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Generalized Hopf manifolds with flat local Kaelher metrics

SORIN DRAGOMIR⁽¹⁾ AND RENATA GRIMALDI⁽²⁾

RÉSUMÉ. — On donne un résultat du type B.Y. Chen et M. Okumura (voir [3]) sur la courbure scalaire d'une sous-variété M d'une variété de Vaisman (c'est-à-dire une variété localement conformément Kaelhérienne ayant la forme de Lee parallèle et les métriques locales Kaelhériennes plates). Si M est une sous-variété de Cauchy-Riemann Levi-plate (d'une variété de Vaisman), alors on calcule les courbures sectionnelles complexes de M .

ABSTRACT. — We give a B.Y. Chen and M. Okumura (see [3]) type result on the scalar curvature of a submanifold M of a Vaisman manifold (i.e. a locally conformal Kähler manifold having a parallel Lee form and flat local Kähler metrics). If M is a Levi-flat Cauchy-Riemann submanifold (of a Vaisman manifold), the complex sectional curvatures of M are estimated.

1. Introduction and statement of results

Let (M, g, J) be a Hermitian manifold of complex dimension n , with the complex structure J and the Hermitian metric g . It is *locally conformal Kähler* (l.c.K.) if there exists an open covering $(U_i)_{i \in I}$ of M and a family $(f_i)_{i \in I}$ of real valued smooth functions $f_i \in C^\infty(U_i)$ such that each $g_i = \exp(-f_i)g$ is a Kähler metric on U_i , $i \in I$.

The local 1-forms df_i of a l.c.K. manifold M are known to glue up to a globally defined (closed) 1-form ω on M , namely the *Lee form*.

A l.c.K. manifold is a *generalized Hopf* (g.H.) manifold if its Lee form is parallel with respect to the Riemannian connection of (M, g) . Typical

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examples of g.H. manifolds are products $S \times \mathbb{R}$ between a Sasaki manifold S and the real line, see [10], p. 614.

Let M be a g.H. manifold. It is said to be a *Vaisman manifold* if the local Kaehler metrics $g_i, i \in I$, of M are flat. Each complex Hopf manifold $CH^n = W/G_d, W = C^n - \{0\}, G_d = \{d^m I : m \in z\}, d \in C - \{0\}, |d| \neq 1$, is a Vaisman manifold in a natural way. Indeed, let $g_0 = |z|^{-2} \delta_{ij} dz^i \otimes d\bar{z}^j$, where $|z|^2 = \delta_{ij} z^i \bar{z}^j$ and (z^1, \dots, z^n) are the natural complex analytic coordinates on W . Note that g_0 is G_d -invariant, thus giving rise to a (globally defined) l.c.K. metric on CH^n .

Let M be a Vaisman manifold. Since the Lee form ω is parallel, its norm is constant; set $\|\omega\| = 2c, c \in \mathbb{R} - \{0\}$. The local structure of Vaisman manifolds is completely understood due to a deep result of I. Vaisman, (thus justifying our terminology), i.e. the theorem 3.8. in [12], p. 277, asserting that the universal covering of M is W with the metric $\rho^2 g_0, \rho = \frac{1}{c}$.

The curvature form of a Vaisman manifold is expressed by

$$(1.1) \quad R(X, Y)Z = \frac{1}{4} \{ [\omega(X)Y - \omega(Y)X] \omega(Z) + [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)] B \} + \frac{1}{4} \|\omega\|^2 \{ g(Y, Z)X - g(X, Z)Y \}$$

for any tangent vector fields X, Y, Z on M , see (2.1) of [13], p. 441. Here $B = \omega^\#$ is the *Lee field* of M , while $\#$ denotes raising of indices with respect to g . As a consequence of (1.1) one obtains the following results :

THEOREM 1. — *Let M be an n -dimensional submanifold of a Vaisman manifold. If the scalar curvature ρ of M is subject to :*

$$(1.2) \quad \rho \geq (n - 2) \|h\|^2 + (n - 2)(n - 1)c^2 + 2(n - 1)A$$

at a point $x \in M$ for some $A \in \mathbb{R}$ then the sectional curvatures of M are $\geq A$ at the point x .

If \bar{M} is a Vaisman manifold and $j : M \rightarrow \bar{M}$ the given immersion of M in \bar{M} , then h denotes the second fundamental form of j . Let ω_0 be the Lee form of \bar{M} and $\omega = j^* \omega_0$. Since ω is closed, the distribution $\text{Ker}(\omega)$ is integrable thus defining a canonical foliation \mathcal{F} on M , see also [4].

THEOREM 2. — *Let M be a Levi flat Cauchy-Riemann submanifold of a Vaisman manifold. Let $p \in G_2(M), p \subseteq D_{\pi(p)}, J(p) = p$. Then the complex*

sectional curvature k_C of M verifies :

$$(1.3) \quad k_C(p) \leq c^2 - \omega_0(h(X, X))$$

for any $X \in p, \|X\| = 1$. The equality holds if and only if p is tangent to some leaf of \mathcal{F} passing through x and $h_x = 0$ on $p \times p$.

Here $\pi : G_2(M) \rightarrow M$ denotes the Grassman bundle of all 2-planes tangent to M . Also D stands for the Levi distribution of the C.R. submanifold M , (i.e. D_x is the maximal holomorphic subspace of $T_x(M)$, $x \in M$).

For other results concerning the geometry of (the second fundamental form of) submanifolds in l.c.K. manifolds see [4], [5], [6], [7], [8].

2. Scalar curvature of submanifolds in Vaisman manifolds

Let M be an n -dimensional submanifold of a Vaisman manifold $(\overline{M}, \overline{g}, J)$. By (2.8) in [4], p. 203, the Gauss equation of M in \overline{M} is given by :

$$(2.1) \quad \begin{aligned} R(X, Y)Z &= A_{h(Y, Z)}X - A_{h(X, Z)}Y + \\ &+ \frac{1}{4}\{\omega(x)Y - \omega(Y)X\}\omega(Z) + \\ &+ [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)]B + \\ &+ \frac{1}{4}\|\omega_0\|^2\{g(Y, Z)X - g(X, Z)Y\} \end{aligned}$$

for any tangent vector fields X, Y, Z on M . Here $g = j^*\overline{g}$. Moreover A_ξ is the Weingarten operator (associated with the normal section ξ). Suitable contraction of indices in (2.1) leads to the expression of the Ricci tensor of (M, g) , i.e.

$$(2.2) \quad \begin{aligned} R_{jk} &= h_{jk}^a \text{Trace}(A_a) - g^{is} h_{ik}^a h_{js}^b \delta_{ab} + \\ &+ c^2(n-2)g_{jk} - \frac{n-2}{4}\omega_j\omega_k \end{aligned}$$

Indices i, j, k, \dots run from 1 to n , while a, b, c, \dots from 1 to $\text{codim}(M) = 2m - n$. Further contraction of indices in (2.2) gives :

$$(2.3) \quad \rho = n^2\|H\|^2 - \|h\|^2 + c^2(n-1)(n-2)$$

Here ρ, H denote respectively the scalar curvature of (M, g) and the mean curvature vector (i.e. $H = \frac{1}{n} \text{Trace}(h)$) of the given immersion j .

Let $k : G_2(M) \rightarrow I\mathbf{R}$ be the sectional curvature of (M, g) . Let $p \in G_2(M)$ and $\{X, Y\}$ an orthonormal basis in p . By (2.1) one obtains :

$$(2.4) \quad k(p) = \bar{g}(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2 + c^2 - \frac{1}{4}\{\omega(X)^2 + \omega(Y)^2\}$$

At this point we may prove our Theorem 1. To this end, let $x \in M$ and (U, x^i) be normal coordinates at x .

Substitution from (2.3) into(1.2) furnishes :

$$(2.5) \quad n^2\|H\|^2 \geq (n-1)\|h\|^2 + 2(n-1)A$$

Let $\xi_a, 1 \leq a \leq 2m-n, \dim(\bar{M}) = 2m$, be an orthonormal frame in the normal bundle $T(M)^\perp$ of the given immersion. For simplicity, we may choose ξ_1 to be collinear with H at x (if $H_x \neq 0$, and arbitrary if $H_x = 0$ occurs). Let $X_i = \frac{\partial}{\partial x^i}, 1 \leq i \leq n$. We set $h(X_i, X_j) = h_{ij}^a \xi_a$. Also $h_{ja}^i = g^{ik} h_{jk}^a, h_a^{ij} = g^{jk} g^{is} h_{ks}^a$. Clearly $h_{ij}^a = h_{ji}^a$. All computations are carried out at x (where $g_{ij} = \delta_{ij}$) so that $h_{ji}^a = h_{ja}^i = h_a^{ji}$ at x . Let us put $h_{ij} = h_{ij}^1$. Then :

$$(2.6) \quad n^2\|H\|^2 = \left(\sum_{i=1}^n h_{ii} \right)^2$$

Substitution from (2.6) into (2.5) gives :

$$(2.7) \quad \left(\sum_{i=1}^n h_{ii} \right)^2 \geq (n-1) \left\{ \sum_{i=1}^n (h_{ii})^2 + \sum_{i \neq j} (h_{ij})^2 + \sum_{\substack{a \geq 2 \\ 1 \leq i, j \leq n}} (h_{ij}^a)^2 \right\} + 2(n-1)A$$

since $\|h\|^2 = h_{ji}^a h_a^{ji}$. We shall need the following :

LEMMA .— (B.Y. Chen and M. Okumura, [3])

Let a_1, \dots, a_n, b be real numbers, $n > 1$, with the property :

$$\left(\sum_{i=1}^n a_i \right)^2 \geq (n-1) \sum_{i=1}^n (a_i)^2 + b$$

Then for any $i \neq j$ one has $2a_i a_j \geq \frac{b}{n-1}$.

At this point we may use (2.7) and the Lemma (for $a_i = h_{ii}$) such as to yield :

$$(2.8) \quad \begin{aligned} h_{ii}h_{jj} - (h_{ij})^2 &\geq \\ &\geq \sum_{a=2}^{2m-n} \{ |h_{ii}^a h_{jj}^a| + (h_{ij}^a)^2 \} + A \end{aligned}$$

for any $i \neq j$. Set $\omega_i = \omega(X_i)$. Then $\omega_i^2 + \omega_j^2 \leq \sum_{i=1}^n \omega_i^2 = \|\omega\|^2 = 4c^2$. Let $p_{ij} \in G_2(M)$ be spanned by $X_i, X_j, i \neq j$. Finally, using (2.4) and (2.8) we have :

$$\begin{aligned} k(p_{ij}) &= \sum_{a=1}^{2m-n} (h_{ii}^a h_{jj}^a - (h_{ij}^a)^2) + \\ &\quad + c^2 - \frac{1}{4}(\omega_i^2 + \omega_j^2) \geq A \end{aligned}$$

Q.E.D.

This extends Theorem 4.1. in [1], p. 55, to the case of submanifolds in Vaisman manifolds.

3. Cauchy-Riemann submanifolds of Vaisman manifolds

Let $(\overline{M}, \overline{g}, J)$ be a Vaisman manifold of complex dimension m and M a real n -dimensional Cauchy-Riemann (C.R.) submanifold of \overline{M} . That is M carries a pair of orthogonal (complementary) distributions D, D^\perp such that D is holomorphic, i.e. $J_x(D_x) = D_x, x \in M$, while D^\perp is totally-real, i.e. $J_x(D_x^\perp) \subseteq T_x(M)^\perp, x \in M$. See also [15], p. 83. Hereafter D is called the *Levi distribution* of M . Moreover, if D is integrable, the C.R. submanifold M is said to be *Levi flat*.

Let $B_0 = \omega_0^\#$ and $A_0 = -JB_0$ be the Lee, respectively the *anti-Lee vector* fields of \overline{M} . Also $\theta_0 = \omega_0 \circ J$ will denote the *anti-Lee form*.

Let X, ξ be respectively a tangent vector field on M and a normal section. We set $PX = \tan(J, X), FX = \text{nor}(JX), t\xi = \tan(J\xi), f\xi = \text{nor}(J\xi)$.

Here \tan_x, nor_x denote the natural projections associated with the direct sum decomposition $T_x(\overline{M}) = T_x(M) \oplus T_x(M)^\perp$, for any $x \in M$. Note that

P is D -valued. Also $F = 0$ on D . Moreover the following identities hold :

$$(3.1) \quad \begin{aligned} F \circ P &= 0, f \circ F = 0 \\ P^2 + t \circ F &= -I \\ t \circ f &= 0, P \circ t = 0 \\ f^2 + F \circ t &= -I \end{aligned}$$

Let $B = \tan(B_0)$, $B^\perp = \text{nor}(B_0)$, $A = \tan(A_0)$ and $A^\perp = \text{nor}(A_0)$. Note that :

$$(3.2) \quad A = -PB - tB^\perp, A^\perp = -FB - fB^\perp$$

The complex structure J is not parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of \bar{M} . Nevertheless \bar{M} admits a significant almost complex connection \bar{D} , namely the *Weyl connection*, i.e.

$$(3.3) \quad \begin{aligned} \bar{D}_X Y &= \bar{\nabla}_X Y - \frac{1}{2} \{ \omega_0(X)Y + \omega_0(Y)X - \\ &\quad - \bar{g}(X, Y)B_0 \} \end{aligned}$$

Since $\bar{D}J = 0$, (3.3) yields :

$$(3.4) \quad \begin{aligned} \bar{\nabla}_X JY &= J\bar{\nabla}_X Y + \\ &\quad + \frac{1}{2} \{ \theta_0(Y)X + \omega_0(Y)JX - \\ &\quad - \bar{\Omega}(X, Y)B_0 - \bar{g}(X, Y)A_0 \} \end{aligned}$$

Here $\bar{\Omega}$ denotes the Kaehler 2-form of \bar{M} . By (3.4) and the Gauss formula (1.10) of [1], p. 38, one obtains :

$$(3.5) \quad \begin{aligned} \nabla_X JY &= P\nabla_X Y + th(X, Y) + \\ &\quad + \frac{1}{2} \{ \theta(Y)X + \omega(Y)JX - \\ &\quad - \Omega(X, Y)B - g(X, Y)A \} \end{aligned}$$

$$(3.6) \quad \begin{aligned} h(X, JY) &= fh(X, Y) + F\nabla_X Y - \\ &\quad - \frac{1}{2} \{ \Omega(X, Y)B^\perp + g(X, Y)A^\perp \} \end{aligned}$$

for any $X, Y \in D$. Here ∇ denotes the Levi-Civita connection of (M, g) and $\theta = j^*\theta_0$, $\Omega = j^*\bar{\Omega}$.

Let us denote by k_C the restriction of the sectional curvature k of M to the holomorphic 2-planes $p \in G_2(M)$, $J(p) = p$, with the property $p \subseteq D_x$, $x = \pi(p)$, $x \in M$. Then k_C is called the *complex sectional curvature* of the C.R. submanifold M .

At this point we may prove our Theorem 2. As the Levi distribution D is integrable, one has $F\nabla_X Y = F\nabla_Y X$, for any $X, Y \in D$. By (3.5)-(3.6) one obtains :

$$(3.7) \quad h(JX, JY) = -h(X, Y) - g(X, Y)B^\perp$$

for any $X, Y \in D$. Let us apply (2.4) for the 2-plane $p \in G_2(M)$ spanned by $\{X, JX\}$, $X \in D_x$, $\|X\| = 1$, $x = \pi(p)$. It follows :

$$k_C(p) = \bar{g}(h(X, X), h(JX, JX)) - \|h(X, JX)\|^2 + \quad (3.8) \\ + c^2 - \frac{1}{4}\{\omega(X)^2 + \theta(X)^2\}$$

Finally (3.7)-(3.8) lead to (1.3). If equality holds, then $h(X, X) = 0$, $\omega(X) = 0$, $\omega(JX) = 0$, (and actually (1.3) reads $k_C(p) = c^2$). The converse is obvious, Q.E.D.

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