

OULD AHMED-IZID-BIH ISSELKOU

**Critical boundary constants and Pohozaev identity**

*Annales de la faculté des sciences de Toulouse 6<sup>e</sup> série*, tome 10,  
n° 2 (2001), p. 347-359

[http://www.numdam.org/item?id=AFST\\_2001\\_6\\_10\\_2\\_347\\_0](http://www.numdam.org/item?id=AFST_2001_6_10_2_347_0)

© Université Paul Sabatier, 2001, tous droits réservés.

L'accès aux archives de la revue « Annales de la faculté des sciences de Toulouse » (<http://picard.ups-tlse.fr/~annales/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## Critical boundary constants and Pohozaev identity <sup>(\*)</sup>

OULD AHMED-IZID-BIH ISSELKOU <sup>(1)</sup>

*à mes deux filles Kénizé et Maöna*

**RÉSUMÉ.** — La première partie de ce travail concerne le problème,

$$(P\epsilon) \begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ dans } B_1, \\ u > 0 \text{ dans } B_1, \\ u = \epsilon \text{ sur } \partial B_1, \end{cases}$$

où  $B_1 = \{x \in \mathbb{R}^n, \|x\| < 1\}$ ,  $n \geq 3$  et  $\epsilon > 0$ .

On démontre qu'il existe une constante critique  $\epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$ , telle que le problème  $(P\epsilon)$  admet exactement deux solutions  $u_{\epsilon 1}$  et  $u_{\epsilon 2}$  ( $u_{\epsilon 1} < u_{\epsilon 2}$ ) si  $0 < \epsilon < \epsilon^*$ , une solution unique si  $\epsilon = \epsilon^*$  et n'admet pas de solution si  $\epsilon > \epsilon^*$ . Toutes ces solutions seront données explicitement. Il est démontré que quand  $\epsilon \rightarrow 0$ ,

$$\frac{u_{\epsilon 1}(x) - \epsilon}{\epsilon} \rightarrow 0 \text{ sur } \overline{B_1} \text{ et } \frac{u_{\epsilon 2}(x) - \epsilon}{\epsilon} \rightarrow \|x\|^{2-n} - 1, \text{ sur } \overline{B_1} \setminus \{O\}.$$

Au cours de la seconde partie, on s'intéresse au problème

$$(Q\epsilon) \begin{cases} \Delta u + f(x, u) = 0 \text{ dans } \Omega, \\ u > 0 \text{ dans } \Omega, \\ u = \epsilon \text{ sur } \partial\Omega, \end{cases}$$

où  $\Omega$  est un domaine borné, régulier et étoilé par rapport à l'origine,  $f$  est continue et dépend asymptotiquement de  $u$  "comme  $u^\alpha$ ",  $1 < \alpha$  et  $\alpha \neq \frac{n+2}{n-2}$ . Différents résultats d'existence de constante au bord critique  $\epsilon^*$  pour le problème  $(Q\epsilon)$  sont donnés.

**ABSTRACT.** — The first part of this work deals with the problem

$$(P\epsilon) \begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = \epsilon \text{ on } \partial B_1. \end{cases}$$

(\*) Reçu le 21 mai 2000, accepté le 29 janvier 2001

(1) Faculté des Sciences et Techniques, B.P. 5026 Nouakchott, Mauritanie and The Abdus Salam ICTP, Trieste, Italy.

E-mail: isselkou@univ-nkc.mr

where  $B_1 = \{x \in \mathbb{R}^n, \|x\| < 1\}$ ,  $n \geq 3$  and  $\epsilon > 0$ .

We show that there exists a critical constant  $\epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$ , such that the problem  $(P\epsilon)$  admits two solutions  $u_{\epsilon 1}$  and  $u_{\epsilon 2}$  ( $u_{\epsilon 1} < u_{\epsilon 2}$ ) if  $0 < \epsilon < \epsilon^*$ , only one solution if  $\epsilon = \epsilon^*$  and no solution if  $\epsilon > \epsilon^*$ . We give all these solutions explicitly. We show that, when  $\epsilon \rightarrow 0$ ,

$$\frac{u_{\epsilon 1}(x) - \epsilon}{\epsilon} \rightarrow 0 \text{ on } \overline{B_1} \text{ and } \frac{u_{\epsilon 2}(x) - \epsilon}{\epsilon} \rightarrow \|x\|^{2-n} - 1, \text{ if } x \in \overline{B_1} \setminus \{O\}.$$

The second part is devoted to the following problem

$$(Q\epsilon) \begin{cases} \Delta u + f(x, u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = \epsilon \text{ on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a regular bounded domain which is starshaped about the origin,  $f$  is continuous and behaves like  $u^\alpha$  in the second variable,  $1 < \alpha$  and  $\alpha \neq \frac{n+2}{n-2}$ . We give different existence results for a boundary critical datum  $\epsilon^*$  for  $(Q\epsilon)$ .

---

## 1. The Sobolev Exponent Growth

Let us consider the problem

$$(P\epsilon) \begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } B_1, \\ u \geq 0 \text{ in } B_1, \\ u = \epsilon \text{ on } \partial B_1. \end{cases}$$

In [9], it is shown that there exists a critical boundary datum  $\epsilon^*$  such that  $\forall 0 < \epsilon < \epsilon^*$ ,  $(P\epsilon)$  admits -at least- one  $C^2$ -solution. There is no solution when  $\epsilon > \epsilon^*$ . It is known that  $(P0)$  does not admit a nontrivial solution (see [13]). According to [5], every regular solution of  $(P\epsilon)$  is spherically symmetric. Let  $u_\epsilon$  ( $\epsilon > 0$ ), be a solution of  $(P\epsilon)$ , then

$$v_\epsilon = \frac{u_\epsilon - \epsilon}{\epsilon}$$

is a solution of

$$(A\lambda) \begin{cases} \Delta w + \lambda(1+w)^{\frac{n+2}{n-2}} = 0 \text{ in } B_1, \\ w > 0 \text{ in } B_1, \\ w = 0 \text{ on } \partial B_1, \end{cases}$$

where

$$\lambda = \epsilon^{\frac{4}{n-2}}.$$

The maximum principle implies

$$v_\epsilon > 0, \text{ in } B_1.$$

It is known (see [11],[4] and [3]) that there exists a constant  $\lambda^*$  such that  $(A\lambda)$  admits just two  $C^3$ -spherically symmetric solutions when  $0 < \lambda < \lambda^*$ , only one solution for  $\lambda = \lambda^*$  and no solution if  $\lambda > \lambda^*$ . We give here the value of  $\lambda^*$  and the explicit solutions for every  $\lambda \leq \lambda^*$ . We show that when  $\lambda \rightarrow 0$ , the “small” solution tends to  $v = 0$  the trivial solution of  $(A0)$  and the “big” one tends to  $H(x) = \|x\|^{2-n} - 1$ ,  $x \neq O$ .

PROPOSITION 1

$$\lambda^* = (\epsilon^*)^{\frac{4}{n-2}} = \frac{n(n-2)}{4}.$$

*Proof.* — For

$$0 < \epsilon < \epsilon^* = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

let  $u_\epsilon$  be a regular  $(P\epsilon)$  solution. As in [13], let us put

$$g(x) = \sum_{i=1}^n x_i D_i v_\epsilon \nabla v_\epsilon \quad (\text{where } v_\epsilon = \frac{u_\epsilon - \epsilon}{\epsilon}),$$

and use the Divergence Theorem, to get

$$\begin{aligned} \epsilon^{\frac{4}{n-2}} \left\{ \left(1 - \frac{1}{2}n\right) \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} v_\epsilon(x) dx + \frac{n-2}{2} \int_{B_1} \left[ (1 + v_\epsilon(x))^{\frac{2n}{n-2}} - 1 \right] dx \right\} \\ + \frac{1}{2} \int_{\partial B_1} [x \cdot \nu] \|\nabla v_\epsilon(x)\|^2 ds = \int_{\partial B_1} [x \cdot \nabla v_\epsilon(x)] [\nabla v_\epsilon(x) \cdot \nu] ds. \end{aligned}$$

From the identity

$$\int_{B_1} [1 + v_\epsilon(x)]^{\frac{2n}{n-2}} dx = \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} v_\epsilon(x) dx + \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} dx,$$

we infer that

$$\begin{aligned} (*) \quad \epsilon^{\frac{4}{n-2}} \left\{ \frac{2-n}{2} \int_{B_1} dx + \frac{n-2}{2} \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} dx \right\} \\ + \frac{1}{2} \int_{\partial B_1} [x \cdot \nu] \|\nabla v_\epsilon(x)\|^2 ds = \int_{\partial B_1} [x \cdot \nabla v_\epsilon(x)] [\nabla v_\epsilon(x) \cdot \nu] ds. \end{aligned}$$

Using again the Divergence Theorem, we get

$$\epsilon^{\frac{4}{n-2}} \int_{B_1} (1 + v_\epsilon)^{\frac{n+2}{n-2}} dx = - \int_{B_1} \Delta v_\epsilon(x) dx = - \int_{\partial B_1} \frac{\partial v_\epsilon(x)}{\partial \nu} ds,$$

where  $\nu$  denotes the unit outward normal to  $\partial B_1$ .

The Maximum Principle implies that

$$\frac{\partial v_\epsilon}{\partial \nu} < 0 \text{ on } \partial B_1.$$

As  $v_\epsilon$  is spherically symmetric and vanishes on  $\partial B_1$ , we get

$$x \cdot \nabla v_\epsilon(x) = x \cdot \nu \frac{\partial v_\epsilon}{\partial \nu}, \text{ on } \partial \Omega, \text{ and}$$

$$\frac{\partial v_\epsilon(x)}{\partial \nu} = -\|\nabla v_\epsilon(x)\| = l, \text{ on } \partial B_1,$$

where  $l$  is a constant on  $\partial B_1$ . Using the fact that  $x \cdot \nu = 1$  on  $\partial B_1$ , we obtain from (\*)

$$\epsilon^{\frac{4}{n-2}} \frac{2-n}{2} \int_{B_1} dx - \frac{n-2}{2} l \int_{\partial B_1} ds = \frac{1}{2} l^2 \int_{\partial B_1} ds.$$

This equation is equivalent to

$$|S_1| l^2 + (n-2) |S_1| l + \epsilon^{\frac{4}{n-2}} (n-2) |B_1| = 0.$$

$$|B_1| = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}, \text{ and } |S_1| = n|B_1|,$$

where  $|B_1|$  is the Lebesgue measure of the unit ball of  $\mathfrak{R}^n$  and  $|S_1| = n|B_1|$  is the surface measure of the unit sphere.

We obtain the following equation in  $l = v'_\epsilon(1)$

$$(1) \quad nl^2 + n(n-2)l + \epsilon^{\frac{4}{n-2}}(n-2) = 0.$$

When

$$0 < \epsilon < \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

this equation admits two negative solutions

$$l_1(\epsilon) = \frac{n(2-n) + \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}$$

$$l_2(\epsilon) = \frac{n(2-n) - \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}.$$

When

$$\epsilon = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

equation (1) admits a unique negative solution

$$l_1 = \frac{2-n}{2},$$

and no real solution if

$$\epsilon > \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}}.$$

So it is clear that

$$\lambda^* = (\epsilon^*)^{\frac{4}{n-2}} \leq \frac{n(n-2)}{4}.$$

The proof will be complete, if one shows that

$$\forall 0 < \epsilon < \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

there exists just two regular solutions of  $(P\epsilon)$ . Let us recall that the problem

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } \mathfrak{R}^n,$$

admits the radial solutions (see [12])

$$u_\lambda(\|x\|) = \lambda^{\frac{n-2}{4}} (n(n-2))^{\frac{n-2}{4}} (\lambda^2 + \|x\|^2)^{\frac{2-n}{2}}, \lambda > 0.$$

Let us put

$$\phi_\lambda = u_{\lambda|_{B_1}}.$$

$$\max_{\lambda > 0} \phi_\lambda(1) = \phi_1(1) = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}}.$$

It is immediate to verify that

$$\phi_\lambda(1) = \phi_{\frac{1}{\lambda}}(1), \forall \lambda > 0.$$

$$\phi_\lambda \neq \phi_{\frac{1}{\lambda}}, \forall \lambda \neq 1.$$

In particular,

$$\phi'_\lambda(1) \neq \phi'_{\frac{1}{\lambda}}(1), \forall \lambda \neq 1.$$

As the function

$$\lambda \rightarrow \phi_\lambda(1) = \lambda^{\frac{n-2}{2}} (n(n-2))^{\frac{n-2}{4}} (1 + \lambda^2)^{\frac{2-n}{2}},$$

is continuous on  $[1, \infty[$ , with

$$\lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} (n(n-2))^{\frac{n-2}{4}} (1 + \lambda^2)^{\frac{2-n}{2}} = 0,$$

we obtain two distinct solutions of  $(P\epsilon)$ , when

$$0 < \epsilon < \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

and one solution when

$$\epsilon = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

which is  $\phi_1$ .

So the proof of Proposition 1 is complete.

Let us study the behavior of solutions when  $\epsilon \rightarrow 0$ . Let  $u_{\epsilon 1}$  and  $u_{\epsilon 2}$  be the two solutions of  $(P\epsilon)$ , with

$$u'_{\epsilon 1}(1) = l_1(\epsilon) = \frac{n(2-n) + \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n},$$

and

$$u'_{\epsilon 2}(1) = l_2(\epsilon) = \frac{n(2-n) - \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}.$$

$$\forall 0 < \epsilon < \epsilon^*, \exists! \lambda(\epsilon) > 1,$$

such that

$$u_{\epsilon 1} = \phi_{\lambda(\epsilon)} \quad \text{and} \quad u_{\epsilon 2} = \phi_{\frac{1}{\lambda(\epsilon)}}.$$

$$\lambda(\epsilon) = \frac{[n(n-2)]^{\frac{1}{2}} + \sqrt{n(n-2) - 4\epsilon^{\frac{4}{n-2}}}}{2\epsilon^{\frac{2}{n-2}}}.$$

Let us put

$$\psi_{\epsilon 1} = \frac{u_{\epsilon 1} - \epsilon}{\epsilon} \quad \text{and} \quad \psi_{\epsilon 2} = \frac{u_{\epsilon 2} - \epsilon}{\epsilon}.$$

PROPOSITION 2

- (i)  $\psi_{\epsilon 1} \rightarrow \psi_1 = 0$ , in  $C^1(\overline{B_1})$ , as  $\epsilon \rightarrow 0$ .
- (ii)  $\psi_{\epsilon 2}(x) \rightarrow \psi_2 = \|x\|^{2-n} - 1$ , in  $C^1_{loc}(\overline{B_1} \setminus \{O\})$ , as  $\epsilon \rightarrow 0$ .

*Proof.* — Let us remark the following

$$l_1(\epsilon) \rightarrow 0, \text{ as } \epsilon \rightarrow 0, \text{ and } l_2(\epsilon) \rightarrow 2 - n, \text{ as } \epsilon \rightarrow 0.$$

We give here a direct proof, using the explicit knowledge of  $\psi_{\epsilon i}$ ,  $i \in \{1, 2\}$ . As we have seen, for every  $0 < \epsilon < \epsilon^*$ , there exists a unique

$$\lambda(\epsilon) = \frac{[n(n-2)]^{\frac{1}{2}} + \sqrt{n(n-2) - 4\epsilon^{\frac{4}{n-2}}}}{2\epsilon^{\frac{2}{n-2}}} > 1,$$

$$(\lambda(\epsilon) \rightarrow \infty \text{ as } \epsilon \rightarrow 0)$$

such that we have

$$\psi_{\epsilon 1}(r) = \frac{[\lambda(\epsilon)]^{\frac{n-2}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ ([\lambda(\epsilon)]^2 + r^2)^{\frac{2-n}{2}} - ([\lambda(\epsilon)]^2 + 1)^{\frac{2-n}{2}} \right\}}{[\lambda(\epsilon)]^{\frac{n-2}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ [\lambda(\epsilon)]^2 + 1 \right\}^{\frac{2-n}{2}}},$$

$$\psi_{\epsilon 2}(r) = \frac{[\lambda(\epsilon)]^{\frac{2-n}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ ([\lambda(\epsilon)]^{-2} + r^2)^{\frac{2-n}{2}} - ([\lambda(\epsilon)]^{-2} + 1)^{\frac{2-n}{2}} \right\}}{[\lambda(\epsilon)]^{\frac{2-n}{2}} (n(n-2))^{\frac{n-2}{4}} \left\{ [\lambda(\epsilon)]^{-2} + 1 \right\}^{\frac{2-n}{2}}}.$$

We finally get

$$\psi_{\epsilon 1}(r) = \left( \frac{[\lambda(\epsilon)]^2 + 1}{[\lambda(\epsilon)]^2 + r^2} \right)^{\frac{n-2}{2}} - 1; \quad \psi_{\epsilon 2}(r) = \left\{ \frac{1 + [\lambda(\epsilon)]^2}{1 + [\lambda(\epsilon)]^2 r^2} \right\}^{\frac{n-2}{2}} - 1.$$

It is immediate to verify that

$$\psi_{\epsilon 1}(\|x\|) = \left( \frac{[\lambda(\epsilon)]^2 + 1}{[\lambda(\epsilon)]^2 + \|x\|^2} \right)^{\frac{n-2}{2}} - 1 \rightarrow 0, \text{ in } C^1(\overline{B_1}), \text{ as } \epsilon \rightarrow \infty$$

and

$$\psi_{\epsilon 2}(\|x\|) = \left\{ \frac{1 + [\lambda(\epsilon)]^2}{1 + [\lambda(\epsilon)]^2 \|x\|^2} \right\}^{\frac{n-2}{2}} - 1 \rightarrow \|x\|^{2-n} - 1, \text{ on } \overline{B_1} \setminus \{O\}.$$

*Remark 1.* — According to Theorem 1.1 in [7], it is, in general, false that every positive solution  $u$  in  $B_1$  of  $\Delta u + u^\alpha = 0$ , is a restriction of a positive solution  $v$  of this problem in  $\mathfrak{R}^n$ .



## 2. Nonlinearities with Noncritical Growth

### 2.1. The Subcritical Behavior

We deal here with the following problem

$$(Q\epsilon) \begin{cases} \Delta u + f(x, u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = \epsilon \text{ on } \partial\Omega, \end{cases}$$

Let us suppose the following

(i)  $\Omega$  is a bounded regular domain of  $\mathbb{R}^n$ , which is starshaped about the origin.

(ii)  $f \in C^0(\bar{\Omega} \times \mathbb{R}_+, \mathbb{R}_+)$ ,

(iii) there exist positive constants  $c_1, \gamma, \alpha$  and a positive function  $a \in C^0(\bar{\Omega})$  such that

$$1 < \gamma \leq \alpha < \frac{n+2}{n-2}; \quad c_1 t^\gamma \leq f(x, t), \quad \forall x \in \bar{\Omega}, t > 0,$$

$$\lim_{t \rightarrow \infty} \frac{f(x, t)}{t^\alpha} = a(x) \text{ and } f(x, t) = o(t) \text{ near } t = 0, \text{ uniformly in } x \in \bar{\Omega}.$$

**PROPOSITION 3.** — *Under the previous hypotheses on  $\Omega$  and  $f$ , there exists a positive constant  $\epsilon^*(\Omega, f)$ , such that for every  $0 \leq \epsilon \leq \epsilon^*(\Omega, f)$ , the problem  $(Q\epsilon)$  admits, at least, one solution  $u_\epsilon \in C^{1,\delta}(\bar{\Omega})$ ,  $0 \leq \delta < 1$ . There is no bounded solution of  $(Q\epsilon)$  if  $\epsilon > \epsilon^*(\Omega, f)$ .*

*Proof.* — The proof is nearly the same as in (Theorem 1 in [9]). The only difference is that the subsolutions and supersolutions are considered as elements of  $H_0^1(\Omega) \cap L^\infty(\Omega)$ , and the inequalities are in the sense of duality  $H^{-1}(\Omega)$ ,  $H_0^1(\Omega)$ .

Let us recall the main steps for this proof.

1. We use the hypothesis (iii) and Théorème 3.1 in [2], to show that the problem  $(Q\epsilon)$  admits -at least- one solution  $u \in H_0^1(\Omega)$ , when  $\epsilon$  is “small” enough. Using the  $L^p$ -estimates (see [1]), we infer that  $u \in W^{2,p}(\Omega)$ ,  $\forall p > 1$ . One can use embedding results (see [8]) to deduce that  $u \in C^{1,\alpha}(\bar{\Omega})$ .

2. We show that if  $(Q\bar{\epsilon})$  admits a solution, so does  $(Q\epsilon)$  for every  $\epsilon \leq \bar{\epsilon}$ .

3. Using the a priori estimate in [6], we show that  $(Q\epsilon)$  does not admit a solution, if  $\epsilon$  is great enough.

From these steps, we infer that the set  $I$  of  $\epsilon$ , for which  $(Q\epsilon)$  admits a solution, is a bounded interval.

4. Let  $\epsilon^*(\Omega, f)$  be the upper bound of  $I$ . The blow-up argument used in [6], can be applied to show that there exists no increasing sequence  $(\epsilon_j)$  in  $I$ , such that

$$\lim_{j \rightarrow \infty} \epsilon_j = \epsilon^*(\Omega, f), \text{ with } \lim_{j \rightarrow \infty} \max_{x \in \Omega} u_{\epsilon_j}(x) = \infty.$$

This last a priori  $L^\infty$ -estimate of the solutions  $u_\epsilon$  near  $\epsilon^*(\Omega, f)$ , leads to a solution of  $(Q\epsilon^*(\Omega, f))$ .

*Remark 2.* — When  $\Omega = B_r = \{x \in \mathbb{R}^n; \|x\| < r\}$  and  $f(x, u) = u^\alpha$ , then

1. every solution of  $(Q\epsilon)$  is spherically symmetric (see [5]),
2.  $\epsilon^*(B_r, \alpha) = r^{\frac{2}{1-\alpha}} \epsilon^*(B_1, \alpha)$ , (see [10]).

## 2.2. The Supercritical Growth Case

Let us consider the following problem

$$(T\epsilon) \begin{cases} \Delta u + a(x)u^\beta = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = \epsilon \text{ on } \partial\Omega. \end{cases}$$

We suppose that

- (i)  $\Omega$  is a bounded regular domain, which is starshaped about the origin.
- (ii)  $a \in C^0(\overline{\Omega}, \mathbb{R}_+^*)$  and  $\beta > \frac{n+2}{n-2}$ .

Under appropriate hypotheses, the following problem

$$(P) \begin{cases} \Delta u + a(x)u^\beta = 0 \text{ in } \mathbb{R}^n, \\ u > 0 \text{ in } \mathbb{R}^n, \\ u \in C^2(\mathbb{R}^n), \end{cases}$$

admits solutions (see [14]).

**PROPOSITION 4.** — *Let us suppose that the problem (P) admits a solution, then under hypotheses (i) and (ii), there exists a positive constant  $\epsilon^*(\Omega, a)$  such that  $(T\epsilon)$  admits, at least, one solution  $u_\epsilon \in C^{1,\delta}(\overline{\Omega})$ ,  $0 \leq \delta < 1$ , when  $0 < \epsilon < \epsilon^*(\Omega, a)$ . There is no  $L^\infty$ -solution of  $(T\epsilon)$  for  $\epsilon > \epsilon^*(\Omega, a)$ .*

*Proof.* — The proof is similar to the demonstration of Theorem 2 in [9].

**Remark 3.** — *The hypothesis concerning the existence of a solution of (P) is justified by the critical growth case (see section 1).*

**The  $\epsilon^*(\Omega, a)$ -limit case.**

Before dealing with this case, let us state the following lemma.

**LEMMA 1.** — *Under the hypotheses (i) and (ii), assume that  $(u_j)$  is a sequence of  $C^2(\overline{\Omega})$  – functions and  $(\epsilon_j)$  is a real sequence, such that*

$$(P_j) \begin{cases} \Delta u_j + a(x)u_j^\beta = 0 \text{ in } \Omega, \\ u_j > 0 \text{ in } \Omega, \\ u = \epsilon_j > 0 \text{ on } \partial\Omega. \end{cases}$$

*Then, if the real sequence  $(\epsilon_j)$  is bounded in  $\mathfrak{R}$ , so is  $(u_j)$  in  $H_0^1(\Omega)$ .*

*Proof.* — Using Pohozaev Identity, we get

$$\begin{aligned} (1 - \frac{n}{2}) \int_{\Omega} \|\nabla u_j(x)\|^2 dx + \frac{1}{2} \int_{\partial\Omega} (x.\nu) \|\nabla u_j(x)\|^2 ds + \frac{n}{\beta + 1} \int_{\Omega} a(x)u_j^{\beta+1}(x) dx \\ - \int_{\partial\Omega} (x.\nu)a(x)\epsilon_j^{\beta+1} dx = \int_{\partial\Omega} (x.\nabla u_j(x))(\nabla u_j(x).\nu) ds. \end{aligned}$$

Using the Green’s first identity, we get

$$\int_{\Omega} a(x)u_j^{\beta+1} dx = \int_{\Omega} \|\nabla u_j(x)\|^2 dx - \int_{\partial\Omega} \epsilon_j \frac{\partial u_j(x)}{\partial \nu} ds.$$

So we infer that

$$\begin{aligned} (*) \quad \left(1 - \frac{n}{2} + \frac{n}{\beta + 1}\right) \int_{\Omega} \|\nabla u_j(x)\|^2 dx = \int_{\partial\Omega} (x.\nabla u_j(x))(\nabla u_j(x).\nu) ds \\ - \frac{1}{2} \int_{\partial\Omega} (x.\nu) \|\nabla u_j(x)\|^2 ds + \int_{\partial\Omega} x.\nu a(x)\epsilon_j^{\beta+1} ds + \frac{n}{\beta + 1} \int_{\partial\Omega} \epsilon_j \frac{\partial u_j(x)}{\partial \nu} ds. \end{aligned}$$

Using the maximum principle, and the fact that  $u_j = \epsilon_j$ , on  $\partial\Omega$ , we obtain

$$\begin{aligned} (1 - \frac{n}{2} + \frac{n}{\beta + 1}) \int_{\Omega} \|\nabla u_j(x)\|^2 dx = \frac{1}{2} \int_{\partial\Omega} \|\nabla u_j(x)\|^2 x.\nu ds \\ + \int_{\partial\Omega} x.\nu a(x)\epsilon_j^{\beta+1} ds - \frac{n}{\beta + 1} \int_{\partial\Omega} \epsilon_j \|\nabla u_j(x)\| ds. \end{aligned}$$

As  $\Omega$  is regular and starshaped, we get

$$\begin{aligned} \left(1 - \frac{n}{2} + \frac{n}{\beta + 1}\right) \int_{\Omega} \|\nabla u_j(x)\|^2 dx &\geq c_0 \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds \\ &\quad - \frac{n}{\beta + 1} \int_{\partial\Omega} \epsilon_j \|\nabla u_j(x)\| ds - c_1, \end{aligned}$$

where,

$$c_0 = \frac{1}{2} \min_{x \in \partial\Omega} x \cdot \nu > 0 \text{ and } \int_{\partial\Omega} x \cdot \nu a(x) \epsilon_j^{\beta+1} ds \leq c_1.$$

As,

$$\beta > \frac{n+2}{n-2} \iff 1 - \frac{n}{2} + \frac{n}{\beta+1} < 0,$$

we get

$$\int_{\Omega} \|\nabla u_j(x)\|^2 dx \leq c_2 \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds + c_3 \int_{\partial\Omega} \|\nabla u_j(x)\| ds + c_4,$$

where  $c_2 < 0 < c_3$  and  $c_i, i = 2, \dots, 4$  are constants not depending on  $j$ . Using Hölder's Inequality, we obtain

$$\begin{aligned} \int_{\Omega} \|\nabla u_j(x)\|^2 dx &\leq c_2 \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds + c_5 \left( \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds \right)^{\frac{1}{2}} + c_4 \\ &\leq \sup_{t \in \mathfrak{R}} c_2 t^2 + c_5 t + c_4 < \infty. \end{aligned}$$

Let us put  $v_j = u_j - \epsilon_j$ , then  $v_j \in H_0^1(\Omega)$  and

$$\|\nabla v_j\|_{L^2(\Omega)} = \|\nabla u_j\|_{L^2(\Omega)}.$$

Using Poincaré Inequality, we get

$$\exists c_0 > 0 ; \|u_j - \epsilon_j\|_{H_0^1(\Omega)} \leq c_0, \forall j.$$

As

$$u_j^2(x) = [u_j(x) - \epsilon_j + \epsilon_j]^2 \leq 2 \left\{ [u_j(x) - \epsilon_j]^2 + \epsilon_j^2 \right\},$$

and  $\epsilon_j$  is bounded in  $\mathfrak{R}$ , this completes the proof of Lemma 1.

*Remark 4.* — The a priori estimate in Lemma 1 remains true for nonlinearities such that, there exist constants  $c$  and  $\gamma$ , with

$$c + uf(x, u) \leq \gamma F(x, u), \text{ where } F(x, u) = \int_0^u f(x, t) dt, \gamma > 2^* = \frac{2n}{n-2}.$$

PROPOSITION 5. — Under the hypotheses of Proposition 4, if  $a \in C^{0,\delta}(\overline{\Omega})$ ,  $0 < \delta \leq 1$ , then  $(T\epsilon^*(\Omega, a))$  admits a solution.

*Proof.* — Let  $(\epsilon_j)$  be an increasing real sequence such that

$$0 < \epsilon_j < \epsilon_{j+1} < \lim_{i \rightarrow \infty} \epsilon_i = \epsilon^*(\Omega, a).$$

For every  $j$ , let  $u_j$  be the solution of  $(T\epsilon_j)$  (see Proposition 4). As  $u_j \in C^2(\overline{\Omega})$ , one can use Lemma 1 to obtain

$$\exists c > 0, \|u_j\|_{H^1(\Omega)} \leq c, \quad \forall j.$$

Then, up to a subsequence,  $u_j \rightharpoonup u$  in  $H^1(\Omega)$ -weak,  $u_j \rightarrow u$  in  $L^2(\Omega)$  — strong and  $u_j \rightarrow u$ , a.e. in  $\Omega$ . One can multiply  $(P_j)$  by  $u_j$  to verify that

$$a(x)u_{\epsilon_j}^\beta \in L^{\frac{\beta+1}{\beta}}(\Omega).$$

By using the  $L^p$ -estimates and a bootstrap argument, one can show that  $u$  is a solution of  $(T\epsilon^*(\Omega, a))$ .

PROPOSITION 6. — Let  $u$  be a spherically symmetric  $L_{loc}^\infty(\mathbb{R}^n)$ — solution of

$$\begin{cases} \Delta u + u^\beta = 0 & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n, \end{cases}$$

then  $u \in C^2(\mathbb{R}^n)$  and  $u \in L^p(\mathbb{R}^n)$ ,  $\forall p > \frac{n(\beta-1)}{2}$ .

*Proof.* — Let us choose  $r > 0$ , such that  $u(r) < \infty$ . As  $u \in L^\infty(B_r)$ , one can use the  $L^p$ -estimates (see [1]) to infer that  $u \in W^{2,p}(B_r)$ ,  $\forall p > 1$ . We infer that (see [8])

$$u \in C^{1,\delta}(\overline{B_r}), \quad \forall 0 < \delta < 1.$$

From the previous line, we see that  $u^\beta \in C^{0,\delta}(\overline{B_r})$ ,  $\forall 0 < \delta < 1$ . So we can use the Schauder Estimates to deduce that  $u \in C^{2,\delta}(\overline{B_r})$ . We can use Proposition 4, to infer that

$$\exists \epsilon^*(B_r, \beta) \text{ such that } u(r) \leq \epsilon^*(B_r, \beta).$$

It is easy to verify( see [10]) that

$$\epsilon^*(B_r, \beta) \leq \epsilon^*(B_1, \beta)r^{\frac{2}{1-\beta}},$$

so we deduce that, if  $p > \frac{n(\beta-1)}{2}$ , then  $u \in L^p(\mathbb{R}^n)$ .

*Acknowledgments.* — The author would like to thank Prof. A. Bahri and the Mathematics Department of Rutgers University (USA) for hospitality during the fall 1998.

## Bibliography

- [1] AGMON (S.), DOUGLIS (A.) and NIRENBERG (L.). — *Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations satisfying General Boundary Value Conditions I*. Comm. Pure Appl. Math., 12, pp. 623-727 (1959).
- [2] BOCCARDO (L.), MURAT (F.) and PUEL (J.P.). — *Quelques Opérateurs Quasi-linéaires*. C. R. Acad. Sc. Paris, t. 307, Série I, pp. 749-752 (1988).
- [3] BREZIS (H.) and NIRENBERG (L.). — *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponent*. Comm. Pure Appl. Math. 36, pp. 437-477 (1983).
- [4] CRANDALL (M. G.) and RABINOWITZ (P.H.). — *Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems*. Arch. Rational Mech. Anal. 58, pp.207-218 (1975).
- [5] GIDAS (B.), NI (W.-M.) and NIRENBERG (L.). — *Symmetry and Related Properties via the Maximum Principle*. Comm. Math. Phys. 68, pp.209-243 (1979).
- [6] GIDAS (B.) and SPRUCK (J.). — *A Priori Bounds for Positive Solutions of Nonlinear Elliptic Equations*. Comm. Partial Differential Equations 6, pp.883-901 (1981).
- [7] GIDAS (B.) and SPRUCK (J.). — *Global and Local Behavior of Positive Solutions of Nonlinear Elliptic Equations*. Comm. Pure Appl. Math., Vol.34, pp.525-598 (1981).
- [8] GILBARG (D.) and TRUDINGER (N.S.). — *Elliptic Partial Differential Equations of Second Order*. Springer Verlag (1977).
- [9] ISSELKOU (O.A.-I.-B.). — *A Critical Value for the Boundary Datum of a Dirichlet's Problem*. Funkcialaj Ekvacioj 41, pp.207-214 (1998).
- [10] ISSELKOU (O.A.-I.-B.). — *Donnée au Bord Critique pour un Problème de Dirichlet*. Revue URED No 8 and 9 (Dakar, 1999).
- [11] JOSEPH (D.D.) and LUNDGREN (T.S.). — *Quasilinear Dirichlet Problems Driven by Positive Sources*. Arch. Rational Mech. Anal. 49, pp. 241-269 (1973).
- [12] LOEWNER (C.) and NIRENBERG (L.). — *Partial Differential Equations Invariant under Conformal or Projective Transformation*. Contribution to Analysis (L. Ahlfors ed.), Academic Press, New York, pp. 245-272 (1974).
- [13] POHOZAEV (S.I.). — *Eigenfunctions of the Equations  $\Delta u + \lambda f(u) = 0$* . Soviet Math. Dokl. 6, pp.1408-1411 (1965).
- [14] POHOZAEV (S.I.). — *On Entire Solutions of Semilinear Elliptic Equations*. Research Note Math. 266, Pitman, pp.56-69, London (1992).