as we will show below.

The jet manifold  $J^1(\mathbf{R}, M)$  is fibred over  $\mathbf{R} \times M$ ,  $\mathbf{R}$  and  $M$  with projection maps  $\pi$ ,  $\pi_1$ , and  $\pi_2$ . We notice that  $J^1(\mathbf{R}, M)$  can be identified with  $\mathbf{R} \times TM$  in a very natural way. Therefore we transport the geometric structures defined on  $TM$  to  $J^1(\mathbf{R}, M)$  like the almost tangent structure  $J$  and the Liouville vector field  $C$  on  $TM$ . We may define a new tensor field  $\tilde{J}$  of type (1.1) on  $J^1(\mathbf{R}, M)$  by

$$\tilde{J} = J - C \otimes dt, \quad (1)$$

which is locally characterized by

$$\tilde{J}(\partial/\partial t) = -C; \quad \tilde{J}(\partial/\partial x^i) = \partial/\partial x^i; \quad \tilde{J}(\partial/\partial y^i) = 0 \quad (2)$$

where  $(t, x, y)$  are local coordinates for  $J^1(\mathbf{R}, M)$ .

Hence  $\tilde{J}$  has rank  $m$  and satisfies  $(\tilde{J})^2 = 0$ . We define the adjoint of  $\tilde{J}$ ,  $\tilde{J}^*$ , as the endomorphism of the exterior algebra  $\Lambda(J^1(\mathbf{R}, M))$  of  $J^1(\mathbf{R}, M)$  locally given by

$$\tilde{J}^*(dt) = 0, \quad \tilde{J}^*(dx^i) = 0, \quad \tilde{J}^*(dy^i) = dx^i - y^i dt. \quad (3)$$

Like in the autonomous situation we associate to  $\tilde{J}$  operators  $i_{\tilde{J}}$  and  $d_{\tilde{J}}$  on the algebra  $\Lambda(J^1(\mathbf{R}, M))$  by

$$\left. \begin{aligned} i_{\tilde{J}}\omega(X_1, \dots, X_r) &= \sum_{\ell=1}^r \omega(X_1, \dots, \tilde{J}X_\ell, \dots, X_r), \\ d_{\tilde{J}} &= i_{\tilde{J}}d - di_{\tilde{J}}, \end{aligned} \right\} \quad (4)$$

and so we have

$$\left. \begin{aligned} i_{\tilde{J}}(df) &= \tilde{J}^*(df), \text{ for all } f \text{ on } J^1(\mathbf{R}, M) \\ i_{\tilde{J}}(dt) &= i_{\tilde{J}}(dx^i) = 0; \quad i_{\tilde{J}}(dy^i) = dx^i - y^i dt \end{aligned} \right\} \quad (5)$$

$$\left. \begin{aligned} d_{\tilde{J}}f &= \frac{\partial f}{\partial y^i} (dx^i - y^i dt) \\ d_{\tilde{J}}(dt) &= d_{\tilde{J}}(dx^i) = 0; \quad d_{\tilde{J}}(dy^i) = -d(dx^i - y^i dt) = dy^i \wedge dt. \end{aligned} \right\} \quad (6)$$

In the following we will set

$$\theta^i = dx^i - y^i dt, \quad 1 \leq i \leq m. \quad (7)$$

Also, it is not hard to see that a vector field  $\xi$  on  $J^1(\mathbf{R}, M)$  is a semispray iff  $\theta^i(\xi) = 0$  and  $dt(\xi) = 1$ ,  $1 \leq i \leq m$ . In such a case  $\xi$  is locally given by

$$\xi = \partial/\partial t + y^i \partial/\partial x^i + \xi^i \partial/\partial y^i. \quad (8)$$

Furthermore, a vector field  $\xi$  on  $J^1(\mathbf{R}, M)$  is a semispray iff  $J\xi = C$  and  $\tilde{J}\xi = 0$ .

Let  $\xi$  be a semispray on  $J^1(\mathbf{R}, M)$ . A curve  $s$  in  $M$  is called a *path* of  $\xi$  if its canonical prolongation is an integral curve of  $\xi$ .

Let  $s$  be a curve in  $M$  locally given by  $(x^i(t))$ . Then  $\tilde{s}^1(t) = (t, x^i(t), \dot{x}^i(t))$  and so  $s$  is a path of  $\xi$  if and only if satisfies the following non-autonomous system of differential equations

$$\frac{d^2 x^i}{dt^2} = \xi^i \left( t, x, \frac{dx}{dt} \right), \quad 1 \leq i \leq m$$

where  $\xi$  is given by (8).

### 3. Semisprays and dynamical connections

The tensor fields  $J$  and  $\tilde{J}$  on  $J^1(\mathbf{R}, M)$  permit us to give a characterization of a kind of connections for the fibration  $\pi : J^1(\mathbf{R}, M) \rightarrow \mathbf{R} \times M$ .

DEFINITION (1).— *By a dynamical connection on  $J^1(\mathbf{R}, M)$  we mean a tensor field  $\Gamma$  of type (1.1) on  $J^1(\mathbf{R}, M)$  satisfying*

$$J\Gamma = \tilde{J}\Gamma = \tilde{J}, \quad \Gamma\tilde{J} = -\tilde{J}, \quad \Gamma J = -J. \quad (9)$$

By a straightforward computation from (9) we deduce that the local expressions of  $\Gamma$  are

$$\left. \begin{aligned} \Gamma(\partial/\partial t) &= -y^i \partial/\partial x^i + \Gamma^i \partial/\partial y^i, \\ \Gamma(\partial/\partial x^i) &= \partial/\partial x^i + \Gamma^j_i \partial/\partial y^j, \\ \Gamma(\partial/\partial y^i) &= -\partial/\partial y^i. \end{aligned} \right\} \quad (10)$$

The functions  $\Gamma^i = \Gamma^i(t, x, y)$ ,  $\Gamma^j_i = \Gamma^j_i(t, x, y)$  will be called the *components* of the connection  $\Gamma$ . From (10) we easily deduce that

$$\Gamma^3 - \Gamma = 0 \text{ and } \text{rank}(\Gamma) = 2m.$$

This type of polynomial structure is called  $f(3, -1)$ -structure in the literature (see [YI]). Now, we can associate to  $\Gamma$  two canonical operators  $\underline{\ell}$  and  $\underline{m}$  given by

$$\underline{\ell} = \Gamma^2, \quad \underline{m} = -\Gamma^2 + I.$$

Then we have

$$\underline{\ell}^2 = \underline{\ell}, \quad \underline{m}^2 = \underline{m}, \quad \underline{\ell}\underline{m} = \underline{m}\underline{\ell} = 0, \quad \underline{\ell} + \underline{m} = I, \quad (11)$$

and  $\underline{\ell}$  and  $\underline{m}$  are complementary projectors. From (11) we deduce that  $\underline{\ell}$  and  $\underline{m}$  are locally given by

$$\begin{aligned} \underline{\ell}(\partial/\partial t) &= -y^i \partial/\partial x^i - (\Gamma^i + y^j \Gamma_j^i) \partial/\partial y^i; \quad \underline{\ell}(\partial/\partial x^i) = \\ &= \partial/\partial x^i; \quad \underline{\ell}(\partial/\partial y^i) = \partial/\partial y^i; \quad \underline{m}(\partial/\partial t) = \\ &= \partial/\partial t + y^i \partial/\partial x^i + (\Gamma^i + y^j \Gamma_j^i) \partial/\partial y^i; \quad \underline{m}(\partial/\partial x^i) = \\ &= \underline{m}(\partial/\partial y^i) = 0. \end{aligned} \quad (12)$$

If we put  $\mathcal{L} = I m \underline{\ell}$ ,  $\mathcal{M} = I m \underline{m}$ , then we have that  $\mathcal{L}$  and  $\mathcal{M}$  are complementary distributions on  $J^1(\mathbf{R}, M)$ , that is,

$$T(J^1(\mathbf{R}, M)) = \mathcal{M} \oplus \mathcal{L}.$$

From (12) we deduce that  $\mathcal{L}$  is  $2m$ -dimensional and is locally spanned by  $\{\partial/\partial x^i, \partial/\partial y^i\}$ .  $\mathcal{M}$  is one-dimensional, globally spanned by the vector field  $\xi = \underline{m}(\partial/\partial t)$ . Taking into account the local expression of  $\xi$ , we deduce that  $\xi$  is a semispray which will be called the *canonical semispray associated to the dynamical connection*  $\Gamma$ .

Furthermore, we have  $\Gamma^2 \underline{\ell} = \underline{\ell}$  and  $\Gamma \underline{m} = 0$ . Thus  $\Gamma$  acts on  $\mathcal{L}$  as an *almost product structure* and trivially on  $\mathcal{M}$ . Since  $\mathcal{M} = \ker \Gamma$ ,  $\Gamma$  is said to be an  $f(3, -1)$ -structure on  $J^1(\mathbf{R}, M)$  of *rank*  $2m$  and *parallelizable kernel*. Moreover,  $\Gamma/\mathcal{L}$  has eigenvalues  $+1$  and  $-1$ . From (10) the eigenspaces corresponding to the eigenvalue  $-1$  are the vertical subspaces  $V_z$ ,  $z \in J^1(\mathbf{R}, M)$ . Recall that for each  $z \in J^1(\mathbf{R}, M)$ ,  $V_z$  is the set of all tangent vectors to  $J^1(\mathbf{R}, M)$  at  $z$  which are projected to 0 by  $T\pi$ . Thus  $V$  is a distribution given by  $z \mapsto V_z$ . The eigenspace at  $z \in J^1(\mathbf{R}, M)$  corresponding to the eigenvalue  $+1$  will be denoted by  $H_z$  and called the *strong-horizontal subspace* at  $z$ . We have a canonical decomposition

$$T_z(J^1(\mathbf{R}, M)) = \mathcal{M}_z \oplus H_z \oplus V_z,$$

and obviously,

$$T(J^1(\mathbf{R}, M)) = \mathcal{M} \oplus H \oplus V, \quad (13)$$

where  $H$  is the distribution  $z \mapsto H_z$ .

Let us put  $H'_z = \mathcal{M}_z \oplus H_z$ ;  $H'_z$  will be called the *weak-horizontal subspace at  $z$* . Then we have the following decompositions

$$T_z(J^1(\mathbf{R}, M)) = H'_z \oplus V_z, \quad z \in J^1(\mathbf{R}, M)$$

and

$$T(J_1(\mathbf{R}, M)) = H' \oplus V, \quad (14)$$

where  $H' : x \rightarrow H'_x$  is the corresponding distribution.

We notice that  $\mathcal{L}, \mathcal{M}, H$  and  $H'$  may be considered as vector bundles over  $J^1(\mathbf{R}, M)$ ; the bundles  $H$  and  $H'$  will be called *strong* and *weak-horizontal bundles*, respectively. Thus, from (14)  $\Gamma$  defines a connection on the fibration  $\pi : J^1(\mathbf{R}, M) \rightarrow \mathbf{R} \times M$  with horizontal bundle  $H'$  (see ROUX [R] and de LEON & RODRIGUES [DLR 1]). But *not every connection on the fibration  $\pi : J^1(\mathbf{R}, N) \rightarrow \mathbf{R} \times M$  arises in this way*.

A vector field  $X$  on  $J^1(\mathbf{R}, M)$  which belongs to  $H$  (resp.  $H'$ ) will be called a *strong* (resp. *weak*) horizontal vector field. From (14), we have that the canonical projection  $\pi : J^1(\mathbf{R}, M) \rightarrow \mathbf{R} \times M$  induces an isomorphism

$$\pi_* : H'_z \rightarrow T_{\pi(z)}(\mathbf{R} \times M), \quad z \in J^1(\mathbf{R}, M).$$

Then, if  $X$  is a vector field on  $\mathbf{R} \times M$ , there exists a unique vector field  $X^{H'}$  on  $J^1(\mathbf{R}, M)$  which is weak-horizontal and projects to  $X$ . The projection of  $X^{H'}$  to  $H$  will be denoted by  $X^H$ .

From (10) and by a straightforward computation, we obtain

$$\begin{aligned} (\partial/\partial t)^{H'} &= \partial/\partial t + (\Gamma^j + \frac{1}{2} y^i \Gamma^j_i) \partial/\partial y^j \\ (\partial/\partial x^i)^{H'} &= \partial/\partial x^i + \frac{1}{2} \Gamma^j_i \partial/\partial y^j. \end{aligned} \quad (15)$$

Then, if we put  $H_i = (\partial/\partial x^i)^{H'}$  and  $V_i = \partial/\partial y^i$ , one deduces that  $\{\xi, H_i, V_i\}$  is a local basis of vector fields on  $J^1(\mathbf{R}, M)$ . In fact,  $\mathcal{M} = \langle \xi \rangle$ ,  $H = \langle H_i \rangle$ , and  $V = \langle V_i \rangle$ ;  $\{\xi, H_i, V_i\}$  is called an *adapted basis* to the  $f(3, -1)$ -structure  $\Gamma$ . In terms of  $\{\xi, H_i, V_i\}$  (15) becomes

$$(\partial/\partial t)^{H'} = \xi - y^i H_i, \quad (\partial/\partial x^i)^{H'} = H_i.$$

Therefore, we obtain

$$(\partial/\partial t)^H = -y^i H_i, \quad (\partial/\partial x^i)^H = H_i.$$

If  $X = \tau \partial/\partial t + X^i \partial/\partial x^i$  is a vector field on  $R \times M$ , we have

$$X^H = (X^i - \tau y^i) H_i \quad (16)$$

(compare with CRAMPIN, PRINCE and THOMPSON [CPT]). Finally, we notice that the dual local basis of 1-forms of the adapted basis  $\{\xi, H_i, V_i\}$  is given by  $\{dt, \theta^i, \psi^i\}$ , where  $\theta^i = dx^i - y^i dt$ , and  $\psi^i = -(\Gamma^i + \frac{1}{2} y^j \Gamma_j^i) dt - \frac{1}{2} \Gamma_j^i dx^j + dy^i$ . This fact can be shown by a straightforward computation.

Let  $\xi$  be a semispray on  $J^1(\mathbf{R}, M)$  and suppose that  $\xi$  is locally expressed by

$$\xi = \partial/\partial t + y^i \partial/\partial x^i + \xi^i \partial/\partial y^i. \quad (17)$$

Then a simple computation in local coordinates shows that

$$\left. \begin{aligned} [\xi, \partial/\partial t] &= -\frac{\partial \xi^j}{\partial t} \partial/\partial y^j, \\ [\xi, \partial/\partial x^i] &= -\frac{\partial \xi^j}{\partial x^i} \partial/\partial y^j, \\ [\xi, \partial/\partial y^i] &= -\frac{\partial}{\partial x^i} - \frac{\partial \xi^j}{\partial y^i} \partial/\partial y^j. \end{aligned} \right\} \quad (18)$$

PROPOSITION (1).— *Let  $\Gamma = -\mathcal{L}_\xi \tilde{J}$ . Then  $\Gamma$  is a dynamical connection on  $J^1(\mathbf{R}, M)$  whose associated semispray is, precisely,  $\xi$ .*

*Proof.*— In fact from (18) we have

$$\left. \begin{aligned} \Gamma(\partial/\partial t) &= -y^i \partial/\partial x^i - \left( y^j \frac{\partial \xi^i}{\partial y^j} - \xi^i \right) \partial/\partial y^i, \\ \Gamma(\partial/\partial x^i) &= \partial/\partial x^i + \frac{\partial \xi^j}{\partial y^i} \partial/\partial y^j, \\ \Gamma(\partial/\partial y^i) &= -\partial/\partial y^i. \end{aligned} \right\} \quad (19)$$

Now, from (19) we easily deduce that  $\Gamma$  is a dynamical connection on  $J^1(\mathbf{R}, M)$ . Furthermore, taking into account (12), we have that the associated semispray to  $\Gamma$  is, precisely,  $\xi$ .



From Proposition (1) we may observe that the theory of dynamical connections on  $J^1(\mathbf{R}, M)$  is more simpler than the theory of connections on  $TM$ .

Let  $\Gamma$  be a dynamical connection on  $J^1(\mathbf{R}, M)$ .

DEFINITION (2).— *A curve  $u : \mathbf{R} \rightarrow M$  is called a path of  $\Gamma$  if the canonical prolongation  $j^1u$  of  $u$  to  $J^1(\mathbf{R}, M)$  is a weak-horizontal curve.*

Now, we shall find the differential equations for the paths of  $\Gamma$  (the dots meaning time derivatives).

If  $u : \mathbf{R} \rightarrow M$  is locally given by  $t \mapsto (x^i(t))$ , then we have  $j^1u(t) = (t, x^i(t), \dot{x}^i(t))$ . Hence,

$$\dot{j}^1 u(t) = \frac{\partial}{\partial t} + \frac{dx^i}{dt} \frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2} \frac{\partial}{\partial y^i}.$$

Therefore,  $u$  is a path of  $\Gamma$  if and only if  $\psi^i(\dot{j}^1 u(t)) = 0$ ,  $1 \leq i \leq m$ , that is  $u$  satisfies the following system of differential equations :

$$\frac{d^2x^i}{dt^2} = \Gamma^i \left( t, x, \frac{dx}{dt} \right) + \Gamma_j^i \left( t, x, \frac{dx}{dt} \right) \frac{dx^j}{dt}. \quad (20)$$

Let  $\xi$  be the associated semispray of  $\Gamma$ . Then  $\xi$  is locally given by

$$\xi = \partial/\partial t + y^i \partial/\partial x^i + \xi^i \partial/\partial y^i,$$

where  $\xi^i = \Gamma^i + y^j \Gamma_j^i$ ,  $1 \leq i \leq m$ .

From (20) it is clear that the paths of  $\Gamma$  and  $\xi$  satisfy the same system of differential equations. Then we have

PROPOSITION (2).— *A dynamical connection and its associated semispray on  $J^1(\mathbf{R}, M)$  have the same paths.*

#### 4. Dynamical connections and non-autonomous regular Lagrangian equations

Suppose that a non-autonomous regular Lagrangian  $L$  is given, that is,  $L$  is a non-degenerate real function on  $J^1(\mathbf{R}, M) = \mathbf{R} \times TM$ . Then it is

well-known that an extremal for  $L$  is a curve  $s : \mathbf{R} \rightarrow M$  (or a section of  $(\mathbf{R} \times M, p, \mathbf{R})$ ) such that

$$(\tilde{s})^*(i_X dL \wedge dt) = 0 \quad (21)$$

for all vertical vector fields on  $\mathbf{R} \times TM$ . Also, it is known that (21) is equivalent to

$$(\tilde{s}^2)^*(i_X d\Omega_L) = 0, \quad (22)$$

for all  $\pi_1$ -vertical vector fields on  $J^1(\mathbf{R}, M)$ . In (22)  $\Omega_L$  is the POINCARÉ-CARTAN canonical form on  $J^1(\mathbf{R}, M)$  locally given by

$$\Omega_L = L(t, x, y)dt + \frac{\partial L}{\partial y^i} \theta^i,$$

where  $\theta^i$  is defined in (7) of section 2.

In terms of the tensor field  $\tilde{J}$  and  $J$  and the Liouville vector field  $C$  on  $J^1(\mathbf{R}, M)$ , the POINCARÉ-CARTAN form takes the following expression :

$$\Omega_L = L dt + \frac{\partial L}{\partial y^i} \theta^i = L dt + d\tilde{J}L,$$

or equivalently,

$$\begin{aligned} \Omega_L &= L dt + \frac{\partial L}{\partial y^i} dx^i - y^i \frac{\partial L}{\partial y^i} dt = \left( L - y^i \frac{\partial L}{\partial y^i} \right) dt + d_J L \\ &= (L - CL)dt + d_J L = d_J L - E_L dt; E_L = CL - L. \end{aligned}$$

Thus

$$\Theta_L = d\Omega_L = dd\tilde{J}L + dL \wedge dt$$

or

$$\Theta_L = dd_J L - dE_L \wedge dt.$$

A straightforward computation in local coordinates shows that

$$\Theta_L \wedge \cdots \wedge \Theta_L = \pm \det \left( \frac{\partial^2 L}{\partial y^j \partial y^i} \right) dx^1 \wedge \cdots \wedge dx^m \wedge dy^1 \wedge \cdots \wedge dy^m$$

and if  $L$  is a non-autonomous regular Lagrangian we deduce that  $\Theta_L$  is a contact form on  $J^1(\mathbf{R}, M)$ . Consequently, the *characteristic* bundle of  $\Theta_L$

$$R_{\Theta_L} = \{v \in T(J^1(\mathbf{R}, M)); i_v \Theta_L = 0\}$$

has one-dimensional fibers, that is, they are a line-bundle over  $J^1(\mathbf{R}, M)$ . Let us recall here that a vector field  $X$  on  $J^1(\mathbf{R}, M)$  is *characteristic* if  $X$  is a section of  $R_{\Theta_L}$ , that is,  $i_X \Theta_L = 0$ . The following result can be compared with the corresponding one for autonomous Lagrangian [see [DLR 1]].

**PROPOSITION (3).**— *Let  $L$  be a non-autonomous regular Lagrangian on  $J^1(\mathbf{R}, M)$  and  $\xi$  a characteristic vector field which satisfies  $i_\xi dt = 1$ . Then  $\xi$  is a semispray on  $J^1(\mathbf{R}, M)$  whose paths are the solutions of the Lagrange equations*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad 1 \leq i \leq m.$$

*We call  $\xi$  the Lagrange vector field for  $L$ .*

**THEOREM (1).**— *Let  $L$  be a non-autonomous regular Lagrangian on  $J^1(\mathbf{R}, M)$  and let  $\xi$  be a Lagrange vector field for  $L$ . Then there exists a dynamical connection  $\Gamma$  on  $J^1(\mathbf{R}, M)$  whose paths are the solutions of the Lagrange equations. This connection is given by  $\Gamma = -\mathcal{L}_\xi \tilde{J}$ .*

*Proof.*— From Proposition (1) we deduce that  $\Gamma = -\mathcal{L}_\xi \tilde{J}$  is a dynamical connection whose associated semispray is precisely  $\xi$ . Thus the theorem follows directly from Proposition (2) and (3).

Finally, let us remark that the results of CRAMPIN, PRINCE and THOMPSON [CPT] can be re-obtained in terms of  $\Gamma$ . In fact, with the notation of Section 3 we have a local basis of vector fields on  $J^1(\mathbf{R}, M)$  given by  $\{\xi, H_i, V_i\}$  where  $H_i$  is given by

$$H_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \frac{\partial \xi^j}{\partial y^i} \frac{\partial}{\partial y^j}.$$

Thus the corresponding dual basis is  $\{dt, \theta^i, \psi^i\}$ , where

$$\psi^i = - \left( \xi^i - \frac{1}{2} y^j \frac{\partial \xi^i}{\partial y^j} \right) dt - \frac{1}{2} \frac{\partial \xi^i}{\partial y^j} dx^j + dy^i.$$

The significance of this dual basis is that the form  $\Theta_L$  can be re-written as follows

$$\Theta_L = \frac{\partial^2 L}{\partial y^i \partial y^j} \theta^i \wedge \psi^j$$

and so the semispray  $\xi$  is uniquely determined by the equations

$$i_\xi \theta^i = i_\xi \psi^i = 0, \quad i_\xi dt = 1.$$

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