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# $L^p$-ESTIMATES FOR HYPERBOLIC OPERATORS APPLICATIONS TO $\square u = u^k$

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**Résumé** : Nous obtenons dans ce travail des estimations $L^p - L^p$ pour l'équation des ondes non homogène. Le problème est complètement résolu en dimension d'espace $n \leqslant 3$. Dans le cas $n \geqslant 4$ nous avons une réponse positive si $\left|\frac{1}{p} - \frac{1}{2}\right| \leqslant \frac{1}{n-1}$. Finalement, nous utilisons ces estimations pour montrer l'existence et unicité de solutions faibles pour certains problèmes non linéaires.

**Summary** : $L^p - L^p$ estimates are obtained in this paper for the non-homogeneous wave equation. The problem is completely solved for space dimension $n \leqslant 3$. A positive answer is given for dimension $n \geqslant 4$ and $\left|\frac{1}{p} - \frac{1}{2}\right| \leqslant \frac{1}{n-1}$. These estimates are used in order to prove existence and uniqueness of weak solutions for some non-linear problems.

## ABSTRACT

We present some results on non-linear hyperbolic equations obtained by means of some information on the linear Cauchy problem. Those results for the linear case are essentially :

THEOREM. *Let*

$$\begin{cases} \square u \equiv u_{tt} - \Delta_x u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

*where* $f \in L^p(R^n)$, $g \in L_1^p(R^n)$, *then for each* $t \in [0,\infty)$ *we have*

$$(1) \qquad \| u(t,x) \|_{L^p(R^n)} \leq C_p(t) \left( \| f \|_{L^p(R^n)} + \| g \|_{L_1^p(R^n)} \right)$$

*if*

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{n-1}$$

*The result is sharp.*

The homogeneity of Fourier multipliers implies $C_p(t) \approx \max(1, |t|)$.

Proof and extension to other kind of hyperbolic operators can be seen in J. C. Peral [3] and for related estimates of the type $(L^p, L^q)$ Littman [1], [2] and Strichartz [8] [9].

In § 1 we will solve partially a problem posed by Littman in [1].

In § 2 we give existence and uniqueness results for certain non-linear hyperbolic equations $\Box u = F(u)$. Those results are in the context of the $L^p$ spaces.

Finally in § 3 we deal with the case $|F(u)| = u^k$.

Everything will be expressed in terms of either the wave equations of the Klein-Gordon equations.

It is not hard to generalize these results to some others more general hyperbolic equations.

## 1. - A PRIORI $L^p$-ESTIMATES FOR THE NON-HOMOGENEOUS EQUATIONS

It has been proved by Littman in [1] that an estimates of the type

$$\int_{R^n} \int_t |v(t,x)|^p \, dtdx \leq C_p(T) \int_{R^n} \int_t \Box |v(t,x)|^p \, dtdx$$

with $v \in C_0^\infty(R^{n+1})$ and say support $(v) \subset (0,T) \times R^n$ is false if $p > \frac{2n}{n-3}$.

For the same problem we get the following positive results.

THEOREM. *Let*

$$\begin{cases} \Box u(t,x) = F(t,x) \\ u(0,x) = 0 \\ u_t(0,x) = 0 \end{cases}$$

*where* $F \in L^p([0,T] \times R^n)$ *and* $\left|\frac{1}{p}-\frac{1}{2}\right| \leqslant \frac{1}{n-1}$. *Then*

$$\|u\|_{L^p([0,T]\times R^n)} \leqslant C_p(T)\, \|F\|_{L^p([0,T]\times R^n)}.$$

*Proof. B*y the Duhamel principle we know that the solution of the problem is expressed as

$$u(t,x) = \int_0^t v(t,\tau,x)d\tau$$

where $v(t,\tau,x)$ solves the following problem

$$\begin{cases} \Box v(t,\tau,x) = 0 \\ v(\tau,\tau,x) = 0 \\ v_t(\tau,\tau,x) = F(\tau,x) \end{cases}$$

The estimates (1) gives for each t

$$\left(\int_{R^n} |v(t,\tau,x)|^p\, dx\right)^{1/p} \leqslant C_p(t-\tau)\left(\int_{R^n} |F(\tau,x)|^p\, dx\right)^{1/p}$$

if $\left|\frac{1}{p}-\frac{1}{2}\right| \leqslant \frac{1}{n-1}$. But, $C_p(t-\tau) \leqslant C_p(t),\ \ 0 \leqslant \tau \leqslant t$.

By applying the Minkowski integral inequality, for each t we get

$$\left(\int_{R^n} |u(t,x)|^p\, dx\right)^{1/p} \leqslant \int_0^t \int_{R^n} |v(t,\tau,x)|^p\, dx\Big)^{1/p} d\tau \leqslant$$

$$\leqslant \int_0^t C_p(t)\left(\int_{R^n} |F(\tau,x)|^p\, dx\right)^{1/p} d\tau \leqslant C_p(t)t^{1/g}\left(\int_0^t \int_{R^n} |F(\tau,x)|^p\, dxd\right)^{1/p}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Then we get

$$\int_0^T \int_{R^n} |u(t,x)|^p\, dxdt \leqslant \left(\int_0^T |C_p(t)|^p\, t^{p/q}\, dt\right) \|F\|_{L^p([0,T]\times R^n)}$$

This implies

$$\|u\|_{L^p([0,T]\times R^n)} \leqslant C'_p(T)\, \|F\|_{L^p([0,T]\times R^n)}$$

where

$$C'_p(T) \cong T^2$$

because $C_p(t) \approx |t|$ is the norm of Fourier multiplier $\dfrac{\sin t|\xi|}{|\xi|}$.

*Remark.* For $n \leqslant 3$ our result gives the range $1 \leqslant p \leqslant \infty$ . For $n \geqslant 4$ there is a range where the problem is open, that is $\dfrac{2(n-1)}{n-3} \leqslant p \leqslant \dfrac{2n}{n-3}$ and the conjugates.

We can prove the same kind of estimates for more general equations such a Klein-Gordon equation for example. See J.C. Peral [3] .

## 2. - NON-LINEAR CAUCHY PROBLEM

Assume

$$F : [0,\infty) \times R^n \times R \to R$$

and i) $F(t,x,0) = 0$

ii) $|F(t,x,u) - F(t,x,v)| \leqslant K|u-v|$

if we consider

$$\begin{cases} \square u = F(t,x,u) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $f \in L^p(R^n)$ ; $g \in L^p_1(R^n)$ and $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leqslant \dfrac{1}{n-1}$ then we can give the following theorem.

THEOREM 1. *Let* F *be satisfying i) and ii) then given any* $T > 0$, *The Cauchy problem*

$$\begin{cases} \square u = F(t,x,u) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

*where* $g \in L^p_1(R^n)$, $f \in L^p(R^n)$ *and* $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leqslant \dfrac{1}{n-1}$ *, has unique weak solution* $u \in L^p(]0,T[ \times R^n)$ *and*

$$\|u\|_{L^p((0,T) \times R^n)} \leqslant C_p(T)\,(\,\|g\|_{p,1} + \|f\|_p)$$

*Proof.* With fixed $T > 0$ consider $u_1$ solution of the problem

$$\begin{cases} \Box u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

and, in general for $k \in \mathbb{N}$ $u_k$ solution of the problem

$$\begin{cases} \Box u = F(t,x,u_{k-1}) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

As a consequence of the result in § 1, for each $k \in N$ we have

$$u_k \in L^\infty((0,T), L^p(R^n)) \text{ and } u_k \in L^p((0,T) \times R^n).$$

we will prove that $\{u_k\}_{k \in N}$ is a Cauchy sequence in the first space.

In fact $u_k - u_{k-1}$ verifies :

$$\begin{cases} \Box (u_k - u_{k-1}) = F(t,x,u_{k-1}) - F(t,x,u_{k-2}) \\ (u_k - u_{k-1})\,(0,x) = 0 \\ (u_k - u_{k-1})_t\,(0,x) = 0 \end{cases}$$

then if

$$d_k(t) = \left(\int_{R^n} |u_k(t,x) - u_{k-1}(t,x)|^p\, dx\right)^{1/p}$$

we get

$$d_k(t) \leqslant C_p(t)K \int_0^t d_{k-1}(s)ds \leqslant C_p(T)k \int_0^t d_{k-1}(s)ds = M(T) \int_0^t d_{k-1}(s)$$

Therefore

$$d_k(t) \leqslant \frac{TM(T)^k}{k!} \sup_{s\in [0,T]} d_o(s)$$

and, if $k < \ell$, for each $t \in (0,T)$ we have

$$\| u_k(t.) - u_\ell(t.) \|_{L^p(R^n)} \leqslant \sum_{j=k}^{\ell} d_j(t) \leqslant \left( \sum_{k=\ell} \frac{(M(T)T)^k}{k!} \right) \sup_{s\in(0,T)} d_o(s)$$

Then $u_k \to u \in L^\infty((0,T), L^p(R^n))$ and as A is a continuous operator, u is a solution of our problem.

By using the theorem locally is not hard to prove that u is the unique solution. In others words, we consider the operator

$$L^p((0,T) \times R^n) \to L^p((0,T) \times R^n)$$

such that $Av = u$, is solution of the linear problem

$$\begin{cases} \Box u = F(t,x,v) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

Then for small T, A is contractive. Also it is clear that the convergence is in the space

$$L^p((0,T) \times R^n)$$

and this finishes the proof.

An important example included in the preceding theorem is the sine-Klein-Gordon equations

$$\Box u + m^2 u = \sin u$$

and for the case $n = 3$, there exists a unique solution for every p, $1 \leqslant p \leqslant \infty$.

The following Gronwall lemma, shows the behaviour of the constant C(T).

LEMMA. *Let* $\alpha(t)$ *and* $\beta(t)$ *be positive and continuous functions on some interval* $[0,T]$. *If*

$$y(t) \leqslant \alpha(t) + \beta(t) \int_0^t y(\tau)\, d\tau$$

*then*

$$y(t) \leqslant \alpha(t) + \beta(t) \int_0^t \alpha(s) \exp\left( \int_s^t \beta(u) du \right) ds$$

Then if we consider

$$y(t) = \left( \int_{R^n} |u(t,x)|^p \, dx \right)^{1/p}$$

where u is the solution for the non-linear Cauchy problem and

$$\alpha(t) = C_p(t) \left( \| g \|_{p,1} + \| f \|_p \right)$$

$$\beta(t) = KC_p(t)$$

we get for each t

$$\| u(t,.)\|_p \leqslant C_p(t) \exp \int_0^t KC_p(s) \; ( \| f\|_p + \| g\|_{p,1})$$

where $C_p(t) \approx \max(1,t)$ as in the linear case. Then

$$\| u\|_{L^p((0,T) \times R^n)} \leqslant (\| f\|_p + \| g\|_{p,1})\left(\int_0^T |C_p(t)|^p \exp \left\{pk \int_0^t C_p(s)ds\right\} dt\right)^{1/p}$$

Observe that in the case $|F(t,x,u)| \leqslant H$, $C_p(t)$ behaves like $t^\alpha$ for some $\alpha(p)$.
This is the case of the sine-Klein-Gordon equations.

## 3. - A RESULT OF EXISTENCE AND UNIQUENESS IN $L^\infty$ FOR $\Box u + m^2u = u^k$

For the problem

$$\begin{cases} \Box u + m^2u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \;\; x \in R^3 \end{cases}$$

existence results are know in the case of finite energy ; see references in Segal [5] and Strauss [6].

Uniqueness is known in the case $k < 5$. See also [7]. We get results of existence and uniqueness based in the estimates of § 1 using the fact that for $n = 3$ such estimates are valid for $1 \leqslant p \leqslant \infty$.

Consider

$$\begin{cases} \Box u + m^2u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $g \in L_1^\infty(R^3)$ and $f \in L^\infty(R^3)$ and in addition assume

$$\| f\|_\infty + \| g\|_{\infty,1} \leqslant \frac{1}{4}$$

Define $u_o$ as the solution of the linear problem

$$\begin{cases} \Box u + m^2u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

Then for each t we have

$$\| u_o(t,.) \|_{L^\infty(R^3)} \leqslant t \; \| f \|_\infty + \max(t,1) \; \| g \|_{\infty,1} .$$

Besides if $t \in [0,1]$ we get

$$\| u_o(t,.) \|_{L^\infty(R^3)} \leqslant 1/4$$

On the other hand let $u_n$ be the solution of the linear problem

$$\begin{cases} \Box u + m^2 u = \pm u_{n-1}^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

The result of § 1 implies

$$\| u_n(t,.) \|_{L^\infty(R^3)} \leqslant \frac{1}{4} + \left(\frac{1}{2}\right)^{k-1} \int_0^t \| u_{n-1}(\tau,.) \|_{L^\infty(R^3)} d\tau$$

if $t \in [0,1]$ and by recursion we deduce :

$$\| u_n(t,.) \|_{L^\infty(R^3)} \leqslant \frac{1}{2} \quad \text{if} \quad t \in [0,1]$$

Let's define

$$d_n(t) = \| u_n(t,.) - u_{n-1}(t,.) \|_{L^\infty(R^3)} , \; t \in [0,1]$$

then we obtain

$$d_n(t) \leqslant C_k \int_0^t d_{n-1}(\tau) d\tau$$

where $C_k$ is the bound of the following expression

$$P_{k-1}(u,v) = u^{k-1} + u^{k-2} v + \dots + v^{k-1}$$

for

$$u = \| u_{n-1} \|_{L^\infty([0,1] \times R^3)}$$

and

$$v = \| u_{n-2} \|_{L^\infty([0,1] \times R^3)}$$

Therefore we have $C_k \leqslant k(\frac{1}{2})^k < 1$ if $k \geqslant 2$.

Obviously

$$d_n(t) \leqslant \frac{(C_k t)^n}{n} \; \frac{1}{2},$$

and then $\{u_n\}_{n \in N}$ is a Cauchy sequence in

$$L^\infty([0,1], L^\infty(R^3)) \text{ and in } L^\infty([0,1] \times R^3).$$

Since $C_k \leqslant 1$ the following existence and uniqueness theorem has been proved

THEOREM 2. *Let* $f \in L^\infty(R^3)$ *and* $g \in L_1^\infty(R^3)$ *such that* $\|f\|_\infty + \|g\|_{\infty,1} \leqslant \frac{1}{4}$ *then*

$$\begin{cases} \Box u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

*has unique solution in* $B_{1/2}(0) = \left\{ u \in L^\infty([0,1] \times R^3) \; ; \; \|u\|_\infty \leqslant \frac{1}{2} \right\}$.

A classical theorem of uniqueness in the following (see Strauss [6] ).

THEOREM 3. *Consider*

$$\begin{cases} \Box u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

*where* $g \in L_1^\infty(R^3)$ *and* $f \in L^\infty(R^3)$. *If the problem has a solution* $u \in L^\infty((0,T) \times R^3)$ *then it is the unique solution in such space.*

Several observations should be made at this point. The theorem 3 gives a result on uniqueness with independence of how big or small are the $L^\infty$-norms of the data. However for the theorem 2 we need the data to be small.

From both theorems we get the following corollary :

COROLLARY. *Under the hipothesis of the Theorem 2, then there is a unique solution in* $L^\infty([0,1] \times R^3)$.

Finally if $\alpha = \|f\|_\infty + \|g\|_{\infty,1} > \frac{1}{4}$ results on existence and uniqueness can be given locally in t.

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