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Annales de la faculté des sciences de Toulouse 6^e série, tome 2, n^o 3
(1993), p. 429-440

http://www.numdam.org/item?id=AFST_1993_6_2_3_429_0

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On a theorem of Enriques – Swinnerton-Dyer^(*)

ALEXEI N. SKOROBOGATOV⁽¹⁾

RÉSUMÉ. — On propose ici une nouvelle démonstration de l'énoncé classique suivant : chaque surface sur le corps k qui, sur la clôture algébrique de k devient isomorphe à un plan projectif avec quatre points en position générale éclatés, a un point rationnel. Nous retrouvons toutes ces surfaces comme les "quotients" d'une variété de Grassmann $G(3,5)$ par rapport à l'action de tores maximaux du groupe linéaire $GL(5)$.

ABSTRACT. — We propose a new proof of the following classical statement: every surface over a field k , which over an algebraic closure of k becomes isomorphic to the projective plane with four points in general position blown-up, has a rational point. In fact all such surfaces can be obtained as "quotients" of a Grassmannian variety $G(3,5)$ by the action of maximal tori of the general linear group $GL(5)$.

1. Introduction

Let k be a perfect field. The aim of this note is to give a new proof of the following statement formulated by Enriques [3] in 1897 and proved by Swinnerton-Dyer [11] in 1970.

THEOREM . — *Any del Pezzo k -surface of degree 5 has a k -point.*

(*) Reçu le 12 mai 1992

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Partially supported by the Russian fundamental research foundation (project 93-012-458)

This statement is usually used to prove the k -rationality of such a surface. The proof of [11] is indirect, so it appears that the present proof, which is conceptually very simple, is of some interest⁽²⁾.

Recall that a del Pezzo surface of degree 5 is defined as a k -form of the projective plane \mathbb{P}^2 with 4 points in general position blown-up (we shall call this a split del Pezzo surface of degree 5; “general position” simply means that no three points are collinear). In fact, we prove that for any del Pezzo k -surface X of degree 5 there exists a maximal k -torus $T \subset GL(5)$, such that X is isomorphic to the orbit space of T on the set of semistable points of the natural action of T on the Grassmannian $G(3, 5)$. If k is infinite we have plenty of semistable k -points since semistable points form a dense open subset of $G(m, n)$. For arbitrary k the existence of a k -point on X follows from a simple statement known as the lemma of Lang – Nishimura ([6], [9]).

The action of the group of diagonal matrices of $GL(n)$ on $G(m, n)$ is briefly discussed in Section 2. In Section 3 we show that for $m = 3$ and $n = 5$ the corresponding space of (semi)stable orbits is no other than \mathbb{P}^2 with four points blown-up, a fact probably well known to experts (cf. [2]). Note that the automorphism group of the split del Pezzo surface of degree 5 is precisely the Weyl group $W(A_4)$, isomorphic to the group of permutations on five elements. We prove the main theorem in Section 4 (Theorem 4.4) by combining these geometric facts with a formal argument in Galois cohomology.

2. Torus action on Grassmannians : the split case

Let $V = k \oplus \cdots \oplus k$, $\dim(V) = n$, be the vector space with a fixed decomposition into the direct sum of one-dimensional subspaces. Let $SL(n)$ be the group of linear transformations of V with determinant 1, and $D \subset SL(n)$ be the subgroup of diagonal matrices. Consider the Grassmannian $G(m, n)$ of m -dimensional subspaces of V with the natural (right) action of $SL(n)$. The restriction of this action to D was studied in a series of papers by I. M. Gelfand and his colleagues (see, for example, [4]). Let us recall some of their constructions. In what follows $A \subset B$ means that $A \subseteq B$ and $A \neq B$.

⁽²⁾ After this paper has been sent to a journal, the author became aware that N. I. Shepherd-Barron has recently obtained another simple proof of this theorem (“The rationality of quintic del Pezzo surfaces – A short proof.” Bull. London Math. Soc. 24 (1992), pp. 249-250).

Let $I_n := \{1, 2, \dots, n\}$. Choose $e_i \in V$ to be a vector whose i -th coordinate is non-zero, and all the other coordinates are zeros. For $I \subseteq I_n$ define $V_I \subseteq V$ as the subspace generated by $e_i, i \in I$. Let f be a function from the subsets of I_n to non-negative integers. Define a constructible algebraic set $U_f \subset G(m, n)$ whose points are the subspaces $S \subset V$ such that $\dim(S \cap V_I) = f(I)$ for all $I \subseteq I_n$. We have a decomposition $G(m, n) = \bigcup_f U_f$. Obviously, U_f are D -invariant. The unique dense open set $U_0 = U_{f_0}$ parametrizes the subspaces S in general position with respect to all V_I . It is given by

$$f_0(I) = \max\{0, m + \#I - n\}.$$

It is often simpler to work not with f but with another function defined by:

$$r(I) = m - f(I_n \setminus I).$$

Let $S \subset V$ be the subspace corresponding to a point of $G(m, n)$.

Choose a basis in S , and decompose it with respect to the coordinate system $V = k \oplus \dots \oplus k$. Let M be the resulting matrix. One checks that for a subset $I \subseteq I_n$ the value $r(I)$ is the rank of the submatrix of M of size $(m \times \#I)$ consisting of the columns with numbers in I (see, e.g. [4, (1.1)]). In particular, the function $r_0(I) = m - f_0(I_n \setminus I) = \min\{\#I, m\}$ describes the matrices whose every m columns are linearly independent.

We are interested in “the quotient” of $G(m, n)$ by D . For this reason we consider stable and semistable points of $G(m, n)$ with respect to the ample sheaf $\mathcal{O}(1)$ corresponding to the Plücker embedding. (This makes sense because $SL(n)$ acts linearly on V , thus $\mathcal{O}(1)$ has an $SL(n)$ -linearization, see [8, Chap. 4, § 4].)

LEMMA 2.1. — *The set $G(m, n)^s$ (resp. $G(m, n)^{ss}$) of stable (resp. semistable) points of $G(m, n)$ with respect to D and $\mathcal{O}(1)$ is the union of U_f for f satisfying $f(I) < (m/n) \#I$ (resp. $f(I) \leq (m/n) \#I$) for all $I \subset I_n$.*

Proof. — This follows from the proof of [8, Prop. 4.3]. \square

If m and n are coprime then $(m/n) \#I$ is never an integer for $\#I < n$, and the lemma implies that $G(m, n)^s = G(m, n)^{ss}$.

The condition of stability can be reformulated as follows:

$$r(I) > (m/n) \#I \quad \text{for all nonempty } I \subseteq I_n. \quad (1)$$

This implies that M does not contain a zero column.

By geometric invariant theory [8, (1.10)] there exists a quasiprojective scheme Y , and a morphism $\phi : G(m, n)^{\text{ss}} \rightarrow Y$ satisfying $\phi(xt) = \phi(x)$, $t \in D$, which is the universal categorical quotient [8, Def. 0.5]. According to the remark following the proof of [8, (1.11)], Y is proper over k . Moreover, there is an open set $Y' \subseteq Y$ such that $\phi^{-1}(Y') = G(m, n)^{\text{s}}$, and $\phi : G(m, n)^{\text{s}} \rightarrow Y'$ is the universal geometric quotient [8, Def. 0.6]. Y' has the property that every fibre $\phi^{-1}(y)$, $y \in Y'$, is an orbit of D . Note that up to an isomorphism, Y and Y' do not depend on the choice of a decomposition $V = k \oplus \cdots \oplus k$, or, equivalently, on the choice of a split maximal torus $D \subset SL(n)$.

LEMMA 2.2. — Let $\varepsilon \in GL(n)$ be a diagonal matrix $[\varepsilon_i \delta_{ij}]$, $\varepsilon_i \in k^*$.

Define the decomposition $I_n = \bigcup_{r=1}^p J_r$ such that $\varepsilon_i = \varepsilon_j$ if and only if $\{i, j\} \subseteq J_r$ for some r . A subspace $S \subset V$ is ε -invariant if and only if

$$S = \bigoplus_{r=1}^p (S \cap V_{J_r}).$$

Let $i : SL(n) \rightarrow PGL(n)$ be the canonical isogeny such that $\text{Ker}(i)$ is the center of $SL(n)$. Let $T := i(D)$.

COROLLARY 2.3. — Let $x \in U_f \subset G(m, n)$. Then the stabilizer of x in T is trivial if and only if there does not exist a decomposition

$$I_n = \bigcup_{r=1}^p J_r, \quad p \geq 2, \quad \text{such that} \quad \sum_{r=1}^p f(J_r) = m.$$

In particular, this is true for the points of $G(m, n)^{\text{s}}$.

PROPOSITION 2.4. — The restriction of ϕ to $G(m, n)^{\text{s}} \rightarrow Y'$ endows $G(m, n)^{\text{s}}$ with the structure of a Y' -torsor under T . In particular, Y' is smooth.

Proof. — If $\#I = m$ the condition $r(I) = m$ defines an invariant dense open set $Z_I \subset G(m, n)$ (given by the non-vanishing of the corresponding determinant, or in other words, the corresponding Plücker coordinate). These form an open covering of $G(m, n)$. Let us construct a family of invariant open subsets of Z_I such that each of them is a trivial torsor under T .

In fact, we shall use the constructions of chapter 3 of the book [8]. Assume $I = \{1, \dots, m\}$. Define an R -partition of $\{1, \dots, m\}$ as an ordered set of subsets S_1, \dots, S_{n-m} which cover $\{1, \dots, m\}$, and such that ([8, Def. 3.3]):

$$\#(S_i \cap (S_{i-1} \cup \dots \cup S_1)) = 1 \quad \text{for } i = 2, \dots, n - m.$$

To each R -covering we associate an open set $Z_R \subseteq Z_I$ defined as the intersection of Z_I with all Z_J 's such that

$$J = I \cup \{m + j\} \setminus \{i\}, \quad \text{where } i \in S_j.$$

One then checks similarly to [*loc. cit.*] that

$$Z_R \cong T \times \mathbf{A}^{(m-1)(n-m-1)}.$$

It is not hard to verify that the union of Z_R 's for all possible permutations of I_n coincides with the subset of $G(m, n)$ consisting of points satisfying the condition of corollary 2.3 (cf. [8, Prop. 3.3]). These two facts imply the proposition. \square

COROLLARY 2.5. — *Let m and n be coprime. Then*

$$G(m, n)^s = G(m, n)^{ss},$$

and $Y = Y'$ is a smooth projective variety.

Remark 2.6. — Let N be the normalizer of D in $SL(n)$, then the Weyl group $W = W(A_{n-1}) := N/D$ of the root system A_{n-1} is the symmetric group Σ_n permuting the components of the decomposition $V = k \oplus \dots \oplus k$. It acts on D , and thus on T . Clearly $G(m, n)^s$ and $G(m, n)^{ss}$ are invariant under N , thus W acts by automorphisms on Y and Y' .

The following trivial remark will be important in what follows. The group Σ_n of permutations of the components of the decomposition $V = k \oplus \dots \oplus k$ is naturally a subgroup of $GL(n)$. This makes it possible to identify W with a subgroup of $GL(n)$. As such, it naturally acts on $G(m, n)$. This action preserves $G(m, n)^s$ and $G(m, n)^{ss}$, and the corresponding morphisms to Y and Y' are W -equivariant.

3. Del Pezzo surfaces of degree 5: the split case

DEFINITION 3.1. — *A split del Pezzo surface of degree 5 is defined as the blowing-up of \mathbb{P}^2 in points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ and $(1 : 1 : 1)$.*

Note that we could as well define a split del Pezzo surfaces of degree 5 as the blowing-up of four points in \mathbb{P}^2 , no three of them collinear. Indeed, $PGL(3)$ acts transitively on such quadruples. By the universal property of blowing-up [5, II.7.15], there is a unique isomorphism of the corresponding blowings-up extending this action.

PROPOSITION 3.2. — *Let $(m, n) = (3, 5)$, then $Y = Y'$ is a split del Pezzo surface of degree 5.*

Proof. — The stability condition (1) implies that every two columns are not proportional. Let $I \subset I_5$, $\#I = 3$. The condition that the columns of M with numbers in I are linearly independent defines a dense open set $Z_I^s = Z_I \cap G(3, 5)^s$. It is D -invariant, so its image $\phi(Z_I^s)$ is also open. Define a dense open set $Z \subset G(3, 5)^s$ as the intersection of the Z_I^s 's for all possible three-element subsets of $\{1, 2, 3, 4\}$. Now let $S \subset V$ be the subspace corresponding to a point of Z . From the way we defined Z it follows that:

- every three out of the first four columns of M are linearly independent;
- no two columns are proportional.

Changing the basis, and multiplying the columns of M by non-zero numbers (this is the action of D), we can arrange that M is of the following form:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & x \\ 0 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 1 & z \end{bmatrix}$$

Here x, y, z are uniquely determined up to multiplication by a common non-zero constant. Conversely, taking any point

$$(x : y : z) \in \mathbb{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)\}$$

one checks immediately that the corresponding matrix M satisfies the stability condition (1), and so the space generated by its rows defines a point in $G(3, 5)^s$. Thus the map which sends S to $(x : y : z) \in \mathbb{P}^2$ is an

isomorphism of $\phi(Z) \subset Y$ onto $\mathbf{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)\}$. By Corollary 2.5, Y is a smooth projective surface, and this isomorphism extends to a birational morphism $\sigma : Y \rightarrow \mathbf{P}^2$ (Zariski's Main Theorem [5, V.5.2]).

Let us denote $L_I = Y \setminus \phi(Z_I^s)$, $I \subset I_5$, $\#I = 3$. We now prove that:

- (a) $L_I \cap L_J = \emptyset$ if and only if $\#(I \cup J) = 4$;
- (b) every L_I is isomorphic to \mathbf{P}^1 .

It follows from (a) and (b) that $Y \setminus \phi(Z)$ is the disjoint union of four smooth proper curves of genus 0. Thus σ^{-1} is the blowing-up of the above four points in \mathbf{P}^2 (cf. [5, V.5.4]), and the proposition will be proved.

Note that the stability condition (1) has it that $r(K) = 3$ for any 4-element subset $K \subset I_5$. To prove (a) one checks that $\#(I \cup J) = 4$ and $r(I) = r(J) = r(I \cap J) = 2$ automatically imply that $r(I \cup J) = 2$, which is not possible.

In order to prove (b) we can assume by symmetry that $I = \{3, 4, 5\}$. Then $L_{\{3, 4, 5\}}$ is covered by the following open sets:

$$\begin{aligned} A &= L_{\{3, 4, 5\}} \setminus (L_{\{1, 2, 3\}} \cup L_{\{1, 2, 4\}}), \\ B &= L_{\{3, 4, 5\}} \setminus (L_{\{1, 2, 4\}} \cup L_{\{1, 2, 5\}}), \\ C &= L_{\{3, 4, 5\}} \setminus (L_{\{1, 2, 3\}} \cup L_{\{1, 2, 5\}}). \end{aligned}$$

Choose a point in $\phi^{-1}(A)$, and a basis in the corresponding vector space S . Let M be the matrix obtained by decomposing this basis with respect to the standard basis of $V = k \oplus \dots \oplus k$. We have $r(\{1, 2, 3\}) = 3$, $r(\{1, 2, 4\}) = 3$. It follows from (a) that $r(\{1, 3, 4\}) = 3$, $r(\{2, 3, 4\}) = 3$. This means that every three out of the first four columns of M are linearly independent. On the other hand, the last three columns are linearly dependent. Now changing the basis, and multiplying the columns of M by non-zero numbers, we can arrange that M is of the following form:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & x \end{bmatrix}$$

Here $x \in k$ is uniquely defined, and any $x \neq 1$ would do. This proves that A is isomorphic to \mathbf{P}^1 minus two points. We leave to the reader the routine verification that A, B, C glue together to produce \mathbf{P}^1 . This completes the proof of the proposition. \square

From now on we fix the notation Y for the split del Pezzo surface of degree 5. Recall that Y contains precisely 10 exceptional curves of the first kind (see, e.g., [7, Chap. 4]).

COROLLARY 3.3. — *(of the proof) The genus zero curves L_I are exceptional curves of the first kind on Y . There are 10 of these, therefore every exceptional curve of the first kind on Y coincides with L_I for some $I \subset I_5$, $\#I = 3$.*

Proof. — The curves L_I for $I \subset \{1, 2, 3, 4\}$ can be smoothly blown down as it follows from the proof of Proposition 3.2. By symmetry, the same is true for any L_I . \square

The following statement seems to be well known to experts (cf. [2, VII]).

PROPOSITION 3.4. — *The natural map*

$$\nu : \text{Aut}(Y) \rightarrow \text{Aut}(\text{Pic}(Y))$$

is an isomorphism onto the group of automorphisms of $\text{Pic}(Y)$ leaving invariant the canonical class $K_Y \in \text{Pic}(Y)$ and the scalar product (\cdot, \cdot) given by the intersection index. The group $\nu(\text{Aut}(Y))$ is isomorphic to the Weyl group $W = W(A_4)$, implying $\text{Aut}(Y) \cong W$.

Proof. — We know from Remark 2.6 that W acts on Y . We prove that $\text{Ker}(\nu) = 1$, $\text{Im}(\nu) \cong W$. Indeed, let $\alpha \in \text{Ker}(\nu)$, then α fixes the classes $[L_I] \in \text{Pic}(Y)$ of exceptional curves of the first kind. Since L_I is the only curve in its class of linear equivalence, L_I is α -invariant. By the proof of Proposition 3.2, the complement in Y to the union of L_I , for $I \subset \{1, 2, 3, 4\}$, is isomorphic to $\mathbb{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)\}$. Thus α defines a birational automorphism of \mathbb{P}^2 , which is in fact biregular by Zariski's Main Theorem. It follows that α comes from an element of $PGL(3)$ fixing the four points as above. Thus α must be the identity map. Next we consider $\nu(\text{Aut}(Y))$. This group fixes the canonical class $K_Y \in \text{Pic}(Y)$. On the other hand, the scalar product (\cdot, \cdot) given by the intersection index, is also $\nu(\text{Aut}(Y))$ -invariant. The restriction of (\cdot, \cdot) to the orthogonal complement K_Y^\perp is negative definite, and the elements with norm -2 form a root system A_4 [7, IV]. By [7, IV.1] the subgroup of $\text{Aut}(\text{Pic}(Y))$ leaving invariant K_Y and (\cdot, \cdot) is the Weyl group $W = W(A_4)$. Thus $\nu(\text{Aut}(Y)) \subseteq W$. By Remark 2.6, $\nu(\text{Aut}(Y))$ contains $\nu(W) \cong W$, implying that $\text{Aut}(Y) \cong W$. \square

4. Del Pezzo surfaces of degree 5 and Galois cohomology

Let us recall some standard facts on forms and Galois cohomology [10, 1.5; 2.1; 3.1]. Let X be a variety over k . We denote by \bar{k} the algebraic closure of k , $\bar{X} := X \times_k \bar{k}$, and $\Gamma := \text{Gal}(\bar{k}/k)$ is the Galois group. The group $\text{Aut}(\bar{X})$ of \bar{k} -automorphisms of \bar{X} is equipped with a continuous invariant action of Γ :

$$a \rightarrow {}^s a = (1 \otimes s)a(1 \otimes s^{-1}), \quad s \in \Gamma.$$

In what follows this action comes from an action of a finite factor of Γ , so we shall make this assumption from now on.

If $k \subseteq K \subseteq \bar{k}$, then $\text{Aut}(X \times_k K)$ is the set of fixed elements of $\text{Aut}(\bar{X})$ with respect to the Galois group $\text{Gal}(\bar{k}/k)$. If K/k is a Galois extension, a 1-cocycle $a \in Z^1(K/k, \text{Aut}(X \times_k K))$ is a continuous map

$$a : \text{Gal}(K/k) \rightarrow \text{Aut}(X \times_k K)$$

such that $a_{st} = a_s \cdot {}^s a_t$. The cocycles a and a' are cohomologous if there exists $b \in \text{Aut}(X \times_k K)$ such that $a'_s = b^{-1} \cdot a_s \cdot {}^s b$. This is an equivalence relation, and the pointed set of orbits is $H^1(K/k, \text{Aut}(X \times_k K))$ (the neutral element comes from the zero cocycle).

A k -variety Z is a K/k -form of X if $Z \times_k K$ is isomorphic to $X \times_k K$. Let $E(K/k, X)$ be the pointed set of such forms considered up to an isomorphism, with the isomorphism class of X as the neutral element. Let K/k be a finite Galois extension. Then there is a canonical injection of pointed sets

$$\theta : E(K/k, X) \rightarrow H^1(K/k, \text{Aut}(X \times_k K)).$$

Let $Z \in E(K/k, X)$, then a 1-cocycle $a \in \theta(Z)$ can be chosen in the following way. Fix an isomorphism

$$\rho : Z \times_k K \xrightarrow{\sim} X \times_k K,$$

and take $a = (a_s)$ to be the function $\text{Gal}(K/k) \rightarrow \text{Aut}(X \times_k K)$ such that the natural action of $\text{Gal}(K/k)$ on $Z \times_k K$ (via the second factor) translates as its twisted action on $X \times_k K$:

$$\rho(1 \otimes s) \rho^{-1}(x) = a_s(1 \otimes s)x, \quad s \in \text{Gal}(K/k), \quad x \in X \times_k K.$$

The cohomology class of a does not depend on ρ .

If X is a quasiprojective k -variety, and K/k is a finite Galois extension, then θ is bijective [10, III.1.3]. In fact, the corresponding form is the quotient scheme $(X \times_k K)/\text{Gal}(K/k)$ with respect to the twisted action of $\text{Gal}(K/k)$.

PROPOSITION 4.1. — *Let X be a quasiprojective k -variety. Assume that $\text{Aut}(X) = \text{Aut}(\overline{X})$, and that this group is finite. Let $\text{Inn}(\text{Aut}(X))$ be the group of inner automorphisms of $\text{Aut}(X)$, and let*

$$\text{Hom}(\Gamma, \text{Aut}(X))/\text{Inn}(\text{Aut}(X))$$

be the set of orbits of $\text{Inn}(\text{Aut}(X))$ on $\text{Hom}(\Gamma, \text{Aut}(X))$ with respect to the natural action. Then there is a canonical bijection of pointed sets

$$\theta : E(\overline{k}/k, X) \xrightarrow{\sim} \text{Hom}(\Gamma, \text{Aut}(X))/\text{Inn}(\text{Aut}(X)).$$

Proof. — Since $\text{Aut}(X) = \text{Aut}(\overline{X})$, this group has a trivial action of Γ . Thus 1-cocycles are no other than homomorphisms, and the equivalence relation of 1-cocycles is just the conjugation. A homomorphism $\Gamma \rightarrow \text{Aut}(\overline{X})$ has a finite image, thus the corresponding form can be recovered as a quotient scheme, and so θ is bijective. \square

DEFINITION 4.2. — *A del Pezzo surface of degree 5 is defined as a \overline{k}/k -form of the split del Pezzo surface of degree 5.*

COROLLARY 4.3. — *There is a natural bijection between the following pointed sets:*

- (i) *the set of isomorphism classes of del Pezzo k -surfaces of degree 5 with the class of the split surface as the neutral element;*
- (ii) *the pointed set $H^1(\Gamma, W)$;*
- (iii) *the pointed set of orbits $\text{Hom}(\Gamma, W)/\text{Inn}(W)$ with the trivial homomorphism as the neutral element.*

Proof. — By Proposition 3.4 we have $\text{Aut}(Y) \cong W$, but we also have $\text{Aut}(\overline{Y}) \cong W$ by the same result, so we are in the situation of Proposition 4.1. \square

THEOREM 4.4. — *Any del Pezzo k -surface of degree 5 has a k -point.*

Proof. — Let us consider a twisted version of the whole set-up of Section 2. Let us identify W with the group Σ_5 of permutational matrices in $GL(5)$. Fix a homomorphism $h : \Gamma \rightarrow W \cong \Sigma_5$. Define the following action of Γ on $V \otimes \bar{k} = \bar{k} \otimes \cdots \otimes \bar{k}$:

$$s(v) = h(s)(1 \otimes s)v, \quad s \in \Gamma, v \in V \otimes \bar{k}. \quad (2)$$

This obviously induces an action of Γ on $G(3, 5) \times_{\bar{k}} \bar{k}$, and thus on $G(3, 5)^s \times_{\bar{k}} \bar{k}$. By the general theory, we can consider the corresponding \bar{k}/k -forms ${}_hG(3, 5)$ and ${}_hG(3, 5)^s$.

The map $\phi : G(3, 5)^s \rightarrow Y$ gives rise to ${}_h\phi : {}_hG(3, 5)^s \rightarrow {}_hY$ (recall that W normalizes the torus D , and hence ϕ is W -equivariant). Clearly ${}_hY$ is a form of Y . Since Σ_5 normalizes the diagonal torus of $GL(5)$, we get from (2) that the corresponding twisted action of Γ on \bar{Y} is given by

$$s(x) = h(s)(1 \otimes s)x, \quad s \in \Gamma, x \in \bar{Y}.$$

Thus ${}_hY$ is a del Pezzo surface of degree 5 whose cohomology class is represented by $h \in \text{Hom}(\Gamma, W)$. It follows from Corollary 4.3 that we obtain all del Pezzo surfaces of degree 5 in this way.

Now let us go back to ${}_hG(3, 5)$. This is a homogeneous space of $GL(5)$ twisted by a cocycle $h : \Gamma \rightarrow W$. Due to the fact that $W \cong \Sigma_5$ naturally lies in $GL(5)$, the cocycle h lifts to a cocycle with coefficients in $GL(5)$. Any such is a coboundary by Hilbert's Theorem 90. It follows that ${}_hG(3, 5)$ is isomorphic to $G(3, 5)$.

If k is infinite, then k -points are Zariski dense on $G(3, 5)$, and so there is a k -point on ${}_hG(3, 5)^s$, and hence on ${}_hY$. Following [11] we may end the proof in the finite field case by referring to a general theorem of Weil [12] that a smooth projective rational surface defined over a finite field k always has a k -point (see also [7, 23.1]). However, a simple general argument is available, which I owe to J.-L. Colliot-Thélène:

LEMMA (Lang [6], Nishimura [9]). — *If $f : X \rightarrow Z$ is a rational map of integral k -varieties, where Z is proper and X has a smooth k -point, then Z has a k -point.*

Applying this with $X = G(3, 5)$ and $Z = {}_hY$ we prove the theorem. \square

One can interpret ${}_hG(3, 5)^S$ as an “almost universal” torsor on ${}_hY$: it is a torsor under the algebraic k -torus dual to the Γ -module $K_{\overline{Y}}^\perp$. (Recall that a universal torsor is a torsor under the dual torus of the whole Picard group $\text{Pic}(\overline{Y})$, see the details in [1].) Thus it is not surprising that in our proof k -points are first traced on ${}_hG(3, 5)^S$: this agrees with the philosophy of the descent theory [1] that the universal torsors over a rational variety in a certain sense “untwist” its arithmetic.

Acknowledgments

I would like to thank V.V. Batyrev for driving my attention to the connection between rational surfaces and homogeneous varieties of type G/P . I am grateful to V. V. Serganova, J.-L. Colliot-Thélène and J.-J. Sansuc for many warm discussions.

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