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H^∞ -extensibility and finite proper holomorphic surjections^(*)

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RÉSUMÉ. — Soit $\theta : X \rightarrow Y$ une application bornée, propre, holomorphe, et surjective entre deux espaces de Banach analytiques. Si Y possède la propriété de prolongement pour les fonctions H^∞ , on montre que X la possède également. Réciproquement, si X possède cette propriété et X ne contient pas un ensemble analytique compact de dimension positive, alors toute application holomorphe d'un domaine de Riemann D étalé sur un Banach avec image dans Y peut être prolongée comme une application Gâteaux-holomorphe sur chaque prolongement H^∞ de D ; de surcroît, le prolongement est holomorphe dans le complémentaire d'une hypersurface.

ABSTRACT. — Let $\theta : X \rightarrow Y$ be a finite proper holomorphic surjection, where X and Y are Banach analytic spaces. It is shown that if Y has the holomorphic H^∞ -extension property, so has X . Conversely if X has the holomorphic H^∞ -extension property, where X does not contain a compact analytic set of positive dimension, then every holomorphic map from a Riemann domain D over a Banach space into Y can be extended Gateaux-holomorphically on every H^∞ -extension of D . Moreover the extension is holomorphic outside a hypersurface.

The extension of holomorphic maps from a Riemann domain D over a Stein manifold to its envelope of holomorphy \widehat{D}_∞ for the Banach algebra of bounded holomorphic functions $H^\infty(D)$ has been investigated by some authors.

For holomorphic maps with values in finite dimensional complete C -spaces, the problem was considered by Sibony [6], Hirschowitz [3], and recently by Nguyen van Khue and Bui Dac Tac [4]. The aim of the present paper is to consider the problem in the infinite dimensional case.

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Let X be a Banach analytic space in the sense of Douady [1]. As in the finite dimensional case, we define the Carathéodory pseudodistance C_X on X by the formula

$$C_X(x, y) = \text{Sup} \{ |f(x) - f(y)| : |f| \leq 1, f \in H^\infty(X) \} .$$

We say that X is a C -space if C_X is a distance defining the topology of X .

Let (D, p, B) and (D', q, B) be Riemann domains over a Banach space B . D' is called a H^∞ -extension of D if there is a holomorphic map $e : D \rightarrow D'$ such that $p = q \cdot e$ and for every bounded holomorphic function f on D , there exists a bounded holomorphic function f' on D' such that $f = f' \cdot e$.

A Banach analytic space X is said to be a space having the holomorphic H^∞ -extension property (for short, the HEH^∞ -property) if for every holomorphic map g from a Riemann domain D over a Banach space into X there exists a holomorphic map g' from D' into X such that $g = g' \cdot e$, where D' is a H^∞ -extension of D and D' is a C -space. In this case we say also that g can be extended to a holomorphic map g' on D' .

The main result of this note is the following.

THEOREM 1. — *Let θ be a finite proper holomorphic map from a Banach analytic space X onto a Banach analytic space Y . Then:*

- (i) *if Y has the HEH^∞ -property and $H^\infty(X)$ separates the points of the fibers of θ , then X has the HEH^∞ -property;*
- (ii) *if X has the HEH^∞ -property and X does not contain a compact analytic set of positive dimension, then every holomorphic map from D into Y can be extended Gateaux holomorphically on D' , where D is a Riemann domain over a Banach space, D' is a H^∞ -extension of D and D' is a C -space.*

Moreover, the extension is holomorphic outside a hypersurface.

Let X be a Banach analytic space. We say that an upper semi-continuous function $\varphi : X \rightarrow [-\infty, \infty)$ is plurisubharmonic if for every holomorphic map $\sigma : \Delta \rightarrow X$ the function $\varphi \circ \sigma$ is subharmonic, where Δ is the unit disc.

Let Z be a Banach analytic space. By $F_c(Z)$ we denote the hyperspace of non-empty compact subsets of Z . An upper semi-continuous multivalued function $K : X \rightarrow F_c(Z)$, where X is a Banach analytic space, is called analytic in the sense of Slodkowski [7] if for every plurisubharmonic function

Ψ on a neighbourhood of $\Gamma_{K|_G}$, where G is an open set in X and $\Gamma_{K|_G}$ denote the graph of K on G , the function

$$\varphi(x) = \max\{\Psi(x, z) \mid z \in K(x)\}$$

is plurisubharmonic on G .

LEMMA 1 ([5]).— *Let $K : Y \rightarrow F_c(X)$ be an analytic multivalued function such that $\text{card } K(y) < \infty$ for all $y \in Y$, where Y is a connected Banach analytic space. Assume that U and V are disjoint open subsets of X such that $K(y) \subset U \cup V$ for all $y \in Y$. Then either $K(y) \cap U = \emptyset$ for all $y \in Y$ or $K(y) \cap U \neq \emptyset$ for all $y \in Y$.*

Proof.— Define Ψ on $Y \times (U \cup V)$ by

$$\Psi(y, z) = \begin{cases} 1 & \text{if } z \in U \\ 0 & \text{if } z \in V. \end{cases}$$

Then Ψ is plurisubharmonic on a neighbourhood of the graph of K , so φ is plurisubharmonic on Y , where

$$\begin{aligned} \varphi(y) &= \max\{\Psi(y, z) \mid z \in K(y)\} \\ &= \begin{cases} 0 & \text{if } K(y) \cap U = \emptyset \\ 1 & \text{if } K(y) \cap U \neq \emptyset. \end{cases} \end{aligned}$$

By the plurisubharmonicity of φ and the connectedness of Y , it implies that either $K(y) \cap U = \emptyset$ for all $y \in Y$ or $K(y) \cap U \neq \emptyset$ for all $y \in Y$. The lemma is proved. \square

LEMMA 2.— *Let $K : Y \rightarrow F_c(X)$ be an analytic multivalued function such that $\text{card } K(y) < \infty$ for all $y \in Y$. Then*

$$V_m = \{y \in Y \mid \text{card } K(y) < m\}$$

is closed in Y for every $m \geq 1$.

Proof.— Given a sequence $\{y_n\}$ in V_m , $y_n \rightarrow y^*$, choose disjoint neighbourhoods U_i of x_i , $i = 1, \dots, \ell$, where $\{x_1, \dots, x_\ell\} = K(y^*)$. Take a neighbourhood D of y^* such that

$$K(D) \subset \bigcup_{i=1}^{\ell} U_i.$$

Then by lemma 1, $K(y) \cap U \neq \emptyset$ for all $i = 1, \dots, \ell$ and for all $y \in D$. Hence $m > \text{card } K(y_n) \geq 1$ for sufficiently large n . This implies that $y^* \in V_m$. The lemma is proved. \square

LEMMA 3. — *Let $\theta : X \rightarrow Y$ be a finite proper holomorphic surjection, where X and Y are Banach analytic spaces. Then the multivalued function*

$$K : Y \rightarrow F_c(X)$$

given by

$$K(y) = \theta^{-1}(y)$$

is analytic.

Proof

(i) Consider first the case where $Y = \Delta$, the unit disc in \mathbb{C} .

Since θ is proper, K is upper semi-continuous. Let Ψ be a plurisubharmonic function on a neighbourhood of $\Gamma_{K \upharpoonright G}$, where G is an open subset of Δ . Since θ is a branch covering map [2], there exists a discrete sequence A in Δ such that

$$\theta : X \setminus \theta^{-1}(A) \rightarrow \Delta \setminus A$$

is an unbranched covering map of order $m < \infty$. Let $y_0 \in \Delta \setminus A$ and

$$\theta^{-1}(y_0) = \{x_1, \dots, x_m\}.$$

Take a neighbourhood W of y_0 such that

$$\theta^{-1}(W) = U_1 \cup \dots \cup U_m,$$

where U_j are disjoint, $x_j \in U_j$ and $\theta : W \cong U_j$, $j = 1, \dots, m$. Then the function

$$\varphi(y) = \max_j \max \{ \Psi(y, x) \mid z \in \theta^{-1}(y) \cap U_j \}$$

is subharmonic on $W \cap G$. Since φ is locally bounded on G , it follows that φ is subharmonic on G .

(ii) Consider now the general case where Y is a Banach analytic space.

Let φ be as in (i). Obviously φ is upper semi-continuous because of the upper semi-continuity of K and Ψ . It remains to check that $\varphi \circ h$ is subharmonic on Δ for every holomorphic map $h : \Delta \rightarrow X$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\Delta} & \xrightarrow{\tilde{h}} & X \\ \tilde{\theta} \downarrow & & \downarrow \theta \\ \Delta & \xrightarrow{h} & Y \end{array}$$

where $\tilde{\Delta} = \Delta \times_Y X$. By (i) and by the relation

$$\varphi \circ h(\lambda) = \max \left\{ \Psi(h(\lambda), z) \mid \theta(z) = h(\lambda) \right\}$$

it follows that $\varphi \circ h$ is subharmonic on Δ . The lemma is proved. \square

Let X and D be Banach analytic spaces. A finite proper holomorphic surjection $\pi : X \rightarrow D$ is called a branch covering map if it satisfies the following:

- (i) there is a closed subset A of D which is a removable for bounded holomorphic germs on $D \setminus A$;
- (ii) $\pi : X \setminus \pi^{-1}(A) \rightarrow D \setminus A$ is a local biholomorphism and $\text{card } \pi^{-1}(z)$ is constant on every connected component of $D \setminus A$.

LEMMA 4. — *Let θ be a finite proper holomorphic map from a Banach analytic space X onto an open set D in a Banach space B . Then θ is a branch covering map.*

Proof. — Without loss of generality we may assume that D is convex. For each $n \geq 1$ put

$$F_n = \{y \in D \mid \text{card } \theta^{-1}(y) < n\}.$$

By lemma 2 and lemma 3, F_n is closed in D . Applying the Baire theorem to $D = \bigcup_1^\infty F_n$, we can find n_0 such that $\text{Int } F_{n_0} \neq \emptyset$. Put

$$m = \max \{ \text{card } \theta^{-1}(y) \mid y \in \text{Int } F_{n_0} \}.$$

Since $\theta : \theta^{-1}(E \cap D) \rightarrow E \cap D$ is a branch covering map for every finite dimensional subspace E of B [2], by the connectedness of $D \cap E$ for all subspace E of B , $\dim E < \infty$, we have

$$\begin{aligned} \sup\{\text{card } \theta^{-1}(y) \mid y \in D\} &= \\ &= \sup\{\text{card } \theta^{-1}(y) \mid y \in D \cap E, E \subset B, \dim E < \infty\} = m. \end{aligned}$$

Put

$$Z = \{y \in D \mid \text{card } \theta^{-1}(y) < m\}.$$

Then Z is closed in D , and from the finiteness and properness of θ it follows that

$$\theta : X \setminus \theta^{-1}(Z) \rightarrow D \setminus Z$$

is an unbranched covering map. It remains to show that Z is removable for bounded holomorphic germs. Let h be a bounded holomorphic function on $U \setminus Z$, where U is an open subset of D . Then for every finite dimensional space E of B such that

$$\sup\{\text{card } \theta^{-1}(y) \mid y \in E \cap D\} = m,$$

$h|_{U \setminus Z}$ can be extended holomorphically on U . From the relation

$$\begin{aligned} D = \bigcup \left\{ E \cap D \mid E \subset B, \dim E < \infty, \right. \\ \left. \sup\{\text{card } \theta^{-1}(y) \mid y \in D \cap E\} = m \right\}, \end{aligned}$$

it follows that h can be extended to a bounded Gateaux-holomorphic function \hat{h} on U . By the boundedness of \hat{h} , we deduce that \hat{h} is holomorphic on U . The lemma is proved. \square

LEMMA 5. — Let $\theta : X \rightarrow D$, where D is a C -manifold, be a branch covering map. Denote by $\text{SH}^\infty(X)$ and $\text{SH}^\infty(D)$ the spectra of Banach algebras $H^\infty(X)$ and $H^\infty(D)$, respectively. Let $\hat{\theta} : \text{SH}^\infty(X) \rightarrow \text{SH}^\infty(D)$ be the map induced by θ . Then

$$\hat{\theta} : \hat{\theta}^{-1}(D) \rightarrow D$$

is also a branch covering map.

Proof. — Obviously $\widehat{\theta} : \widehat{\theta}^{-1}(D) \rightarrow D$ is finite, proper and surjective, since $H^\infty(X)$ is an integral extension of finite degree of $H^\infty(D)$. By lemma 4, it suffices to prove that $\widehat{\theta}^{-1}(D)$ is a Banach analytic space. Let $B(0, r)$ (resp. $B^*(0, r)$) denote the open ball in $H^\infty(X)$ (resp. $(H^\infty(X))^*$) centred at 0 with radius $r > 0$. Consider the holomorphic map

$$F : (D \setminus Z) \times B^*(0, 2) \longrightarrow H^\infty(B(0, 2))$$

given by

$$F(z, w)(h) = w(h)^m + \sigma_{m-1}(h \circ p_1(z), \dots, h \circ p_m(z))w(h)^{m-1} + \dots + \sigma_0(h \circ p_1(z), \dots, h \circ p_m(z)),$$

where z is the branch locus of θ , m the order of θ and $\sigma_0, \dots, \sigma_{m-1}$ are elementary symmetric polynomials in m variables and

$$\theta^{-1}(z) = (p_1(z), \dots, p_m(z)) \quad \text{for } z \in D \setminus Z.$$

Since $\sigma_0, \dots, \sigma_{m-1}$ are bounded holomorphic functions on $D \setminus Z$, it follows that F is holomorphic on $D \times B^*(0, 2)$. We have

$$F^{-1}(0) = \{(z, w) \mid \widehat{\theta}(w) = z\} \cong \widehat{\theta}^{-1}(D).$$

Hence $\widehat{\theta} : \widehat{\theta}^{-1}(D) \rightarrow D$ is a branch covering map. The lemma is proved. \square

LEMMA 6. — *Every Banach space has the HEH^∞ -property.*

Proof. — Let D be a Riemann domain over a Banach space B and D' a H^∞ -extension of D . Let $f : D \rightarrow E$ be a holomorphic map, where E is a Banach space.

For each $x^* \in E^*$, by $\widehat{x^*f}$ we denote the holomorphic extension of x^*f on D' . Since D' is a H^∞ -extension of D , from the open mapping theorem, it follows that

$$\|\widehat{x^*f}\| = \|x^*f\| \quad \text{for all } x^* \in E^*.$$

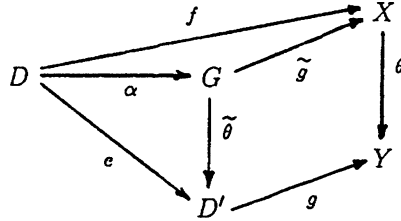
On the other hand, by the uniqueness, $\widehat{x^*f}(z)$ is a continuous linear function on E^* for every $z \in D'$. Thus we can define a bounded map $\widehat{f} : D' \rightarrow E^{**}$ by

$$(\widehat{f}(z))(x^*) = \widehat{x^*f}(z)$$

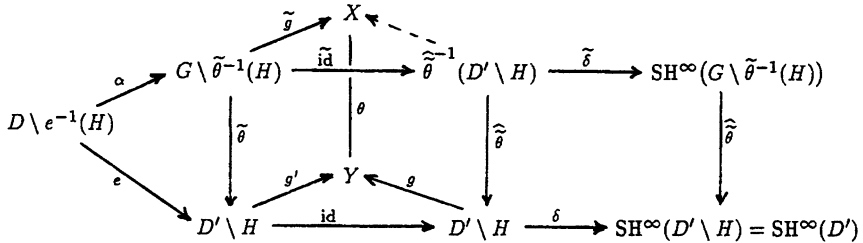
which is separately holomorphic in variables $z \in D'$ and $x^* \in E^*$. From the boundedness of $\widehat{f}(D')$ we deduce that \widehat{f} is holomorphic and $\widehat{f}(D') \subset E$. Obviously \widehat{f} is a holomorphic extension of f on D' . The lemma is proved. \square

Proof of theorem 1

(i) Let first Y have the HEH^∞ -property. Let $f : D \rightarrow X$ be a holomorphic map, where D is a Riemann domain over a Banach space B . By hypothesis, there is a holomorphic map $g : D' \rightarrow Y$ which is a holomorphic extension of θf on D' , where D' is a H^∞ -extension of D . Consider the commutative diagram



where $G = D' \times_Y X$, $\tilde{\theta}$ and \tilde{g} are the canonical projections, α and e are the canonical maps. By lemma 4, $\tilde{\theta}$ is a branch covering map. Let H denote the branch locus of $\tilde{\theta}$. Consider the commutative diagram



where

$$\widehat{\theta} : \text{SH}^\infty(G \setminus \tilde{\theta}^{-1}(H)) \longrightarrow \text{SH}^\infty(D' \setminus H) \cong \text{SH}^\infty(D')$$

is induced by $\tilde{\theta} : G \setminus \tilde{\theta}^{-1}(H) \rightarrow D' \setminus H$. From lemma 5, it follows that

$$\widehat{\theta} : \widehat{\theta}^{-1}(D' \setminus H) \longrightarrow D' \setminus H$$

is a branch covering map. By lemma 6, $(H^\infty(G \setminus \tilde{\theta}^{-1}(H)))^*$ has the HEH^∞ -property.

Since $D' \setminus H$ is also a H^∞ -extension of $D \setminus e^{-1}(H)$, there exists

$$h : D' \setminus H \longrightarrow \left(H^\infty(G \setminus \tilde{\theta}^{-1}(H)) \right)^*$$

which is a holomorphic extension of

$$\tilde{\text{id}}_\alpha : D \setminus e^{-1}(H) \longrightarrow \left(H^\infty(G \setminus \tilde{\theta}^{-1}(H)) \right)^* .$$

From the relation $\widehat{\theta}h = \delta$, where $\delta : D' \setminus H \rightarrow \text{SH}^\infty(D' \setminus H)$ is the canonical map, we have $h(D' \setminus H) \subset \widehat{\theta}^{-1}(D' \setminus H)$. Since $H^\infty(X)$ separates the points of the fibers of θ , there exists a holomorphic mapping $\widehat{g} : \widehat{\theta}^{-1}(D' \setminus H) \rightarrow X$ such that $g\widehat{\theta} = \theta\widehat{g}$. Put

$$f_1 = \widehat{g}h .$$

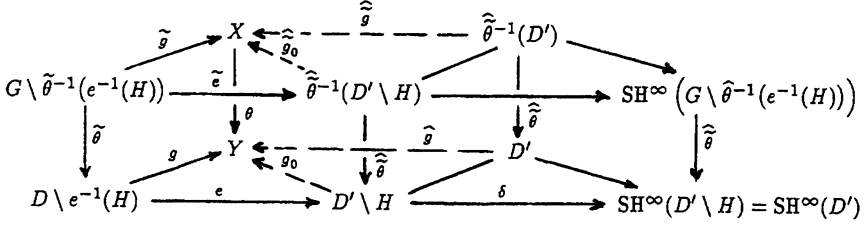
Assume now $z \in H$. Take two neighbourhoods U and V of z and $g(z)$, respectively, such that $g(U) \subset V$ and $\theta^{-1}(V)$ is an analytic set in a finite union W of balls in a Banach space. Then $f_1 : U \setminus H \rightarrow W$ can be extended holomorphically on U . This implies that f_1 and hence f can be extended holomorphically on D' .

(ii) Let X be a space having the HEH^∞ -property and let $g : D \rightarrow Y$ be a holomorphic map, where D is a Riemann domain over a Banach space B . Let D' be a H^∞ -extension of D which is a C -space. Consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{g}} & X \\ \downarrow \tilde{\theta} & & \downarrow \theta \\ D & \xrightarrow{g} & Y \end{array}$$

where $G = D \times_Y X$, $\tilde{\theta}$ and \tilde{g} are the canonical projections.

Obviously $\widehat{\theta} : \text{SH}^\infty(G) \rightarrow \text{SH}^\infty(D')$ is finite, proper and surjective, since $H^\infty(G)$ is an integral extension of finite degree of $H^\infty(D)$ and every bounded holomorphic function on D can be extended to a bounded holomorphic function on D' . By lemmas 4 and 5, $\tilde{\theta}$ and $\widehat{\theta} : \widehat{\theta}^{-1}(D') \rightarrow D'$ are branch covering maps. Let H denote the branch locus of $\widehat{\theta} : \widehat{\theta}^{-1}(D') \rightarrow D'$. Consider the commutative diagram



where δ is the canonical map. Since every bounded holomorphic function on $G \setminus \tilde{\theta}^{-1}(e^{-1}(H))$ can be extended to a bounded holomorphic function on $SH^\infty(G \setminus \tilde{\theta}^{-1}(e^{-1}(H)))$ and the topology of $\tilde{\theta}^{-1}(D' \setminus H)$ is defined by bounded holomorphic functions, it follows that $\tilde{\theta}^{-1}(D' \setminus H)$ is a H^∞ -extension of $G \setminus \tilde{\theta}^{-1}(e^{-1}(H))$ and it is a C -space. By hypothesis, \tilde{g} can be extended to a holomorphic map

$$\hat{g}_0 : \tilde{\theta}^{-1}(D' \setminus H) \longrightarrow X.$$

It is easy to see that $\tilde{e}\tilde{\theta}^{-1}(x) = \tilde{\theta}^{-1}(e(x))$ for every $x \in D \setminus e^{-1}(H)$. This yields

$$\hat{g}_0 \upharpoonright_{\tilde{\theta}^{-1}(e(x))} = \text{const} \quad \text{for all } x \in D \setminus e^{-1}(H).$$

Since $\tilde{\theta} : \tilde{\theta}^{-1}(D' \setminus H) \rightarrow D' \setminus H$ is a branch covering map, it follows that there exists a holomorphic map $\hat{g}_0 : D' \setminus H \rightarrow Y$ such that $\theta \hat{g}_0 = \tilde{g} \tilde{\theta}$.

X does not contain a compact set of positive dimension. By the Hironaka singular resolution theorem, for every finite dimensional subspace E of B such that $q^{-1}(E) \not\subset e(H)$,

$$\hat{g}_0 \upharpoonright_{\tilde{\theta}^{-1}(q^{-1}(E) \setminus H)}$$

can be extended to a holomorphic map $\hat{g}_E : \tilde{\theta}^{-1}(q^{-1}(E)) \rightarrow X$. This yields that $\hat{g}_0 \upharpoonright_{q^{-1}(E) \setminus H}$ can be extended to a holomorphic map $\hat{g}_E : q^{-1}(E) \rightarrow Y$. Thus \hat{g}_0 and hence g can be extended to a Gateaux holomorphic map $\hat{g} : D' \rightarrow Y$ which is holomorphic on $D' \setminus H$. The theorem is proved. \square

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References

- [1] DOUADY (A.) .— *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier, Grenoble, **16**, n° 1 (1966), pp. 1-95.
- [2] FISCHER (G.) .— *Complex Analytic Geometry*, Springer Verlag, Berlin, Lecture Notes in Math. **538** (1976).
- [3] HIRSCHOWITZ (A.) .— *Domaines de Stein et fonctions holomorphes bornées*, Math. Ann. **213** (1975), pp. 185-193.
- [4] KHUE (N.V.) and TAC (B.D.) .— *Extending holomorphic maps in infinite dimensions*, Studia Math. **45** (1990), pp. 263-272.
- [5] RANSFORD (T.J.) .— *Open mapping, inversion and implicit function theorems for analytic multivalued functions*, Proc. London Math. Soc. (3), **49** (1984), pp. 537-562.
- [6] SIBONY (N.) .— *Prolongement des fonctions holomorphes bornées et métriques de Carathéodory*, Invent. Math. **29** (1975), pp. 205-230.
- [7] SŁODKOWSKI (Z.) .— *Analytic set-valued functions and spectra*, Math. Ann. **256** (1981), pp. 363-386.