MOHSEN MASMOUDI

Covariant star-products


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1. Introduction

*-products were defined in [1] by Flato, Fronsdal, Lichnerowicz as a tool for quantizing a classical system, described with a symplectic manifold \((M, \omega)\). Roughly speaking, a \(\ast\)-product is a (formal) deformation of the associative algebra \(C^\infty(M)\) provided with usual (pointwise) product starting with the Poisson bracket. The quantum structure is then the deformed structure on the unchanged space of observables.

Each quantization procedure, when applied on a coadjoint orbit \(M\) of a Lie group \(G\), gives some way to build up unitary irreducible representations of \(G\). To use \(\ast\)-products for such a purpose, we need in fact a particular property, the covariance of the \(\ast\)-product:

\[
[\tilde{X}, \tilde{Y}]_\ast = \{\tilde{X}, \tilde{Y}\} = [\tilde{X}, \tilde{Y}],
\]
if $\tilde{X}$ is, for each $X$ in the Lie algebra $g$ of $G$, the function on $M$ defined by

$$x \mapsto \tilde{X}(x) = \langle x, X \rangle.$$  

(See [2] for a discussion on invariance and covariance properties for $\ast$-products on a coadjoint orbit.)

In this paper, we recall first the theorem of existence of $\ast$-products on a sympletic manifold. This theorem is due to P. Lecomte and M. de Wilde [3]. Some recent new proofs were given by Maeda, Omori and Yoshioka [4] and Lecomte and de Wilde [5]. We expose here that last proof in a slighly different way, which is direct and totally elementary: we build a $\ast$-product by gluing together local $\ast$-products defined on domains of a chart of $M$. That proof follows the idea of Vey, Lichnerowicz, Neroslavsky and Vlassov ([6], [7]) and, of course, Maeda, Onori and Yoshioka. In these approaches, the obstruction to construct $\ast$-product lies in the third cohomology group $H^3(M)$ of the manifold $M$. Lecomte and de Wilde defined formal deformation of the Lie algebra $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$, for such a deformation, the obstruction is in the group $H^3(\mathcal{C}^\infty(M))$ for the adjoint action, which contains strictly $H^3(M)$. Let us finally mention the construction of Maslov and Karasev [8] who found an obstruction in $H^2(M)$ to construct simultaneously a deformation and a representation of the deformed structure on $\mathcal{C}^\infty(M)$. Lecomte and de Wilde proved that all these obstructions can be surrounded, with the use of local conformal vector fields on $M$. We follow here that classical proof, using only local computations and Čech calculus.

Then we use this proof in the case of a coadjoint orbit in the dual $g^*$ of a Lie algebra $g$. More precisely, we consider a point $x_0$ in $g^*$ and suppose there exists in $x_0$ a real polarization. Under this assumption, we prove the existence of a covariant $\ast$-product on the coadjoint orbit of $x_0$, endowed with its canonical symplectic structure.

2. Existence of $\ast$-products on a symplectic manifold

Let $(M, \omega)$ be a symplectic manifold. We denote by $\{\cdot, \cdot\}$ the Poisson bracket defined on $\mathcal{C}^\infty(M)$ by the usual relations:

$$\{u, v\} = X_u v \quad \text{if} \quad i_{X_u} \omega = -dv.$$  

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A $\ast$-product is by definition a formal deformation in the sense of Gerstenhaber [9] of the associative algebra $C^\infty(M)$, i.e. a bilinear map:

$$C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)[[\nu]], \quad (u, v) \mapsto v \ast v = \sum_{r \geq 0} \nu^r C_r(u, v),$$

where $C^\infty(M)[[\nu]]$ is the space of formal power series in the variable $\nu$ with coefficients in $C^\infty(M)$, such that each $C_r$ is a bidifferential operator and:

(i) $C_0(u, v) = uv$, $C_1(u, v) = \{u, v\}$,
(ii) $C_r(u, v) = (-1)^r C_r(v, u)$,
(iii) $C_r(1, u) = 0$, $\forall$ $r > 0$,
(iv) $\sum_{r+s=t} C_r(C_s(u, v), w) = \sum_{r+s=t} C_r(u, C_s(u, w)), \forall$ $t \geq 0$.

With these properties, $\ast$ defines an associative structure on $C^\infty(M)[[\nu]]$, of whom unity is 1 and:

$$[u, v]_\ast = \sum_{r \geq 0} \nu^{2r} C_{2r+1}(u, v) = \frac{1}{2\nu^2} (u \ast v - v \ast u)$$

is a Lie bracket (it satisfies Jacobi identity) and a formal deformation of the Poisson bracket.

On a symplectic vector space $\mathbb{R}^{2n}$ and thus on any domain $U$ of a canonical chart in $M$; there exists $\ast$-products, for instance the Moyal $\ast$-product [1].

**Theorem [3].** On each simplectic manifold $(M, \omega)$, there exists a $\ast$-product.

**Proof.** Let us first choose a locally finite covering $(U_\alpha)_{\alpha \in A}$ of the manifold such that each $U_\alpha$ is the domain of a canonical chart on $M$ and all the intersections:

$$U_{\alpha_1 \ldots \alpha_n} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$$

are contractible. We fix a total ordering $\leq$ on $A$ and a partition of the unity $\psi_\alpha$ subordinated to $(U_\alpha)_{\alpha \in A}$. If $\ast_\alpha$ is a $\ast$-product on $U_\alpha$ and $\text{Der}(\ast_\alpha)$ the space of derivation of $\ast_\alpha$, there exists a canonical linear mapping:

$$\Phi_\alpha : F(U_\alpha) = C^\infty(U_\alpha)/\{\text{constants}\} \longrightarrow \text{Der}(\ast_\alpha), \quad \Phi_\alpha([f])(v) = [f, v]_\alpha.$$
Let us denote by $\text{Conf}(U_\alpha)$ the space of (conformal) vector fields $\xi_\alpha$ on $U_\alpha$ such that:
\[ L_{\xi_\alpha} \omega = \omega \quad \text{on } U_\alpha. \]

$\text{Conf}(U_\alpha)$ is an affine space on $F(U_\alpha)$.

Now we suppose, by induction on $k$, to have, on each $U_\alpha$, a $\star$-product $\star_\alpha$:
\[ u \star_\alpha v = \sum_{r \geq 0} \nu^r C_{r,\alpha}(u, v), \]

with $C_{r,\alpha} = C_{r,\beta}$ on $U_{\alpha\beta}$, for all $r < 2k$ and an affine mapping $D_\alpha$ form $\text{Conf}(U_\alpha)$ into $\text{Der}(\star_\alpha)(D_\alpha(\xi_\alpha + X_f)) = D_\alpha(\xi_\alpha) + \Phi_\alpha([f])$ such that:
\[ D_\alpha(\xi_\alpha) = \nu \partial_\nu + L_{\xi_\alpha} + \sum_{r > 0} \nu^{2r} D_{2r}^\alpha(\xi_\alpha), \]

the $D_{2r}^\alpha(\xi_\alpha)$ being differential operators, vanishing on constants. Of course, these assumptions hold for $k = 1$.

Now it is well known ([6], [7]) that for each $\alpha < \beta$, we can find a differential operator vanishing on constants $H_{\alpha\beta}$ such that, up to order $2k + 2$,
\[ u \star_\alpha' v = \exp \nu^{2k} H_{\alpha\beta}(\exp -\nu^{2k} H_{\alpha\beta} u \star_\alpha \exp -\nu^{2k} H_{\alpha\beta} v) \]

coincide with $u \star_\beta v$. Thus:
\[ (D_\alpha - D_\beta)(\xi_{\alpha\beta}) = \]
\[ = \sum_{r=0}^{k-1} \nu^{2r} \Phi_\alpha([f_{\alpha\beta}^{2r}]) + \nu^{2k} \left( L_{\xi_{\alpha\beta}} H_{\alpha\beta} + 2k H_{\alpha\beta} + \Phi_\alpha([g_{\alpha\beta}(\xi_{\alpha\beta})]) \right) \]

here $[f_{\alpha\beta}^{2r}]$ in $F(U_{\alpha\beta})$ do not depend of $\xi_{\alpha\beta}$ while:
\[ [g_{\alpha\beta}(\xi_{\alpha\beta} + X_f)] = [g_{\alpha\beta}(\xi_{\alpha\beta}) + H_{\alpha\beta} f]. \]

For each $\alpha < \beta$, we choose a vector field $\xi_{\alpha\beta}$ in $\text{Conf}(U_{\alpha\beta})$, a $C^\infty$ function $g_{\alpha\beta}(\xi_{\alpha\beta})$ and put for any $[f]$ in $F(U_{\alpha\beta})$:
\[ g_{\alpha\beta}(\xi_{\alpha\beta} + X_f) = g_{\alpha\beta}(\xi_{\alpha\beta}) + H_{\alpha\beta} f, \]
\[ f_{\alpha\alpha}^{2r} = 0, \quad f_{\beta\alpha}^{2r} = -f_{\alpha\beta}^{2r}, \]
\[ g_{\alpha\alpha} = 0, \quad g_{\beta\alpha} = -g_{\alpha\beta}. \]
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Now, on $U_{\alpha\beta\gamma}$ ($\alpha < \beta < \gamma$), $H_{\alpha\beta\gamma}$ ($= H_{\alpha\beta} + H_{\beta\gamma} + H_{\gamma\alpha}$) can be written as $\Phi([h_{\alpha\beta\gamma}])$. The problem is to choose simultaneously the $C^\infty$ functions $h_{\alpha\beta\gamma}$. As in [3], we choose the unique $C^\infty$ solution of all the equations:

$$L_{\xi_{\alpha\beta\gamma}} h_{\alpha\beta\gamma} + (2k - 1) h_{\alpha\beta\gamma} = -g_{\alpha\beta\gamma}(\xi_{\alpha\beta\gamma}),$$

for each $\xi_{\alpha\beta\gamma}$ in $\text{Con} f(U_{\alpha\beta\gamma})$. $h_{\alpha\beta\gamma}$ is totally antisymmetric in $\alpha$, $\beta$, $\gamma$ and $h_{\alpha\beta\gamma} - h_{\alpha\beta\delta} + h_{\alpha\gamma\delta} - h_{\beta\gamma\delta}$ vanishes on $U_{\alpha\beta\gamma\delta}$. We define then:

$$s_{\alpha\beta} = \sum_{\gamma} h_{\alpha\beta\gamma} \psi_{\gamma} \quad \text{in} \quad C^\infty(U_{\alpha\beta}),$$

$$G_{\alpha\beta} = H_{\alpha\beta} - \{s_{\alpha\beta}, \cdot\},$$

$$K_{\alpha} = \sum_{\beta} G_{\alpha\beta} \psi_{\beta}.$$

$G_{\alpha\beta\gamma}$ vanishes on $C^\infty(U_{\alpha\beta\gamma})$, $K_{\alpha}$ is well defined and, for each $\alpha$,

$$u *_{\alpha} v = \exp \nu^{2k} K_{\alpha}(\exp -\nu^{2k} K_{\alpha} u *_{\alpha} \exp -\nu^{2k} K_{\alpha} v),$$

$$D_{\alpha}'(\xi_{\alpha}) = \exp \nu^{2k} K_{\alpha} \circ D_{\alpha}(\xi_{\alpha}) \circ \exp -\nu^{2k} K_{\alpha}$$

satisfy the induction hypothesis at order $2k + 2$.

If the second Čech cohomology group of $M$ vanishes, there exists a global conformal vector field $\xi$ on $M$ and a global derivation $D'(\xi)$ of the $*$-product therefore we re-find here the proof of [11]. In the general case, our proof by building directly a $*$-product does not need the theorem of [6] which allows to construct $*$-product, starting with particular deformation of the Poisson bracket.

3. Parametrization of coadjoint orbits

Let $G$ be a connected and simply connected Lie group, $g$ its Lie algebra and $g^*$ the dual of $g$. $G$ acts on $g^*$ by the coadjoint action, denoted here by:

$$\langle g \cdot x, X \rangle = \langle x, \text{Ad} g^{-1}(X) \rangle, \forall X \in g, \forall x \in g^*, \forall g \in G.$$

Let $x_0$ be a point of $g^*$ and $M$ its coadjoint orbit $G \cdot x_0$, endowed with the canonical 2-form:

$$\omega_x(X^-, Y^-) = \langle x, [X, Y] \rangle (= B_x(X, Y)), \quad \forall X, Y \in g,$$
here $X^-$ is the vector field defined on $M$ by:

$$X^- f(x) = \frac{d}{dt} f(\exp(-tX \cdot x)) \bigg|_{t=0}.$$ 

From now on, we suppose there exists a real polarization $\mathfrak{h}$ in $x_0$. This means $\mathfrak{h}$ is a maximal isotropic subspace in $\mathfrak{g}$ for the bilinear form $B_{x_0}$, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and, if $G(x_0)$ is the stabilizer of $x_0$, $\text{Ad} G_{x_0}(\mathfrak{h}) \subset \mathfrak{h}$.

If $H_0$ is the analytic subgroup of $G$, with Lie algebra $\mathfrak{h}$, we denote by $H$ the subgroup $G(x_0)H_0$ of $G$. Then $M$ becomes a fibre bundle over $G/H$:

$$\pi : M = G/G(x_0) \longrightarrow G/H.$$ 

In this part, we recall the results of Pedersen [10].

Let $\mathcal{E}^0$ be the subspace $\pi_*(C^\infty(G/H))$. It is an abelian subalgebra of

$$(C^\infty(M), \{\cdot, \cdot\})$. Let $\mathcal{E}^1$ be the algebra:

$$\mathcal{E}^1 = \{u \in C^\infty(M) \text{ such that } \{u, \mathcal{E}^0 \} \subset \mathcal{E}^0 \}.$$ 

For each open subset $V$ in $G/H$, we define $\mathcal{E}^0(V)$ as $\pi_*(C^\infty(V))$ and $\mathcal{E}^1(V)$ as

$$\left\{ u \in C^\infty(\pi^{-1}(V)) \text{ such that } \{u, \mathcal{E}^0(V)\} \subset \mathcal{E}^0(V) \right\}.$$ 

The space $\mathcal{E}^1$ is sometimes called the space of quantizable functions. It is easy to verify that the functions $\tilde{X}$, for $X$ in $\mathfrak{g}$ are in $\mathcal{E}^1$. Now let $m$ be a supplementary space of $\mathfrak{h}$ in $\mathfrak{g}$ and $V$ be a sufficiently small neighborhood in $G/H$ such that $m$ is a supplementary space of $\text{Ad} \, g\mathfrak{h}$ for each $g$ in $G$ such that $g \cdot x_0$ belongs to $V$. Pedersen proved that, if $(X_1, \ldots, X_k)$ is a basis of $m$, then, on $\pi^{-1}(V)$, we can write each function $u$ of $\mathcal{E}^1(V)$ in the form:

$$u \big|_{\pi^{-1}(V)} = \left( \sum_{i=1}^{n} \alpha_i \tilde{X}_i + \alpha_0 \right) \bigg|_{\pi^{-1}(V)},$$ 

where the $\alpha_i$ are in $\mathcal{E}_0(V)$. Moreover, the $\alpha_i$ are uniquely determined on $\pi^{-1}(V)$ by that relation. Now we define a "local" induced representation. First there exists a local character $\chi$ of $H$: let $\mathcal{V}$ be a neighborhood of $0$ in $\mathfrak{h}$ such that $\exp$ is a diffeomorphism on $\mathcal{V}$, we put:

$$\chi(\exp X) = e^{i(x_0, X)} \text{ if } X \in \mathcal{V}.$$ 

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Then, if $\mathcal{U}$ is a neighborhood of 0 in $m$ and $\mathcal{V}$, $\mathcal{U}$ sufficiently small, the neighborhood $\mathcal{G} = \exp(\mathcal{U}) \exp(\mathcal{V})$ of unity in $G$ is diffeomorphic to $\mathcal{U} \times \mathcal{V}$, we choose $V$ to be $\exp(\mathcal{U})H$ and define the local representation $(E, \rho)$ by:

$$E = \{ \phi \in C^\infty \text{ such that } \phi(xh) = \chi(h)^{-1} \phi(x) \text{ if } h \in \exp(V), \ x, \ xh \in \mathcal{G} \}$$

and

$$(\rho(a) \phi)(x) = \phi(a^{-1} x) \text{ if } a, \ x, \ a^{-1} x \in \mathcal{G}.$$ 

Of course, we can identify $E$ with $C^\infty(V)$ by putting, for each $f$ in $C^\infty(V)$,

$$\phi(xh) = \chi(h)^{-1} f(xH) \text{ if } x \in \exp(\mathcal{U}), \ h \in \exp(V).$$

Generally, $\rho$ cannot be extended to a representation of $G$. But infinitesimally,

$$d\rho(X)\phi(x) = \frac{d}{dt} (\rho(\exp tX)\phi)(x)\bigg|_{t=0}$$

is a representation of $g$ on the space $C^\infty(V)$. Moreover, by construction, the $d\rho(X)$ are differential operators of order 1 on $V$, we write:

$$d\rho(X) \in \text{Diff}^1(V).$$

Finally, we call $U$ the set $\pi^{-1}(V)$ and define a map $\delta$ from $\mathcal{E}^1(V)$ to $\text{Diff}^1(V)$ by:

$$\delta(u) = \delta \left( \sum_{i=1}^{k} \tilde{X}_i + \alpha_0 \right) = \sum_{i=1}^{k} \alpha_i \ d\rho(X_i) + a_0.$$

**Theorem [10].** $\delta$ is an isomorphism of Lie algebras between $\mathcal{E}^1(V)$ and $\text{Diff}^1(V)$.

The proof of this in [10], indeed, it is a direct consequence of the fact that $d\rho$ is a representation. Now, we define canonical coordinates on $U$: let $(y_1, \ldots, y_k)$ be a coordinate system on $V$ in $G/H$, we define:

$$q_i = \pi_* y_i, \ p_i = \delta^{-1}(\partial_{y_i}).$$

$(p_i, q_i)$ is a canonical system of coordinates on $U$, the $q_i$ belong to $\mathcal{E}^0(V)$ and the $p_i$ to $\mathcal{E}^1(V)$. Then by construction, we have the following theorem.
THEOREM. — On the intersection of two such chart $U$ and $U'$, the coordinates satisfy:

$$q'_i = Q_i(q), \quad p'_i = \sum_{j=1}^{k} \alpha_{ij}(q)p_j + \alpha_{i0}(q).$$

(*)

Endowed with that atlas, $M$ is an open subset of an affine bundle $L$ over $G/H$, whose transition functions are defined by the relations (*).

Remarks

The functions $\tilde{X}$ being in $\mathcal{E}^1$, they have the following form in our coordinate system:

$$\tilde{X} = \sum_{i=1}^{k} \alpha_i(q)p_i + \alpha_0(q).$$

If $\mathfrak{h}$ satisfies the Pukanszky condition, then $M$ is exactly the bundle $L$.

4. Construction of covariant $\star$-product

We consider now our orbit $M$ as an open submanifold of the fibre bundle $\pi : L \to G/H$. $L$ is canonically polarized with the tangent spaces $T_x L_x$ of its fibres $L_x$. Then we build up a $\star$-product on $L$ as in the second part. We still denote by $\mathcal{E}^0$ (resp. $\mathcal{E}^0(V)$) the space $\pi_* (C^\infty (G/H))$ (resp. $\pi_* (C^\infty (V))$). Moreover, we choose our canonical charts with domain $\pi^{-1}(V_\alpha)$ where $V_\alpha$ is one of the local domains of chart defined in the third part and the partition of unity $\psi_\alpha$ subordinate to $U_\alpha$ in $\mathcal{E}^0$. Finally, we add to our induction hypothesis that, for each $\alpha$, $C_{r,\alpha}$ is vanishes on $\mathcal{E}^1(V_\alpha)$ for $r > 2$ and $C_{2,\alpha}(\mathcal{E}^1(V_\alpha), \mathcal{E}^1(V_\alpha)) \subset \mathcal{E}^0(V_\alpha)$.

If we choose the $(p, q)$ coordinates of the preceeding part on our neighborhood $U_\alpha$ and begin with Moyal product with these coordinates, then for $k = 1$, the induction hypothesis holds.

Now, it is not very difficult to choose $H_{\alpha\beta}$ such that $H_{\alpha\beta}$ vanishes on $\mathcal{E}^1(V_{\alpha\beta})$ (we choose first $H'_{\alpha\beta}$ such that $H'_{\alpha\beta}$ vanishes on $\mathcal{E}^1(V_{\alpha\beta})$), then we prove the existence of a $C^\infty$ function $\varphi_{\alpha\beta}$ such that $H_{\alpha\beta} = H'_{\alpha\beta} + \delta \varphi_{\alpha\beta}$ vanishes on $\mathcal{E}^1(V_{\alpha\beta}))$. With that choice, $H_{\alpha\beta\gamma}$ is a Hamiltonian vector field vanishing on $\mathcal{E}^1(V_{\alpha\beta\gamma})$ so it is identically zero. Hence we can construct directly the family $(K_\alpha)_{\alpha \in \mathcal{A}}$.
Now, because $K_\alpha$ vanishes on $\mathcal{E}^1(V_{\alpha\beta})$, our induction hypothesis is still true for $x'_\alpha$. In this way, we obtain a $\star$-product on $L$, after restriction to $M$, we have a covariant $\star$-product on $M$, since each $\bar{X}$ is in $\mathcal{E}^1$.

**Theorem.** — Let $x_0$ be an element in the dual $g^*$ of a Lie algebra $g$ such that there exists in $x_0$ a real polarization $\mathfrak{h}$. Then on the coadjoint orbit $M$ of $x_0$, there exists a covariant $\star$-product.

Let us recall [2] that for each covariant $\star$-product on $M$, there exists a representation of $G$ into the group of automorphisms of $(C^\infty(M)[[\nu]], \star)$, which is a deformation of the geometric action of $G$ on $M$ and $C^\infty(M)$.

**References**


