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Slant surfaces of codimension two

BANG-YEN CHEN⁽¹⁾ and YOSHIHIKO TAZAWA⁽²⁾

RÉSUMÉ. — Une immersion isométrique d'une variété riemannienne dans une variété presque hermitienne est dite oblique si l'angle de Wirtinger est constant [1]. Le but de cet article est d'étudier et de caractériser les surfaces obliques dans le plan complexe \mathbb{C}^2 utilisant l'application de Gauss. Nos démontrons que chaque surface n'ayant aucun point tangent complexe dans une variété presque hermitienne \tilde{M} de dimension 4 est oblique par rapport à une structure presque complexe bien choisie sur \tilde{M} .

ABSTRACT. — A slant immersion is an isometric immersion from a Riemannian manifold into an almost Hermitian manifold with constant Wirtinger angle [1]. In this article we study and characterize slant surfaces in the complex 2-plane \mathbb{C}^2 via the Gauss map. We also prove that every surface without complex tangent points in a 4-dimensional almost Hermitian manifold \tilde{M} is slant with respect to a suitably chosen almost complex structure on \tilde{M} .

1. Introduction

Let $x : M \rightarrow \tilde{M}$ be an isometric immersion of a Riemannian manifold M with Riemannian metric g into an almost Hermitian manifold \tilde{M} with an almost complex structure \tilde{J} and an almost hermitian metric \tilde{g} . For each nonzero vector X tangent to M at $p \in M$, the angle $\theta(X)$ between $\tilde{J}X$ and the tangent space T_pM of M at p is called the Wirtinger angle of X . The immersion x is said to be *general slant* if the Wirtinger angle $\theta(X)$ is constant (which is independent of the choice of $p \in M$ and $X \in T_pM$). In this case the angle θ is called the *slant angle* of the slant immersion.

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If x is a totally real (or Lagrangian) immersion, then $T_p^\perp M \supseteq \tilde{J}(T_p M)$ for any $p \in M$ where $T_p^\perp M$ is the normal space of M in \tilde{M} at p . Thus a totally real immersion is a general slant immersion with Wirtinger angle $\theta \equiv \pi/2$. If M is also an almost Hermitian manifold with almost complex structure J , then the immersion $x : M \rightarrow \tilde{M}$ is called *holomorphic* (respectively, *anti-holomorphic*) if we have

$$x_*(JX) = \tilde{J}(x_*X) \quad (\text{respectively, } x_*(JX) = -\tilde{J}(x_*X)) \quad (1.1)$$

for any $X \in T_p M$. It is clear that holomorphic and anti-holomorphic immersions are general slant immersions with $\theta \equiv 0$. A general slant immersion which is neither holomorphic nor anti-holomorphic is simply called a *slant immersion* [1]. In paragraph 2 we review the geometry of the Grassmannian $G(2, 4)$ for later use. In paragraph 3 we investigate the relationship between 2-planes in the Euclidean 4-space E^4 and the complex structures on E^4 . By applying the relationship we obtain a pointwise observation concerning slant surfaces. In paragraph 4 we study slant surfaces via their Gauss map. In particular we obtain in this section a new characterization of slant surfaces and we also prove that a non-minimal surface in E^4 can be slant with respect to at most four compatible complex structures on E^4 . In paragraph 5 we prove that every Hermitian manifold is a proper slant surface with any prescribed slant angle with respect to a suitable almost complex structure. In the last section we define doubly slant surfaces and show that their Gaussian and the normal curvatures vanish identically.

2. Geometry of $G(2, 4)$

In the section we recall some results concerning the geometry of the Grassmannian $G(2, 4)$ of oriented 2-planes in E^4 (for details, see [2, 3, 7, 8]).

Let $E^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$ be the Euclidean m -space with the canonical inner product $\langle \cdot, \cdot \rangle$. Let $\{\epsilon_1, \dots, \epsilon_m\}$ be the canonical basis of E^m . Then $\Psi := \epsilon_1 \wedge \dots \wedge \epsilon_m$ gives the canonical orientation of E^m . For each $n \in \{1, \dots, m\}$, the space $\wedge^n E^m$ is an $\binom{m}{n}$ -dimensional real vector space with the inner product, also denote by $\langle \cdot, \cdot \rangle$, defined by

$$\langle X_1 \wedge \dots \wedge X_n, Y_1 \wedge \dots \wedge Y_n \rangle = \det(\langle X_i, Y_j \rangle) \quad (2.1)$$

and be extended linearly. The two vector spaces $\bigwedge^n(E^m)^*$ and $(\bigwedge^n E^m)^*$ are identified in a natural way by

$$\Phi(X_1 \wedge \cdots \wedge X_n) = \Phi(X_1, \dots, X_n) \quad (2.2)$$

for any $\Phi \in \bigwedge^n(E^m)^*$ and any $X_1, \dots, X_n \in E^m$. The Grassmannian $G(n, m)$ of oriented n -planes in E^m was identified with the set $D_1(n, m)$ of unit decomposable n -vectors in $\bigwedge^n E^m$. The identification $\varphi : G(n, m) \rightarrow D_1(n, m)$ is given by $\varphi(V) = X_1 \wedge \cdots \wedge X_n$ for any positive orthonormal basis $\{X_i\}$ of $V \in G(n, m)$.

The star operator $*$: $\bigwedge^2 E^4 \rightarrow \bigwedge^2 E^4$ is defined by

$$(*\xi, \eta)\Psi = \xi \wedge \eta \quad \text{for } \xi, \eta \in \bigwedge^2 E^4. \quad (2.3)$$

If we regard a $V \in G(2, 4)$ as an element in $D_1(2, 4)$ via φ , we have $*V = V^\perp$, where V^\perp is the oriented orthogonal complement of V in E^4 . Since $*$ is a symmetric involution, $\bigwedge^2 E^4$ is decomposed into the following orthogonal direct sum :

$$\bigwedge^2 E^4 = \bigwedge_+^2 E^4 \oplus \bigwedge_-^2 E^4 \quad (2.4)$$

of eigenspaces of $*$ with eigenvalues 1 and -1 , respectively. Denote by π_+ and π_- the natural projections from $\bigwedge^2 E^4$ into $\bigwedge_+^2 E^4$ and $\bigwedge_-^2 E^4$, respectively.

Given a positive orthonormal basis $\{e_1, \dots, e_4\}$ of E^4 , we put

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), & \eta_2 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4), \\ \eta_3 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), & \eta_4 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4), \\ \eta_5 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4), & \eta_6 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3). \end{aligned} \quad (2.5)$$

Then $\{\eta_1, \eta_2, \eta_3\}$ and $\{\eta_4, \eta_5, \eta_6\}$ are orthonormal bases of $\bigwedge_+^2 E^4$ and $\bigwedge_-^2 E^4$ respectively. We shall orient $\bigwedge_+^2 E^4$ and $\bigwedge_-^2 E^4$ so that these two bases are positive.

For any $\xi \in D_1(2, 4)$ we have

$$\pi_+(\xi) = \frac{1}{2}(\xi + *\xi) \quad \text{and} \quad \pi_-(\xi) = \frac{1}{2}(\xi - *\xi). \quad (2.6)$$

If we denote by S_+^2 and S_-^2 the 2-spheres centered at the origin with radius $1/\sqrt{2}$ in $\Lambda_+^2 E^4$ and $\Lambda_-^2 E^4$, respectively, then we have

$$\pi_+ : D_1(2, 4) \rightarrow S_+^2 \quad \text{and} \quad \pi_- : D_1(2, 4) \rightarrow S_-^2 \quad (2.7)$$

and

$$D_1(2, 4) = S_+^2 \times S_-^2. \quad (2.8)$$

3. Complex structures on E^4

Let $\mathbb{C}^2 = (\mathbb{R}^2, \langle \cdot, \cdot \rangle, J_0)$ be the complex plane with the canonical (almost) complex structure J_0 defined by $J_0(a, b, c, d) = (-b, a, -d, c)$. J_0 is an orientation preserving isomorphism. We denote by \mathcal{G} the set of all (almost) complex structures on E^4 which are compatible with the inner product $\langle \cdot, \cdot \rangle$, i.e.,

$$\begin{aligned} \mathcal{G} = \{ & J : E^4 \rightarrow E^4 \mid J \text{ is linear, } J^2 = -\text{Id}, \\ & \langle JX, JY \rangle = \langle X, Y \rangle, \text{ for any } X, Y \in E^4 \}. \end{aligned}$$

For each $J \in \mathcal{G}$, we choose an (orthonormal) J -basis $\{e_1, \dots, e_4\}$ so that $Je_1 = e_2, Je_3 = e_4$. Two J -bases of the same complex structure J have the same orientation. By using the canonical orientation $\Psi = \epsilon_1 \wedge \dots \wedge \epsilon_4$, we divide \mathcal{G} into two disjoint subsets :

$$\mathcal{G}^+ = \{J \in \mathcal{G} \mid J\text{-bases are positive,}\}$$

and

$$\mathcal{G}^- = \{J \in \mathcal{G} \mid J\text{-bases are negative.}\}$$

For each $J \in \mathcal{G}$, there is a unique 2-vector $\zeta_J \in \Lambda^2 E^4$ defined as follows.

Let Ω_J be the Kaehler form of J , i.e.,

$$\Omega_J(X, Y) = \langle X, JY \rangle, \quad (3.1)$$

for any $X, Y \in E^4$. The 2-vector ζ_J associated with J is defined to be the unique 2-vector satisfying

$$\langle \zeta_J, X \wedge Y \rangle = -\Omega_J(X, Y), \quad \text{for any } X, Y \in E^4 \quad (3.2)$$

LEMMA 3.1.. — *The mapping*

$$\zeta : \mathcal{G} \rightarrow \bigwedge^2 E^4 \quad (3.3)$$

defined by $\zeta(J) = \zeta_J$ gives rise to two bijections :

$$\zeta : \mathcal{G}^+ \rightarrow S_+^2(\sqrt{2}) \quad \text{and} \quad \zeta : \mathcal{G}^- \rightarrow S_-^2(\sqrt{2}), \quad (3.4)$$

where $S_+^2(\sqrt{2})$ and $S_-^2(\sqrt{2})$ are the 2-spheres centered at the origin with radius $\sqrt{2}$ in $\bigwedge_+^2 E^4$ and $\bigwedge_-^2 E^4$, respectively.

Proof. — Let $J \in \mathcal{G}$ and $\{e_1, \dots, e_4\}$ be a J -basis. If $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$), then $\{e_1, \dots, e_4\}$ is positive (respectively, negative). Since $\zeta_J = e_1 \wedge e_2 + e_3 \wedge e_4$ by (3.1), ζ maps \mathcal{G}^+ into $S_+^2(\sqrt{2})$ and maps \mathcal{G}^- into $S_-^2(\sqrt{2})$. Their injectivity are clear.

Conversely, for each $\xi \in S_+^2(\sqrt{2})$, we have $\frac{1}{2}\zeta \in S_+^2$. Hence, we can pick an oriented 2-plane V such that $V \in \pi_+^{-1}(\frac{1}{2}\zeta)$. Now we choose a positive orthonormal basis $\{e_1, \dots, e_4\}$ such that $\{e_1, e_2\}$ is a positive basis of V . Let J be the complex structure on E^4 such that $Je_1 = e_2, Je_2 = -e_1, Je_3 = e_4, Je_4 = -e_3$. Then $J \in \mathcal{G}^+$ and $\zeta_J = \xi$.

If $\xi \in S_-^2(\sqrt{2})$, we pick $V \in \pi_-^{-1}(\frac{1}{2}\zeta)$ and define J by $Je_1 = e_2, Je_2 = -e_1, Je_3 = -e_4, Je_4 = e_3$. Then we have a similar result. \square

In the following we identify $\mathcal{G}, \mathcal{G}^+$ and \mathcal{G}^- with $S_+^2(\sqrt{2}) \cup S_-^2(\sqrt{2}), S_+^2(\sqrt{2})$, and $S_-^2(\sqrt{2})$, respectively, via ζ .

For each $V \in G(2, 4)$ and each $J \in \mathcal{G}$, we define

$$\alpha_J(V) = \cos^{-1}(-\Omega_J(V)). \quad (3.5)$$

Then $\alpha_J(V) \in [0, \pi]$. A 2-plane V is said to be a -slant if $\alpha_J(V) = a$.

The relation between $\theta(X)$ defined in paragraph 1 and $\alpha_J(V)$ is as follows. Let x be an isometric immersion of M into $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. If we regard $(T_p \widetilde{M}, \widetilde{g}, \widetilde{J})$ as a complex plane with the induced inner product $\langle \cdot, \cdot \rangle$, then we have

$$\theta(X) = \min\{\alpha_{\widetilde{J}}(T_p M), \pi - \alpha_{\widetilde{J}}(T_p M)\} \quad (3.6)$$

for any non-zero vector $X \in T_p M$.

If M is oriented, then M has a unique complex structure J determined by its orientation and its metric induced from \tilde{g} . With respect to α_J , we have:

$$x \text{ is holomorphic} \Leftrightarrow \alpha_{\tilde{J}}(T_p M) \equiv 0, \quad (3.7)$$

$$x \text{ is anti-holomorphic} \Leftrightarrow \alpha_{\tilde{J}}(T_p M) \equiv \pi, \quad (3.8)$$

$$x \text{ is totally real} \Leftrightarrow \alpha_{\tilde{J}}(T_p M) \equiv \frac{\pi}{2}. \quad (3.9)$$

The argument above also hold for the case $\dim \tilde{M} > 4$. We note that α_J coincides with the angle defined in [6].

LEMMA 3.2

(i) If $J \in \mathcal{G}^+$, then $\alpha_J(V)$ is the angle between $\pi_+(V)$ and ζ_J .

(ii) If $J \in \mathcal{G}^-$, then $\alpha_J(V)$ is the angle between $\pi_-(V)$ and ζ_J .

Proof. — If $J \in \mathcal{G}^+$, then, by (3.2), (3.5) and lemma 3.1, we have

$$\cos(\alpha_J(V)) = -\Omega_J(V) = \langle \zeta_J, V \rangle = \langle \zeta_J, \pi_+(V) + \pi_-(V) \rangle = \langle \zeta_J, \pi_+(V) \rangle$$

which is the cosine of the angle between $\pi_+(V)$ and ζ_J , since $\|\zeta_J\| = \sqrt{2}$ and $\|\pi_+(V)\| = 1/\sqrt{2}$. Similar argument applies to the case $J \in \mathcal{G}^-$. \square

For each $a \in [0, \pi]$ and $J \in \mathcal{G}$, we define

$$G_{J,a} = \{V \in G(2, 4) \mid \alpha_J(V) = a\}, \quad (3.10)$$

i.e., $G_{J,a}$ is the set of all oriented 2-planes in E^4 which are a -slant with respect to J . Also for each $a \in [0, \pi]$ and $V \in G(2, 4)$ we define

$$\mathcal{G}_{V,a} = \{J \in \mathcal{G} \mid \alpha_J(V) = a\}, \quad (3.11)$$

i.e., $\mathcal{G}_{V,a}$ is the set of all compatible complex structure on E^4 with respect to which V is a -slant. We put

$$\mathcal{G}_{V,a}^+ = \mathcal{G}_{V,a} \cap \mathcal{G}^+ \quad \text{and} \quad \mathcal{G}_{V,a}^- = \mathcal{G}_{V,a} \cap \mathcal{G}^-. \quad (3.12)$$

By applying lemma 3.2 we obtain the following generalization of proposition 2 of [3].

PROPOSITION 3.3

- (i) If $J \in \mathcal{G}^+$, then $G_{J,\alpha} = S_{J,\alpha}^+ \times S_-^2$ where $S_{J,\alpha}^+$ is the circle on S_+^2 consisting of 2-vectors which make constant angle α with ζ_J .
- (ii) If $J \in \mathcal{G}^-$, then $G_{J,\alpha} = S_+^2 \times S_{J,\alpha}^-$ where $S_{J,\alpha}^-$ is the circle on S_-^2 consisting of 2-vectors which make constant angle α with ζ_J .
- (iii) Via the identification ζ given in (3.4), $\mathcal{G}_{V,\alpha}^+$ is a circle on $S_+^2(\sqrt{2})$ consisting of 2-vectors in $S_+^2(\sqrt{2})$ which make constant angle α with $\pi_+(V)$. Similarly, $\mathcal{G}_{V,\alpha}^-$ is a circle on $S_-^2(\sqrt{2})$ obtained in a similar way.

For simplicity, for each $V \in G(2, 4)$, we define J_V^+ and J_V^- as the complex structures given by

$$J_V^+ = \zeta^{-1}(\pi_+(V)) \quad \text{and} \quad J_V^- = \zeta^{-1}(\pi_-(V)), \quad (3.13)$$

where ζ is the bijection given in lemma 3.1. It is clear that $J_V^+ \in \mathcal{G}^+$ and $J_V^- \in \mathcal{G}^-$.

4. Slant surfaces and Gauss map

Let $x : M \rightarrow E^4$ be an isometric immersion from an oriented surface M into E^4 . We denote by ∇ and $\tilde{\nabla}$ the Riemannian connections of M and E^4 , respectively. We choose a positive orthonormal local frame field $\{e_1, \dots, e_4\}$ in E^4 such that, restricted to M , the vectors e_1, e_2 give a positive frame field tangent to M (and hence e_3, e_4 are normal to M). Let $\omega^1, \dots, \omega^4$ be the field of dual frame field. The structure equations of E^4 are then given by

$$\tilde{\nabla}e_A = \sum \omega_A^B \otimes e_B, \quad \omega_A^B + \omega_B^A = 0, \quad (4.1)$$

$$d\omega^B = \sum \omega_A^B \otimes \omega^A, \quad (4.2)$$

$$d\omega_B^A = - \sum \omega_C^A \wedge \omega_B^C, \quad A, B, C = 1, 2, 3, 4. \quad (4.3)$$

If we restrict these forms to M , then $\omega^3 = \omega^4 = 0$. From $d\omega^3 = d\omega^4 = 0$, (4.2), and Cartan's lemma, we have

$$\omega_i^r = \sum h_{ij}^r \omega^j, \quad h_{ij}^r = h_{ji}^r, \quad r = 3, 4; \quad i, j = 1, 2. \quad (4.4)$$

The Gauss curvature G and the normal curvature G^D of M in E^4 are given respectively by

$$G = \sum (h_{11}^r h_{22}^r - h_{12}^r h_{12}^r), \quad (4.5)$$

$$G^D = (h_{11}^3 - h_{22}^3)h_{12}^4 - (h_{11}^4 - h_{22}^4)h_{12}^3. \quad (4.6)$$

In the following we denote by $\nu : M \rightarrow G(2, 4)$ the *Gauss map* of the immersion x defined by

$$\nu(p) = (e_1 \wedge e_2)(p), \quad (4.7)$$

where we identify $G(2, 4)$ with $D_1(2, 4)$ consisting of unit decomposable 2-vectors in $\bigwedge^2 E^4$. We define two maps ν_+ and ν_- by

$$\nu_+ = \pi_+ \circ \nu \quad \text{and} \quad \nu_- = \pi_- \circ \nu, \quad (4.8)$$

where π_+ and π_- are the projections given in paragraph 2. Then ν_+ and ν_- map M into S_+^2 and S_-^2 , respectively.

In terms of ν_+ and ν_- , we have the following characterization of slant surfaces in \mathbb{C}^2 .

PROPOSITION 4.1. — *Let $x : M \rightarrow E^4$ be an isometric immersion from an oriented surface M into E^4 . Then x is a slant immersion with respect to a complex structure $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$) if and only if $\nu_+(M)$ is contained in a circle on S_+^2 (respectively, $\nu_-(M)$ is contained in a circle on S_-^2).*

Moreover, x is α -slant with respect to $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$) if and only if $\nu_+(M)$ is contained in a circle $S_{J,\alpha}^+$ on S_+^2 (respectively, $\nu_-(M)$ is contained in a circle $S_{J,\alpha}^-$ on S_-^2), where $S_{J,\alpha}^+$ and $S_{J,\alpha}^-$ are the circles defined in proposition 3.3.

Proof. — If $x : M \rightarrow E^4$ is α -slant with respect to $J \in \mathcal{G}^+$, then by (3.11) and proposition 3.3, we have $\nu(T_p M) \in S_{J,\alpha}^+ \times S_-^2$ for any $p \in M$. Thus $\nu_+(M)$ is contained in a circle $S_{J,\alpha}^+$ on S_+^2 consisting of 2-vectors in S_+^2 which make angle α with ζ_J .

Conversely, if $x : M \rightarrow E^4$ is an immersion such that $\nu_+(M)$ is contained in a circle S^1 on S_+^2 . Let η be a vector in $\bigwedge^2 E^4$ with length $\sqrt{2}$ which is normal to the 2-plane in $\bigwedge^2 E^4$ containing S^1 . Then $\eta \in S_+^2(\sqrt{2})$. By

lemma 3.1, there is a unique $J \in \mathcal{G}^+$ such that $\zeta_J = \eta$. It is clear that S^1 is a $S_{J,a}^+$ for some constant angle a . Therefore, by proposition 3.3, the immersion x is a -slant with respect to $J \in \mathcal{G}^+$. Similar argument applies to the other case. \square

The following lemma was obtained in [1] (given in the proof of theorem 1 of [1]). Here we reprove it by using Gauss map.

LEMMA 4.2. — *Let $x : M \rightarrow E^4$ be an isometric immersion from a surface M into E^4 . Then M is minimal and slant with respect to some $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$) if and only if $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton.*

Proof. — If x is minimal, then both ν_+ and ν_- are anti-holomorphic [5]. In particular, ν_+ and ν_- are open maps if they are not constant maps. However, if x is slant with respect to $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$), then, by proposition 4.1, ν_+ (respectively, ν_-) cannot be an open map. Hence, $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton.

Conversely, if $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton, say $\{\xi\}$. Then $2\xi \in S_+^2(\sqrt{2})$. Thus, by lemma 3.1, there is a unique $J = \zeta^{-1}(2\xi) \in \mathcal{G}^+$ such that $\nu_+(M)$ is contained in $S_{J,0}^+$. Thus x is holomorphic with respect to J (see proposition 2 of [3]), in particular, x is a minimal immersion. Because a singleton $\{\xi\}$ lies in every circle S^1 on S_+^2 through ξ , proposition 4.1 implies that for any $a \in [0, \pi]$, there exists a $J_a \in \mathcal{G}^+$ (respectively, $J_a \in \mathcal{G}^-$), such that x is a -slant with respect to J_a . \square

By applying proposition 4.1 and lemma 4.2 we have the following result concerning non-minimal surfaces.

THEOREM 4.3. — *If $x : M \rightarrow E^4$ is not minimal, then there exist at most two complex structures $\pm J \in \mathcal{G}^+$ and at most two complex structures $\pm J' \in \mathcal{G}^-$ such that x is slant with respect to them.*

Proof. — If x is a non-minimal, a -slant immersion with respect to a complex structure $J \in \mathcal{G}^+$ (respectively, $J' \in \mathcal{G}^-$), then, by proposition 4.1 and lemma 4.2, $\nu_+(M)$ (respectively, $\nu_-(M)$) contains an arc of the circle $S_{J,a}^+$ (respectively, $S_{J',a}^-$).

Thus, $\pm J$ and $\pm J'$ are the only possible complex structures which make x to be slant according to proposition 4.1. \square

From proposition 4.1, lemma 4.2 and theorem 4.3 we have the following.

PROPOSITION 4.4. — *Let $x : M \rightarrow E^4$ be an isometric immersion from a surface M into E^4 . Then the following statements are equivalent:*

- (i) *x is minimal and slant with respect to some complex structure $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$);*
- (ii) *$\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton;*
- (iii) *x is holomorphic with respect to some complex structure $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$);*
- (iv) *for each $a \in [0, \pi]$, there exist $J_a \in \mathcal{G}^+$ (respectively, $J_a \in \mathcal{G}^-$) such that x is a -slant with respect to J_a .*

Theorem 4.3 and proposition 4.4 provide us a clear geometric understanding of lemmas 5 and 6 and theorems 1, 3 and 4 of [1].

5. Slant surfaces in 4-dimensional almost Hermitian manifolds

Let $x : M \rightarrow (\widetilde{M}, J)$ be an immersion of a manifold M into an almost complex manifold (\widetilde{M}, J) . Then a point $p \in M$ is called a *complex tangent point* if the tangent plane of M at p is invariant under the action of J . The purpose of this section is to prove the following.

THEOREM 5.1. — *Let $x : M \rightarrow (\widetilde{M}, g, J)$ be an imbedding from an oriented surface M into an almost Hermitian manifold (\widetilde{M}, g, J) . If x has no complex tangent point, for any prescribed angle $a \in (0, \pi)$, there exists an almost complex structure \widetilde{J} on \widetilde{M} satisfying the following conditions:*

- (i) *$(\widetilde{M}, g, \widetilde{J})$ is an almost Hermitian manifold, and*
- (ii) *x is a -slant with respect to \widetilde{J} .*

Proof. — \widetilde{M} has a natural orientation determined by J . At each point $p \in \widetilde{M}$, $(T_p \widetilde{M}, g_p)$ is a Euclidean 4-space and so we can apply the argument given in paragraphs 2 and 3.

According to (2.4) the vector bundle $\Lambda^2(\widetilde{M})$ of 2-vectors on \widetilde{M} is a direct sum of two vector subbundles.

$$\Lambda^2(\widetilde{M}) = \Lambda_+^2(\widetilde{M}) \oplus \Lambda_-^2(\widetilde{M}). \quad (5.1)$$

We define two sphere-bundles over \widetilde{M} by

$$\begin{aligned} S_+^2(\widetilde{M}) &= \left\{ \xi \in \Lambda_+^2(\widetilde{M}) \mid \|\xi\| = \frac{1}{\sqrt{2}} \right\}, \\ \overline{S}_+^2(\widetilde{M}) &= \left\{ \xi \in \Lambda_-^2(\widetilde{M}) \mid \|\xi\| = \sqrt{2} \right\}. \end{aligned} \quad (5.2)$$

By applying lemma 3.1 we can identify a cross-section

$$\gamma : \widetilde{M} \rightarrow \overline{S}_+^2(\widetilde{M}) \quad (5.3)$$

with an almost complex structure J_γ on \widetilde{M} such that $(\widetilde{M}, g, J_\gamma)$ is an almost Hermitian manifold. In the following we denote by ρ the cross-section corresponding to J and we want to construct another cross-section $\tilde{\sigma}$ to obtain the desired almost complex structure \tilde{J} .

We consider the pull-backs of these bundles via the immersion x , i.e.,

$$\begin{aligned} \Lambda_+^2(M) &= x^*(\Lambda_+^2(\widetilde{M})), \\ S_+^2(M) &= x^*(S_+^2(\widetilde{M})), \\ \overline{S}_+^2(M) &= x^*(\overline{S}_+^2(\widetilde{M})). \end{aligned} \quad (5.4)$$

The tangent bundle TM determines a cross-section $\tau : M \rightarrow S_+^2(M)$ given by

$$\tau(p) = \pi_+(T_p M), \quad \text{for any } p \in M, \quad (5.5)$$

where π_+ is the projection of $\Lambda^2(T_p \widetilde{M})$ onto $\Lambda_+^2(T_p \widetilde{M})$. Note that 2τ is a cross-section of $\overline{S}_+^2(M)$;

$$2\tau : M \rightarrow \overline{S}_+^2(M). \quad (5.6)$$

We denote $x^*\rho$ also by ρ , which gives us another cross-section:

$$\rho = x^*\rho : M \rightarrow \overline{S}_+^2(M). \quad (5.7)$$

Since x has no complex tangent point,

$$\rho(p) \neq \pm 2\tau(p), \quad \text{for any } p \in M. \quad (5.8)$$

Therefore, $\rho(p)$ and $2\tau(p)$ determine a 2-plane in $\Lambda_+^2(T_p \widetilde{M})$ which intersects the circle $(\mathcal{G}_{\tau,a}^+)_p$ at two points, where $(\mathcal{G}_{\tau,a}^+)_p$ is the circle on $(\overline{S}_+^2(M))_p$ determined in proposition 3.3 with $V = T_p M$. Let $\sigma(p)$ be one of the two points which lies on the half-great-circle on $(\overline{S}_+^2(M))_p$ starting from $2\tau(p)$ and passing through $\rho(p)$. Since ρ and τ are differentiable, so is σ . Thus we get the third cross-section:

$$\sigma : M \rightarrow \overline{S}_+^2(M) \quad (5.9)$$

and we want to extend σ to a cross-section $\tilde{\sigma}$ of $\overline{S}_+^2(\widetilde{M})$.

For each $p \in M$, we choose an open neighborhood U_p of p in \widetilde{M} such that $\sigma|_{U_p \cap M}$ can be extended to a cross-section of $\overline{S}_+^2(\widetilde{M})$ on U_p :

$$\sigma_p : U_p \rightarrow \overline{S}_+^2(\widetilde{M})|_{U_p}. \quad (5.10)$$

We identify here the manifold M with its image $x(M)$ via the imbedding. We put

$$U = \bigcup_{p \in M} U_p \quad (5.11)$$

and pick a locally finitely countable refinement $\{U_i\}_{i=1}^\infty$ of the open covering $\{U_p\}_{p \in M}$ of U . For each i we pick a point $p \in M$ such that U_i is contained in U_p and put

$$\sigma_i = \sigma_p|_{U_i}. \quad (5.12)$$

Let $\{\phi_i\}$ be a differentiable partition of unity on U subordinated to the covering $\{U_i\}$. We define a cross-section $\overline{\sigma}$ of $\Lambda_+^2(\widetilde{M})|_U$ by

$$\overline{\sigma} : U \rightarrow \Lambda_+^2(\widetilde{M})|_U \quad ; \quad \overline{\sigma} = \sum \phi_i \sigma_i. \quad (5.13)$$

By the construction of σ_i and $\overline{\sigma}$ we have

$$\overline{\sigma}|_M = \sigma. \quad (5.14)$$

Since the angle $\angle(\overline{\sigma}(p), \rho(p)) < \pi$ for any $p \in M$, we have

$$\overline{\sigma}(p) \neq 0, \quad \angle(\overline{\sigma}(p), \rho(p)) < \pi \text{ for any } p \in M. \quad (5.15)$$

By continuity of $\overline{\sigma}$ we can pick an open neighborhood W of M contained in U such that

$$\overline{\sigma}(q) \neq 0, \quad \angle(\overline{\sigma}(q), \rho(q)) < \pi \text{ for any } q \in W. \quad (5.16)$$

We define a cross-section $\widehat{\sigma}$ of $\overline{S}_+^2(\widetilde{M})|_W$ by

$$\widehat{\sigma} : W \rightarrow \overline{S}_+^2(\widetilde{M})|_W; \quad \widehat{\sigma} = \frac{\overline{\sigma}}{\sqrt{2\|\overline{\sigma}\|}}. \quad (5.17)$$

Then we have $\angle(\hat{\sigma}(q), \rho(q)) < \pi$ for any $q \in M$ too. Finally, we consider the open covering $\{W, \widetilde{M} - M\}$ of \widetilde{M} and local cross-section $\hat{\sigma}$ and $\rho|_{\widetilde{M}-M}$ and repeat the same argument using a partition of unity subordinate to $\{W, \widetilde{M} - M\}$ to get a cross-section $\tilde{\sigma} : \widetilde{M} \rightarrow \widetilde{S}_+^2(\widetilde{M})$ satisfying $\tilde{\sigma}|_M = \sigma$. Now, it is clear that the almost complex structure \tilde{J} corresponding to $\tilde{\sigma}$ is the desired almost complex structure. \square

6. Appendix : double slant surfaces in \mathbb{C}^2

An immersion $x : M^2 \rightarrow E^4$ is called *doubly slant* if it is slant with respect to a complex structure $J \in \mathcal{G}^+$ and at the same time it is slant with respect to another complex structure $\hat{J} \in \mathcal{G}^-$.

Equivalently, the immersion x is doubly slant if and only if there exists a $V \in G(2, 4)$ such that x is slant with respect to both J_V^+ and J_V^- , where J_V^+ and J_V^- are defined by (3.13).

PROPOSITION 6.1. — *If $x : M^2 \rightarrow E^4$ is a doubly slant immersion, then $G = G^D = 0$.*

Proof. — If x is doubly slant, then, by proposition 4.1, we know that $\nu_+(M)$ and $\nu_-(M)$ both lie in some circles on S_+^2 and S_-^2 , respectively. Thus, both $(\nu_+)_*$ and $(\nu_-)_*$ are singular at every point $p \in M$. The result then follows from the following lemma of [7].

LEMMA 6.2. — *For an isometric immersion $x : M^2 \rightarrow E^4$, we have*

$$\text{Jacobian of } \nu_+ = \frac{G^D + G}{2} \quad ; \quad \text{Jacobian of } \nu_- = \frac{G^D - D}{2}.$$

This lemma can be proved in our notation as follows.

From (4.8) we have $\nu_*X = (\tilde{\nabla}_X e_1) \wedge e_2 + e_1 \wedge (\tilde{\nabla}_X e_2)$ for any X tangent to M . Thus, by applying (2.5), (4.1) and (4.4), we find (see [3])

$$\begin{aligned} \nu_*X = \frac{1}{\sqrt{2}} \left\{ (\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 + \right. \\ \left. + (-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6 \right\}. \end{aligned} \tag{6.1}$$

Since $\{\eta_2, \eta_3\}$ is a positive orthonormal basis of $T_{\nu_+(p)}S_+^2$ and $\{\eta_5, \eta_6\}$ is a positive orthonormal basis of $T_{\nu_-(p)}S_-^2$, we obtain

$$\begin{aligned} (\nu_+)_*X &= \frac{1}{\sqrt{2}} \left\{ (\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 \right\}, \\ (\nu_-)_*X &= \frac{1}{\sqrt{2}} \left\{ (-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6 \right\}. \end{aligned} \tag{6.2}$$

By applying (4.4), (4.6), (4.7) and (6.2) we obtain the lemma.

Remark 6.3. — Examples 1 through 6 of [1] are examples of doubly slant surfaces in \mathbb{C}^2 . Here we give another example of doubly slant surfaces.

Example. — For any two nonzero real numbers p and q , we consider the following immersion from $\mathbb{R} \times (0, \infty)$ into \mathbb{C}^2 defined by

$$x(u, \nu) = (p\nu \sin u, p\nu \cos u, \nu \sin qu, \nu \cos qu). \tag{6.3}$$

The slant angles and the ranks of ν, ν_+ and ν_- of those examples can be determined by direct computation. We list them as the following table.

	Slant angles	rank ν	rank ν_+	rank ν_-
Example 1	$a = b = \frac{\pi}{2}$	2	1	1
Example 2	$a = b = \frac{\pi}{2}$	1	1	1
Example 3	$a = b = \frac{\pi}{2}$	2	1	1
Example 4	$a = b \in \left[0, \frac{\pi}{2}\right]$	0	0	0
Example 5	$a \in \left[0, \frac{\pi}{2}\right], b = \frac{\pi}{2}$	2	1	1
Example 6	$a = b \in \left(0, \frac{\pi}{2}\right)$	non-constant	1	1
Example A	$a, b \in \left(0, \frac{\pi}{2}\right)$	1	1	1

Except for example 4, a and b above are the slant angles with respect to $J_0 = J_{\epsilon_1 \wedge \epsilon_2}^+$ and $J_1 = J_{\epsilon_1 \wedge \epsilon_2}^-$ (cf. (3.13)). For example 4, a is the slant angle with respect to J_0 and b is the slant angle with respect to $J_{\epsilon_1 \wedge \epsilon_3}^-$.

In [4] further results on slant immersions have been obtained.

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