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Covariant star-products(*)

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RÉSUMÉ. — On donne une démonstration élémentaire du théorème d'existence de produits-star sur les variétés symplectiques.

On montre l'existence de produit-star covariant sur les orbites coadjointes admettant une polarisation réelle.

ABSTRACT. — We give a direct and elementary proof of the well known existence theorem of $\star$-products on a symplectic manifold.

Looking for covariant $\star$-product on coadjoint orbits, we prove the existence of such a deformation when the orbit admits a real polarization.

1. Introduction

$\star$-products were defined in [1] by Flato, Fronsdal, Lichnerowicz as a tool for quantizing a classical system, described with a symplectic manifold (M, ω). Roughly speaking, a $\star$-product is a (formal) deformation of the associative algebra $C^\infty(M)$ provided with usual (pointwise) product starting with the Poisson bracket. The quantum structure is then the deformed structure on the unchanged space of observables.

Each quantization procedure, when applied on a coadjoint orbit M of a Lie group G, gives some way to build up unitary irreducible representations of G. To use $\star$-products for such a purpose, we need in fact a particular property, the covariance of the $\star$-product:

$$[\widetilde{X}, \widetilde{Y}]_\star = \{\widetilde{X}, \widetilde{Y}\} = [\widetilde{X, Y}],$$

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if $\widetilde{X}$ is, for each X in the Lie algebra $\mathfrak{g}$ of G, the function on M defined by

$$x \mapsto \widetilde{X}(x) = \langle x\,,\, X\rangle\,.$$

(See [2] for a discussion on invariance and covariance properties for $\star$-products on a coadjoint orbit.)

In this paper, we recall first the theorem of existence of $\star$-products on a sympletic manifold. This theorem is due to P. Lecomte and M. de Wilde [3]. Some recent new proofs were given by Maeda, Omori and Yoshioka [4] and Lecomte and de Wilde [5]. We expose here that last proof in a slighty different way, which is direct and totally elementary: we build a $\star$-product by gluing together local $\star$-products defined on domains of a chart of M. That proof follows the idea of Vey, Lichnerowicz, Neroslavsky and Vlassov ([6], [7]) and, of course, Maeda, Onori and Yoshioka. In these approaches, the obstruction to construct $\star$-product lies in the third cohomology group $H^3(M)$ of the manifold M. Lecomte and de Wilde defined formal deformation of the Lie algebra $\big(C^\infty(M), \{\,\cdot\,,\,\cdot\,\}\big)$, for such a deformation, the obstruction is in the group $H^3\big(C^\infty(M)\big)$ for the adjoint action, which contains strictly $H^3(M)$. Let us finally mention the construction of Maslov and Karasev [8] who found an obstruction in $H^2(M)$ to construct simultaneously a deformation and a representation of the deformed structure on $C^\infty(M)$. Lecomte and de Wilde proved that all these obstructions can be surrounded, with the use of local conformal vector fields on M. We follow here that classical proof, using only local computations and Čech calculus.

Then we use this proof in the case of a coadjoint orbit in the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. More precisely, we consider a point x_0 in $\mathfrak{g}^*$ and suppose there exists in x_0 a real polarization. Under this assumption, we prove the existence of a covariant $\star$-product on the coadjoint orbit of x_0, endowed with its canonical symplectic structure.

2. Existence of $\star$-products on a symplectic manifold

Let (M,ω) be a symplectic manifold. We denote by $\{\,\cdot\,,\,\cdot\,\}$ the Poisson bracket defined on $C^\infty(M)$ by the usual relations:

$$\{u,v\} = X_u v \quad \text{if} \quad \mathrm{i}_{X_u}\omega = -\mathrm{d}u\,.$$

A $\star$-*product* is by definition a formal deformation in the sense of Gerstenhaber [9] of the associative algebra $C^\infty(M)$, i.e. a bilinear map:

$$C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)[[\nu]]\,, \quad (u,v) \mapsto v \star v = \sum_{r \geq 0} \nu^r C_r(u,v)\,,$$

where $C^\infty(M)[[\nu]]$ is the space of formal power series in the variable ν with coefficients in $C^\infty(M)$, such that each C_r is a bidifferential operator and:

(i) $C_0(u,v) = uv$, $C_1(u,v) = \{u,v\}$,

(ii) $C_r(u,v) = (-1)^r C_r(v,u)$,

(iii) $C_r(1,u) = 0$, $\forall\, r > 0$,

(iv) $\sum_{r+s=t} C_r\big(C_s(u,v)\,,\, w\big) = \sum_{r+s=t} C_r\big(u\,,\, C_s(u,w)\big)$, $\forall\, t \geq 0$.

With these properties, $\star$ defines an associative structure on $C^\infty(M)[[\nu]]$, of whom unity is 1 and:

$$\left[u,v\right]_\star = \sum_{r \geq 0} \nu^{2r} C_{2r+1}(u,v) = \frac{1}{2\nu}\,(u \star v - v \star u)$$

is a Lie bracket (it satisfies Jacobi identity) and a formal deformation of the Poisson bracket.

On a symplectic vector space $\mathbb{R}^{2n}$ and thus on any domain U of a canonical chart in M; there exists $\star$-products, for instance the Moyal $\star$-product [1].

THEOREM [3].— *On each simplectic manifold (M,ω), there exists a $\star$-product.*

Proof.— Let us first choose a locally finite covering $\left(U_\alpha\right)_{\alpha \in A}$ of the manifold such that each U_α is the domain of a canonical chart on M and all the intersections:

$$U_{\alpha_1 \ldots \alpha_n} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$$

are contractible. We fix a total ordering $\leq$ on A and a partition of the unity ψ_α subordinated to $\left(U_\alpha\right)_{\alpha \in A}$. If $\star_\alpha$ is a $\star$-product on U_α and $\mathrm{Der}(\star_\alpha)$ the space of derivation of $\star_\alpha$, there exists a canonical linear mapping:

$$\Phi_\alpha : F(U_\alpha) = C^\infty(U_\alpha)/\{\text{constants}\} \longrightarrow \mathrm{Der}(\star_\alpha)\,, \quad \Phi_\alpha([f])(v) = \left[f,v\right]_\alpha .$$

Let us denote by $\mathrm{Con}\,f(U_\alpha)$ the space of (conformal) vector fields ξ_α on U_α such that:

$$L_{\xi_\alpha}\omega = \omega \quad \text{on } U_\alpha\,.$$

$\mathrm{Con}\,f(U_\alpha)$ is an affine space on $F(U_\alpha)$.

Now we suppose, by induction on k, to have, on each U_α, a $\star$-product $\star_\alpha$:

$$u \star_\alpha v = \sum_{r\geq 0} \nu^r C_{r,\alpha}(u,v)\,,$$

with $C_{r,\alpha} = C_{r,\beta}$ on $U_{\alpha\beta}$, for all $r < 2k$ and an affine mapping D_α form $\mathrm{Con}\,f(U_\alpha)$ into $\mathrm{Der}(\star_\alpha)\big(D_\alpha(\xi_\alpha + X_f)\big) = D_\alpha(\xi_\alpha) + \Phi_\alpha\big([f]\big)$ such that:

$$D_\alpha(\xi_\alpha) = \nu\partial_\nu + L_{\xi_\alpha} + \sum_{r>0} \nu^{2r} D_\alpha^{2r}(\xi_\alpha)\,,$$

the $D_\alpha^{2r}(\xi_\alpha)$ being differential operators, vanishing on constants. Of course, these assumptions hold for $k=1$.

Now it is well known ([6], [7]) that for each $\alpha < \beta$, we can find a differential operator vanishing on constants $H_{\alpha\beta}$ such that, up to order $2k+2$,

$$u \star'_\alpha v = \exp \nu^{2k} H_{\alpha\beta}(\exp -\nu^{2k} H_{\alpha\beta} u \star_\alpha \exp -\nu^{2k} H_{\alpha\beta} v)$$

coincide with $u \star_\beta v$. Thus:

$$\begin{aligned}(D_\alpha - D_\beta)(\xi_{\alpha\beta}) &= \\ &= \sum_{r=0}^{k-1} \nu^{2r}\Phi_\alpha\big([f^{2r}_{\alpha\beta}]\big) + \nu^{2k}\Big(L_{\xi_{\alpha\beta}} H_{\alpha\beta} + 2k H_{\alpha\beta} + \Phi_\alpha\left(\left[g_{\alpha\beta}(\xi_{\alpha\beta})\right]\right)\Big)\end{aligned}$$

here $[f^{2r}_{\alpha\beta}]$ in $F(U_{\alpha\beta})$ do not depend of $\xi_{\alpha\beta}$ while:

$$\big[g_{\alpha\beta}(\xi_{\alpha\beta} + X_f)\big] = \big[g_{\alpha\beta}(\xi_{\alpha\beta}) + H_{\alpha\beta} f\big]\,.$$

For each $\alpha < \beta$, we choose a vector field $\xi_{\alpha\beta}$ in $\mathrm{Con}\,f(U_{\alpha\beta})$, a C^∞ function $g_{\alpha\beta}(\xi_{\alpha\beta})$ and put for any $[f]$ in $F(U_{\alpha\beta})$:

$$\begin{aligned}&g_{\alpha\beta}(\xi_{\alpha\beta} + X_f) = g_{\alpha\beta}(\xi_{\alpha\beta}) + H_{\alpha\beta} f\,,\\ &f^{2r}_{\alpha\alpha} = 0\,, \quad f^{2r}_{\beta\alpha} = -f^{2r}_{\alpha\beta}\,,\\ &g_{\alpha\alpha} = 0\,, \quad g_{\beta\alpha} = -g_{\alpha\beta}\,.\end{aligned}$$

Now, on $U_{\alpha\beta\gamma}$ ($\alpha < \beta < \gamma$), $H_{\alpha\beta\gamma}$ ($= H_{\alpha\beta} + H_{\beta\gamma} + H_{\gamma\alpha}$) can be written as $\Phi([h_{\alpha\beta\gamma}])$. The problem is to choose simultaneously the C^∞ functions $h_{\alpha\beta\gamma}$. As in [3], we choose the unique C^∞ solution of all the equations:

$$L_{\xi_{\alpha\beta\gamma}} h_{\alpha\beta\gamma} + (2k-1) h_{\alpha\beta\gamma} = -g_{\alpha\beta\gamma}(\xi_{\alpha\beta\gamma}),$$

for each $\xi_{\alpha\beta\gamma}$ in $\operatorname{Con} f(U_{\alpha\beta\gamma})$. $h_{\alpha\beta\gamma}$ is totally antisymmetric in α, β, γ and $h_{\alpha\beta\gamma} - h_{\alpha\beta\delta} + h_{\alpha\gamma\delta} - h_{\beta\gamma\delta}$ vanishes on $U_{\alpha\beta\gamma\delta}$. We define then:

$$\begin{aligned} s_{\alpha\beta} &= \sum_\gamma h_{\alpha\beta\gamma}\psi_\gamma \quad \text{in} \quad C^\infty(U_{\alpha\beta}), \\ G_{\alpha\beta} &= H_{\alpha\beta} - \{s_{\alpha\beta}, \cdot\}, \\ K_\alpha &= \sum_\beta G_{\alpha\beta}\psi_\beta . \end{aligned}$$

$G_{\alpha\beta\gamma}$ vanishes on $C^\infty(U_{\alpha\beta\gamma})$, K_α is well defined and, for each α,

$$\begin{aligned} u \star'_\alpha v &= \exp \nu^{2k} K_\alpha (\exp -\nu^{2k} K_\alpha u \star_\alpha \exp -\nu^{2k} K_\alpha v), \\ D'_\alpha(\xi_\alpha) &= \exp \nu^{2k} K_\alpha \circ D_\alpha(\xi_\alpha) \circ \exp -\nu^{2k} K_\alpha \end{aligned}$$

satisfy the induction hypothesis at order $2k+2$.

If the second Čech cohomology group of M vanishes, there exists a global conformal vector field ξ on M and a global derivation $D'(\xi)$ of the $\star$-product therefore we refind here the proof of [11]. In the general case, our proof by building directly a $\star$-product does not need the theorem of [6] which allows to construct $\star$-product, starting with particular deformation of the Poisson bracket.

3. Parametrization of coadjoint orbits

Let G be a connected and simply connected Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. G acts on $\mathfrak{g}^*$ by the coadjoint action, denoted here by:

$$\langle g \cdot x , X \rangle = \langle x , \operatorname{Ad} g^{-1}(X) \rangle, \forall\, X \in \mathfrak{g}, \ \forall\, x \in \mathfrak{g}^*, \ \forall\, g \in G.$$

Let x_0 be a point of $\mathfrak{g}^*$ and M its coadjoint orbit $G \cdot x_0$, endowed with the canonical 2-form:

$$\omega_x(X^-, Y^-) = \langle x , [X , Y] \rangle \ (= B_x(X,Y)), \quad \forall\, X, Y \in \mathfrak{g},$$

here X^- is the vector field defined on M by:

$$X^- f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp -tX \cdot x)\big|_{t=0} .$$

From now on, we suppose there exists a real polarization $\mathfrak{h}$ in x_0. This means $\mathfrak{h}$ is a maximal isotropic subspace in $\mathfrak{g}$ for the bilinear form B_{x_0}, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and, if $G(x_0)$ is the stabilizer of x_0, $\operatorname{Ad} G_{x_0}(\mathfrak{h}) \subset \mathfrak{h}$.

If H_0 is the analytic subgroup of G, with Lie algebra $\mathfrak{h}$, we denote by H the subgroup $G(x_0)H_0$ of G. Then M becomes a fibre bundle over G/H:

$$\pi : M = G/G(x_0) \longrightarrow G/H .$$

In this part, we recall the results of Pedersen [10].

Let $\mathcal{E}^0$ be the subspace $\pi_*\big(C^\infty(G/H)\big)$. It is an abelian subalgebra of $\big(C^\infty(M), \{\cdot\,,\,\cdot\}\big)$. Let $\mathcal{E}^1$ be the algebra:

$$\mathcal{E}^1 = \big\{u \in C^\infty(M) \text{ such that } \{u, \mathcal{E}^0\} \subset \mathcal{E}^0\big\} .$$

For each open subset V in G/H, we define $\mathcal{E}^0(V)$ as $\pi_*\big(C^\infty(V)\big)$ and $\mathcal{E}^1(V)$ as

$$\Big\{u \in C^\infty\big(\pi^{-1}(V)\big) \text{ such that } \big\{u\,,\, \mathcal{E}^0(V)\big\} \subset \mathcal{E}^0(V)\Big\} .$$

The space $\mathcal{E}^1$ is sometimes called the space of quantizable functions. It is easy to verify that the functions $\widetilde{X}$, for X in $\mathfrak{g}$ are in $\mathcal{E}^1$. Now let $\mathfrak{m}$ be a supplementary space of $\mathfrak{h}$ in $\mathfrak{g}$ and V be a sufficiently small neighborhood in G/H such that $\mathfrak{m}$ is a supplementary space of $\operatorname{Ad} g\mathfrak{h}$ for each g in G such that $g \cdot x_0$ belongs to V. Pedersen proved that, if $(X_1, \ldots, X_k)$ is a basis of $\mathfrak{m}$, then, on $\pi^{-1}(V)$, we can write each function u of $\mathcal{E}^1(V)$ in the form:

$$u\big|_{\pi^{-1}(V)} = \left(\sum_{i=1}^{n} \alpha_i \widetilde{X}_i + \alpha_0\right)\Bigg|_{\pi^{-1}(V)} ,$$

where the α_i are in $\mathcal{E}_0(V)$. Moreover, the α_i are uniquely determined on $\pi^{-1}(V)$ by that relation. Now we define a "local" induced representation. First there exists a local character χ of H: let $\mathcal{V}$ be a neighborhood of 0 in $\mathfrak{h}$ such that exp is a diffeormorphism on $\mathcal{V}$, we put:

$$\chi(\exp X) = e^{i\langle x_0, X\rangle} \quad \text{if } X \in \mathcal{V} .$$

Then, if $\mathcal{U}$ is a neighborhood of 0 in $\mathfrak{m}$ and $\mathcal{V}$, $\mathcal{U}$ sufficiently small, the neighborhood $\mathcal{G} = \exp(\mathcal{U})\exp(\mathcal{V})$ of unity in G is diffeomorphic to $\mathcal{U} \times \mathcal{V}$, we choose V to be $\exp(\mathcal{U})H$ and define the local representation (E, ρ) by:

$$E = \left\{\phi \in C^\infty \text{ such that } \phi(xh) = \chi(h)^{-1}\phi(x) \text{ if } h \in \exp(\mathcal{V}),\ x\,,\ xh \in \mathcal{G}\right\}$$

and

$$\big(\rho(a)\phi\big)(x) = \phi(a^{-1}x) \quad \text{if } a\,,\ x\,,\ a^{-1}x \in \mathcal{G}\,.$$

Of course, we can identify E with $C^\infty(V)$ by putting, for each f in $C^\infty(V)$,

$$\phi(xh) = \chi(h)^{-1}f(xH) \quad \text{if } x \in \exp(\mathcal{U})\,,\ h \in \exp(\mathcal{V})\,.$$

Generally, ρ cannot be extended to a representation of G. But infinitesimally,

$$\mathrm{d}\rho(X)\phi(x) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(\exp tX)\phi\right)(x)\Big|_{t=0}$$

is a representation of $\mathfrak{g}$ on the space $C^\infty(V)$. Moreover, by construction, the $\mathrm{d}\rho(X)$ are differential operators of order 1 on V, we write:

$$\mathrm{d}\rho(X) \in \mathrm{Diff}^1(V)\,.$$

Finally, we call U the set $\pi^{-1}(V)$ and define a map δ from $\mathcal{E}^1(V)$ to $\mathrm{Diff}^1(V)$ by:

$$\delta(u) = \delta\left(\sum_{i=1}^{k}\widetilde{X}_i + \alpha_0\right) = \sum_{i=1}^{k}\alpha_i\,\mathrm{d}\rho(X_i) + a_0\,.$$

THEOREM [10].— *δ is an isomorphism of Lie algebras between $\mathcal{E}^1(V)$ and* $\mathrm{Diff}^1(V)$.

The proof of this in [10], indeed, it is a direct consequence of the fact that $\mathrm{d}\rho$ is a representation. Now, we define canonical coordinates on U: let $(y_1, \ldots, y_k)$ be a coordinate system on V in G/H, we define:

$$q_i = \pi_* y_i\,, \quad p_i = \delta^{-1}(\partial_{y_i})\,.$$

(p_i, q_i) is a canonical system of coordinates on U, the q_i belong to $\mathcal{E}^0(V)$ and the p_i to $\mathcal{E}^1(V)$. Then by construction, we have the following theorem.

Theorem .— *On the intersection of two such chart U and U', the coordinates satisfy:*

$$q'_i = Q_i(q)\,, \quad p'_i = \sum_{j=1}^{k} \alpha_{ij}(q)p_j + \alpha_{i0}(q)\,. \qquad (*)$$

Endowed with that atlas, M is an open subset of an affine bundle L over G/H, whose transition functions are defined by the relations $(*)$.

Remarks

The functions $\widetilde{X}$ being in $\mathcal{E}^1$, they have the following form in our coordinate system:

$$\widetilde{X} = \sum_{i=1}^{k} \alpha_i(q)p_i + \alpha_0(q)\,.$$

If $\mathfrak{h}$ satisfies the Pukanszky condition, then M is exactly the bundle L.

4. Construction of covariant $\star$-product

We consider now our orbit M as an open submanifold of the fibre bundle $\pi : L \to G/H$. L is canonically polarized with the tangent spaces $T_x L_x$ of its fibres L_x. Then we build up a $\star$-product on L as in the second part. We still denote by $\mathcal{E}^0$ (resp. $\mathcal{E}^0(V)$) the space $\pi_*\big(C^\infty(G/H)\big)$ (resp. $\pi_*\big(C^\infty(V)\big)$). Moreover, we choose our canonical charts with domain $\pi^{-1}(V_\alpha)$ where V_α is one of the local domains of chart defined in the third part and the partition of unity ψ_α subordinated to U_α in $\mathcal{E}^0$. Finally, we add to our induction hypothesis that, for each α, $C_{r,\alpha}$ is vanishes on $\mathcal{E}^1(V_\alpha)$ for $r > 2$ and $C_{2,\alpha}\big(\mathcal{E}^1(V_\alpha)\,,\,\mathcal{E}^1(V_\alpha)\big) \subset \mathcal{E}^0(V_\alpha)$.

If we choose the (p,q) coordinates of the preceeding part on our neighborhood U_α and begin with Moyal product with these coordinates, then for $k = 1$, the induction hypothesis holds.

Now, it is not very difficult to choose $H_{\alpha\beta}$ such that $H_{\alpha\beta}$ vanishes on $\mathcal{E}^1(V_{\alpha\beta})$ (we choose first $H'_{\alpha\beta}$ such that $H'_{\alpha\beta}$ vanishes on $\mathcal{E}^1(V_{\alpha\beta})$, then we prove the existence of a C^∞ function $\varphi_{\alpha\beta}$ such that $H_{\alpha\beta} = H'_{\alpha\beta} + \partial\varphi_{\alpha\beta}$ vanishes on $\mathcal{E}^1(V_{\alpha\beta})$). With that choice, $H_{\alpha\beta\gamma}$ is a Hamiltonian vector field vanishing on $\mathcal{E}^1(V_{\alpha\beta\gamma})$ so it is identically zero. Hence we can construct directly the family $\big(K_\alpha\big)_{\alpha\in A}$.

Now, because K_α vanishes on $\mathcal{E}^1(V_{\alpha\beta})$, our induction hypothesis is still true for $\star'_\alpha$. In this way, we obtain a $\star$-product on L, after restriction to M, we have a covariant $\star$-product on M, since each $\widetilde{X}$ is in $\mathcal{E}^1$.

THEOREM .— *Let x_0 be an element in the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ such that there exists in x_0 a real polarization $\mathfrak{h}$. Then on the coadjoint orbit M of x_0, there exists a covariant $\star$-product.*

Let us recall [2] that for each covariant $\star$-product on M, there exists a representation of G into the group of automorphisms of $\left(C^\infty(M)[[\nu]]\,,\star\right)$, which is a deformation of the geometric action of G on M and $C^\infty(M)$.

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