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Algebraic points of abelian functions in two variables


<http://www.numdam.org/item?id=AFST_1982_5_4_2_153_0>
ALGEBRAIC POINTS
OF ABELIAN FUNCTIONS IN TWO VARIABLES

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Résumé : On donne une mesure d'indépendance linéaire pour les coordonnées des points algébriques de fonctions abéliennes de deux variables. On en déduit un analogue abélien du théorème de Franklin-Schneider.

Summary : A linear independence measure is given for the coordinates of algebraic points of abelian functions in two variables. From this an abelian analogue of the Franklin-Schneider theorem is deduced.

Let \( A \) be a simple abelian variety defined over the field of algebraic numbers and let \( \Theta : \mathbb{C}^2 \to A(\mathbb{C}) \) be a normalised theta homomorphism (cf. [12], § 1.2). Let \( \theta_0, \ldots, \theta_\nu \) be entire functions such that \((\theta_0(z), \ldots, \theta_\nu(z))\) forms a system of homogeneous coordinates for the point \( \Theta(z) \) in projective \( \nu \)-space. Put \( f_i = \theta_i / \theta_0 \). Assume that \( \theta_0(0) \neq 0 \); then \( f_i(0) \) is algebraic for all \( i \). A point \( (u_1, u_2) \) in \( \mathbb{C}^2 \) with \( \theta_0(u) \neq 0 \) is by definition an algebraic point of \( \Theta \) if and only if \( f_i(u) \) is algebraic for all \( i \). The field of abelian functions associated with \( \Theta \) is \( \mathbb{C}(f_1, \ldots, f_\nu) \).

If \((u_1, u_2)\) is a non-zero algebraic point of \( \Theta \), the coordinates \( u_1 \) and \( u_2 \) are linearly independent over the algebraic numbers (cf. [12], Théorème 3.2.1); the proof uses the Schneider-Lang criterion (cf. [5], Chapter III, Theorem 1). It is the main purpose of this paper to obtain, by means of Gel'fond's method, a quantitative refinement of this statement.
THEOREM 1. For every compact subset K of $\mathbb{C}^2 \setminus \{0\}$ that contains no zeros of $\partial_q$ there exists an effectively computable $C$ with the following property. Let $u$ be an algebraic point of $\Theta$ that lies in K, and let $\beta$ be an algebraic number. Let $A$ be an upper bound for the (classical) heights of the numbers $f_i(u)$, let $B$ be an upper bound for the height of $\beta$ and take $D := [\mathbb{Q}(f_1(u), \ldots, f_v(u), \beta) : \mathbb{Q}]$; assume $A > B > e$. Then

$$|\beta_1 - u_2| > \exp(-CD^6 \log^2 A \log^4 (DB \log A) \log^{-5}(D \log A)),$$

where $u = (u_1, u_2)$.

The dependence of this lower bound on $B$ was first studied in [3]. Moreover, in an unpublished 1979 investigation, Y.Z. Flicker and D.W. Masser also studied the dependence on $B$ and obtained $\log^4 B$ in the exponent. I wish to thank Dr. Masser for making available to me a report of this study, to which several improvements in the present paper are due.

The proof of Theorem 1 resembles that of Lemma 1 of [1]; in parts where this resemblance is particularly strong, the exposition will be brief. The proof is preceded by a lemma that may be called, in Masser's terminology, a 'safe addition formula' for abelian functions.

LEMMA. There exists an effectively computable $C'$ with the following property. If $w_1$ and $w_2$ are points of $\mathbb{C}^2$ such that $\partial_q(w_1) \neq 0$, $\partial_q(w_2) \neq 0$, $\partial_q(w_1 + w_2) \neq 0$, then for every $i$ in $\{1, \ldots, v\}$ there exist polynomials $f_i^*, \phi_i^*$ of total degree at most $C'$ and a neighbourhood $N$ of $(w_1, w_2)$ such that

$$f_i(z_1 + z_2) = \phi_i^* (f_1(z_1), \ldots, f_v(z_1), f_1(z_2), \ldots, f_v(z_2))$$

for all $(z_1, z_2)$ in $N$; the denominator is non-zero on $N$. The coefficients of these polynomials are algebraic integers in a field of degree at most $C'$. Their size (i.e., the maximum of the absolute values of their conjugates) is also bounded by $C'$.

Proof. Let $(w_1, w_2)$ be any point in $\mathbb{C}^4$. Define $\sigma : \mathbb{C}^4 \to \mathbb{P}^{2+2v}(\mathbb{C})$ by $\sigma(z_1, z_2) := (\omega(z_1), \omega(z_2))$, where $\omega$ is the Segre embedding (cf. [9], (2.12)) of $\mathbb{P}^v(\mathbb{C}) \times \mathbb{P}^v(\mathbb{C})$ into projective space. By the regularity of the addition in $A$, we find projective coordinates for $\omega(z_1 + z_2)$ of the form

$$H_i(\omega(z_1), \omega(z_2)) \quad (0 \leq i \leq v)$$

for all $(z_1, z_2)$ with the property that $\sigma(z_1, z_2)$ lies in a certain Zariski neighbourhood of $\sigma(w_1, w_2)$; here the polynomials $H_i$ have algebraic coefficients. The continuity of $\sigma$ now proves this for all $(z_1, z_2)$ in a neighbourhood of $(w_1, w_2)$. Let $P$ be a fundamental region for $\mathbb{C}^2/\Omega$; covering the compact set $P^2$ with a finite number of these neighbourhoods shows that we can bound the
degrees of the polynomials $H_j$, the sizes of their coefficients, the degree of the field generated by these coefficients and their common denominator independently of $(w_1, w_2)$. In particular, it is no restriction to assume the coefficients to be algebraic integers.

Finally, if $\theta_0(w_1) \neq 0$, $\theta_0(w_2) \neq 0$, $\theta_0(w_1 + w_2) \neq 0$, these also hold on some neighbourhood of $(w_1, w_2)$; hence

$$H_0(\Theta(z_1), \Theta(z_2)) \neq 0$$

on some neighbourhood of $(w_1, w_2)$, which now proves (2).

Proof of Theorem 1. In this proof $c_1, c_2, \ldots$ will denote effectively computable real numbers greater than 1 that depend only on $\Theta$ and $K$. Let $x$ be some large real number; further conditions on $x$ will appear at later stages of the proof. Put $B' := xDB \log A$, $E := 4D^{1/2} \log^{1/2} A$ and assume

$$|\beta u_1 - u_2| \leq \exp(-x^{16}D^6 \log^2 A \log^4 B' \log^{-5} E).$$

This will lead to a contradiction, which will prove (1).

The field $\mathbb{C}(f_1, \ldots, f_w)$ has transcendence degree 2 over $\mathbb{C}$ (cf. [10], § 6); assume, without loss of generality, that $f_1$ and $f_2$ are algebraically independent over $\mathbb{C}$. As in [8], § 4.2, we choose a system $\xi_0, \ldots, \xi_{D-1}$ of generators of $\mathbb{Q}(f_1(u), \ldots, f_w(u), \delta)$ of the form

$$\xi_\delta = f_1^{j_1(\delta)} \cdots f_w^{j_w(\delta)} (u)^{j_{w+1}(\delta)},$$

where the $j_1(\delta)$ are non-negative integers satisfying $j_1(\delta) + \ldots + j_{w+1}(\delta) \leq D-1$. Put

$$L := \{x^8D^3 \log A \log^2 B' \log^{-3} E\}$$

and consider the auxiliary functions

$$F(z) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1, \lambda_2, \delta) \xi_\delta f_1^{\lambda_1} f_2^{\lambda_2} (z, \beta z),$$

$$F_s(z) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1, \lambda_2, \delta) \xi_\delta f_1^{\lambda_1} f_2^{\lambda_2} (z, \beta z - e),$$

where $e := \beta u_1 - u_2$. As $K$ is compact and the zero set of $\theta_0$ is closed, these sets have a distance at least $c_1^{-1}$. The functions $f_1, \ldots, f_w$ are continuous on the set $K'$ of points $z$ satisfying $\text{dist}(z, K) \leq \frac{1}{2} c_1^{-1}$; hence their absolute values are bounded by some $c_2$ on $K'$ and a fortiori on
the ball $U$ with radius $\frac{1}{4}c_{1}^{-1}$ centred at $u$. Now put

$$S := \{x^{3} \pm \log B' \log^{-1} E\}.$$ 

As in § 4 of [6], an application of the box principle shows that there is a subset $V$ of $\{1, \ldots, S\}$ such that $\# V \geq c_{3}^{-1} S$ with the property that $(su_{1}, su_{2})$ and $(su_{1}, s\delta u_{1})$ lie in $U + \Omega$ for all $s$ in $V$, where $\Omega$ is the period lattice of $\Theta$. Put

$$T := \{x^{12} D^5 \log^{2} A \log^{3} B' \log^{-5} E\}$$

and consider the system of linear equations

$$(6) \quad f_{s}^{(t)}(su_{1}) = 0 \quad (s \in V, \quad t = 0, \ldots, T - 1)$$

in the $p(\lambda_{1}, \lambda_{2}, \delta)$.

Take $1 \leq i \leq \nu$. Lemma 7.2 of [6], part of which remains valid without complex multiplication, states that for every integer $s$ there exist polynomials $\Psi_{s,i}, \Psi_{s,i}^{*}$ of total degree $N_{s} \leq c_{4}^{s/2}$ such that, if $\vartheta_{0}(s) \neq 0$, then

$$f_{i}(su) = \frac{\Psi_{s,i}^{*}}{\Psi_{s,i}} \Psi_{s,i} \left( f_{1}(u), \ldots, f_{\nu}(u) \right)$$

and $\Psi_{s,i}(f_{1}(u), \ldots, f_{\nu}(u)) \neq 0$. The coefficients of these polynomials are algebraic numbers in a field of degree at most $c_{5}$, of size at most $c_{6}^{2}$ and with a common denominator at most $c_{7}^{2}$. According to the preceding Lemma, there also exist polynomials $\Phi_{i}, \Phi_{i}^{*}$ of total degree at most $c_{8}$ and a neighbourhood $N$ of the origin such that

$$f_{1}(u + z) = \frac{\Phi_{i}^{*}}{\Phi_{i}} \left( f_{1}(u), \ldots, f_{\nu}(u), f_{1}(z), \ldots, f_{\nu}(z) \right)$$

for all $z$ in $N$, with non-zero denominator, the coefficients are algebraic integers in a field of degree at most $c_{9}$, whose sizes are also bounded by $c_{8}$.

Now define

$$\Phi := \prod_{i=1}^{\nu} \Phi_{i},$$

$$\varphi_{s,i}(z) := \Phi \Psi_{s,i}^{*} \left( f_{1}(u), \ldots, f_{\nu}(u), f_{1}(z), \ldots, f_{\nu}(z) \right) \Psi_{s,i} \left( f_{1}(u + z), \ldots, f_{\nu}(u + z) \right),$$

$$\psi_{s,i}(z) := \Phi \Psi_{s,i}^{*} \left( f_{1}(u), \ldots, f_{\nu}(u), f_{1}(z), \ldots, f_{\nu}(z) \right) \Psi_{s,i} \left( f_{1}(u + z), \ldots, f_{\nu}(u + z) \right).$$
Note that on a neighbourhood of the origin \( \phi_{s,i} \) and \( \psi_{s,i} \) are holomorphic and \( \phi_{s,i} \) is non-zero. As Leibniz’ rule shows that we have found a solution of (6) if we choose the \( p(\lambda_1, \lambda_2, \delta) \) in such a way that

\[
f_{s,t} = 0 \quad (s \in \mathcal{V}, t = 0, \ldots, T-1),
\]

where

\[
f_{s,t} := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1, \lambda_2, \delta) \xi_5 \frac{dt}{dz} \left( \frac{L-\lambda_1}{\psi_{s,1}} \frac{L-\lambda_2}{\psi_{s,2}} (z, \beta z) \right)_{z=0}.
\]

The number of equations in (7) is at most

\[
ST \leq c_9 x^{15} D^6 \log^2 A \log^4 B' \log^{-6} E,
\]

while the number of unknowns is

\[
(L + 1)^2 D \geq c_{10}^{-1} x^{16} D^7 \log^2 A \log^4 B' \log^{-6} E.
\]

From the above estimates it follows that \( \psi_{s,i}^{(z)}(z) \) can be written as a polynomial in \( f_1(u), \ldots, f_p(u), f_1(z), \ldots, f_p(z) \) of total degree at most \( c_{11} \lambda_1^{s^2} \); the coefficients are algebraic numbers in a field of degree at most \( c_{12}, \) whose sizes and common denominator are bounded by \( \lambda_3^{s^2} \). With the aid of Lemma 5.1 of [6] it is now easy to see that the expression

\[
\frac{dt}{dz} \psi_{s,i}^{(z, \beta z)} \bigg|_{z=0}
\]

is a polynomial in \( f_1(u), \ldots, f_p(u) \) of total degree at most \( c_{14} (\lambda_1^{s^2} + t) \); the coefficients are algebraic numbers in a field of degree at most \( c_{16} \) over \( \mathbb{Q}(\beta) \), whose sizes and common denominator are bounded by \( \lambda_3^{s^2} + t \log t + t \log B \) are bounded by \( c_{16} \). A similar statement holds for

\[
\frac{dt}{dz} \psi_{s,i}^{(z, \beta z)} \bigg|_{z=0}
\]

Thus the coefficients of the system of linear equations (7) lie in a field of degree at most \( c_{17} D \) and their size and common denominator are bounded by
According to Lemma 1.3.1 of [11], if $x > 2c_9c_{10}$, this implies the existence of rational integers $p(\lambda_1, \lambda_2, \delta)$, not all zero, such that (7) and thereby (6) hold, while

$$P := \max |p(\lambda_1, \lambda_2, \delta)| \leq \exp(c_{21}^* D^5 \log^2 A \log^4 B' \log^{-5} E).$$

Take $s \in V$, $\eta \in \mathbb{R}$, $z \in \mathbb{C}$ such that $|z - su_1| = \eta$. Then the distance between $(z, \beta z)$ and $(su_1, \beta u_1)$ is bounded by $2B\eta$; if $\eta = (8c_4B)^{-1}$, it follows that $(z, \beta z)$ lies in $U' + \Omega$, where $U'$ is the ball with radius $\frac{1}{2}c_1$ centred at $u$. Similarly $(z, \beta z - se) \in U'$. Note that $U' \subset K'$ and therefore $|f_1(z)| \leq c_2$ for all $z$ in $U'$. Comparison of the definitions of $F$ and $F_s$ now gives

$$\sup_{|z - su_1| = \eta} |F(z) - F_s(z)| \leq pc_{22}^* D^5 S |\epsilon|.$$

By Cauchy’s inequality this implies

$$|F(t)(su_1) - F_s(t)(su_1)| \leq t^{c_{23}^*} B^t pc_{24}^* D^5 S |\epsilon|.$$

If $t \leq T - 1$, it now follows from (6) that

$$(8) \quad |F(t)(su_1)| \leq \exp(-c_{25}^* D^6 \log^2 A \log^4 B' \log^{-5} E).$$

Define the entire function $G$ by

$$G(z) := g(z)F(zu_1),$$

where

$$g(z) := g_{0}^{2L}(zu_1, \beta zu_1).$$

By Lemma 1 of [7], the function $g$ satisfies

$$(9) \quad |g(z)| \leq \exp(c_{26}L |z|^2);$$

also the definition of $V$ gives

$$(10) \quad |g(s)| \geq \exp(-c_{27} LS^2) \quad (s \in V).$$

Formulas (8), (9) and (10) form the starting-point for an extrapolation procedure on $G$, analogous to that in [1], which yields

$$(11) \quad F_s(t)(su_1) = 0 \quad (s \in V, t = 0, ..., T' - 1).$$
II. By Proposition 1.2.3 of [12], the partial derivatives of $f_1, \ldots, f_v$ are polynomials in $f_1, \ldots, f_v'$. Therefore there exist polynomials $P_1, \ldots, P_v$ such that the functions $h_{i,s}$ defined by

$$h_{i,s}(z) := f_i(z + su_1, \beta z + su_2)$$

satisfy

$$h_{i,s}' = P_i(h_{1,s}, \ldots, h_{v,s})$$

and

$$h_{i,s}(0) = f_i(su_1, su_2).$$

Define

$$Q_1(x_1, \ldots, x_v) := \sum_{\lambda_1 = 0}^{L} \sum_{\lambda_2 = 0}^{L} \sum_{\delta = 0}^{D-1} p(\lambda_1, \lambda_2, \delta) x_1^{\lambda_1} x_2^{\lambda_2}.$$ 

As

$$h_{i,s}^{(t)}(0) = \frac{d^t}{dz^t} f_i(z, \beta z - se) \bigg|_{z = su_1},$$

(11) shows

$$\frac{d^t}{dz^t} Q_1(h_{1,s}(z), \ldots, h_{v,s}(z)) \bigg|_{z = 0} = 0 \quad (s \in V, t = 0, \ldots, T' - 1),$$

i.e.

(12) \[ \sum_{s \in V} \text{ord}_{z = 0} Q_1(h_{1,s}(z), \ldots, h_{v,s}(z)) \geq c_3^{-1} S T' \geq c_4^{-1} \frac{1}{28} \chi^2 \log^2 A \log^4 B' \log^{-6} E. \]

Let $Q_2, \ldots, Q_n$ be generators of the ideal of $\mathbb{C}[X_1, \ldots, X_v]$ corresponding to the affine part of $A$. Then

(13) \[ Q_j(f_1(w), \ldots, f_v(w)) = 0 \quad (j = 2, \ldots, n) \]

for every $w$ that is not a zero of $\phi_0$; thus in particular

(14) \[ \text{ord}_{z = 0} Q_j(h_{1,s}(z), \ldots, h_{v,s}(z)) = \infty \quad (s \in V, j = 2, \ldots, n). \]

Put $W := \{ \Theta(z, \beta z) \mid z \in \mathbb{C} \}$. Then $W$, with the addition of $A$, forms a subgroup of $A$; it follows
that the Zariski closure of $W$, with the addition of $A$, forms an algebraic subgroup of $A$. Small values of $z$ are separated, thus $W$ is infinite. As $A$ is simple, this implies that $\overline{W} = A_\mathbb{C}$. Therefore the Zariski closure of

\[ \{ \Theta(z + su_1, \beta z + su_2) \mid z \in \mathbb{C}, \partial_0(z + su_1, \beta z + su_2) \neq 0 \} \]

is also equal to $A_\mathbb{C}$. Now suppose for a moment that

for some $s$ in $V$. By continuity, this implies that (13) also holds if $j = 1$. But that contradicts either the algebraic independence of $f_1$ and $f_2$ or the linear independence of $\xi_0, \cdots, \xi_{D-1}$. Thus

\[ \text{ord}_{z=0} Q_1(h_{1,s}(z), \cdots, h_{\nu,s}(z)) = \infty \quad (s \in \mathbb{V}). \]

The set of common zeros of $Q_2, \ldots, Q_n$ has algebraic dimension two (cf. [9], (2.7)). As, by (14) and (15), $Q_1$ is not in the ideal generated by $Q_2, \ldots, Q_n$, the set of common zeros of $Q_1, \ldots, Q_n$ has algebraic dimension at most one (cf. [9], (1.14)). It is no restriction to assume $n > \nu$. Then the Main Theorem of [2] implies that either

\[ \sum_{s \in \mathbb{V}} \text{ord}_{z=0} Q_1(h_{1,s}(z), \cdots, h_{\nu,s}(z)) \leq C_{29} L^2 + C_{30} L S \exp(C_{31} x^{16} D^6 \log^2 A \log^4 B \log^6 E), \]

which contradicts (12) if $x > C_{28} C_{31}$, or the points $\Theta(su)$ are not all different. As $\Theta$ induces an isomorphism between $\mathbb{C}^2 / \Omega$ and $A_\mathbb{C}$, the equality of $\Theta(su)$ and $\Theta(s'u)$, say, shows that there is an $\omega \in \Omega$ with

\[ su = s'u + \omega. \]

Therefore we have now proved the theorem under the hypothesis

\[ \forall m \leq S m\mathbb{u} \notin \Omega. \]

III. It now remains to prove the theorem in the case where $m\mathbb{u} \in \Omega$ for some $m \leq S$. In particular, let $m$ be the smallest positive integer with this property; then the points $\Theta(y), \Theta(2y), \ldots, \Theta(m\mathbb{u})$ are all different. As before, we can choose a subset $V'$ of \{1, \ldots, m\} such that $V' \geq C_{32}^{-1} m$ with the property that $(su_1, su_2)$ and $(su_1, s\beta u_1)$ lie in $U + \Omega$ for all $s$ in $V'$. Put
where $E$, $B'$ retain their earlier meaning, and let $F$ and $F_s$ be defined again by (4) and (5). Put

$$T := [x^9 mD^4 \log^2 A \log^2 B' \log^{-4} E]$$

and consider the system of linear equations

$$(16) \quad F_s^{(t)}(su_1) = 0 \quad (s \in V', t = 0, \ldots, T-1).$$

By the same method used earlier, it is proved that the coefficients $p(\lambda_1, \lambda_2, \delta)$ may be chosen in such a way that they are not all zero and (16) holds. Now let $V$ be the set of all $s \in \{1, \ldots, S\}$ that differ by a multiple of $m$ from an element of $V'$; here $S$ has the same meaning as before. Then

$$\# V \geq \frac{1}{3} S; \quad \text{as } \mu \text{ is a period of every } f_t, (16) \text{ implies}$$

$$F_s^{(t)}(su_1) = 0 \quad (s \in V, t = 0, \ldots, T-1).$$

Repeating the extrapolation procedure gives

$$F_s^{(t)}(su_1) = 0 \quad (s \in V, t = 0, \ldots, T-1),$$

where $T' := [x^2 T]$. Define $Q_1$ and $h_{i,s}$ as before; then

$$\sum_{s \in V'} \ord_{z=0} Q_1(h_{1,s}(z), \ldots, h_{\nu,s}(z)) \geq \frac{1}{3} \cdot m T R \geq \frac{1}{3} \cdot 11 \cdot m^2 D^4 \log^2 A \log^2 B' \log^{-4} E.$$

Another application of the Main Theorem of [2] gives the desired contradiction. Note that for this special case of the theorem we may replace (1) with

$$|\beta u_1 - u_2| > \exp(-CmD^5 \log^2 A \log^3 (DB \log A) \log^{-4} (D \log A)),$$

which is sharper if $m$ is small compared to $S$.

As a corollary to Theorem 1, an abelian analogue of the Franklin-Schneider theorem is easily obtained. It should be noted that the assumption as to the nature of $\beta$, necessary in the exponential and elliptic versions of this result (cf. [1]) does not occur here.

**THEOREM 2.** For every point $\underline{a}$ in $\mathbb{C}^2 \setminus \{0\}$ such that $\partial_0(\underline{a}) \neq 0$, there exists an effectively computable $C$ with the following property. Let $\alpha_1, \ldots, \alpha_\nu, \beta$ be algebraic numbers, let $A \geq e^6$ be an upper bound for the heights of $\alpha_1, \ldots, \alpha_\nu$, and let $B \geq e$ be an upper bound for the height of $\beta$. 
Then if $D = [Q(a_1, \ldots, a_n) : Q]$ we have

$$\sum_{i=1}^{p} |f_i(a) - a_i| + |\beta a_1 - a_2| > \exp(-C''D^6 \log^2 A \log^4 (DB \log A) \log^{-5}(D \log A)).$$

**Proof.** Let $Q_2, \ldots, Q_n$ be generators of the ideal of $\mathbb{C}[X_1, \ldots, X_p]$ corresponding to the affine part of $A$. If $Q_j(a_1, \ldots, a_n) \neq 0$ for some $j$ with $2 \leq j \leq n$, then the result is trivial, as $Q_j(f_1(a), \ldots, f_p(a)) = 0$. Thus we may assume $(a_1, \ldots, a_n)$ to be on the affine part of $A$. By the smoothness of $A$ at $\Theta(a)$, the matrix of partial derivatives of $(f_1, \ldots, f_p)$ at $a$ has rank 2. Thus there exist $k$ and $\ell$ such that the matrix of partial derivatives of $(f_k, f_\ell)$ at $a$ has rank 2. According to Theorem 7.4 in Chapter I of [4], there are open neighbourhoods $U$ of $a$ and $V$ of $(f_k(a), f_\ell(a))$ such that $(f_k, f_\ell)$ induces a biholomorphic mapping from $U$ onto $V$. If $C''$ is sufficiently large, the negation of (17) implies that $f_\ell(0)$ belongs to $V$ for some $u \in U$ and

$$|a - u| \leq c \exp(-C''D^6 \log^2 A \log^4 (DB \log A) \log^{-5}(D \log A))$$

for some $c$ that depends only on $a$ and $\Theta$. Thus

$$|\beta u_1 - u_2| \leq |\beta a_1 - \beta u_1| + |a_2 - u_2| + |\beta a_1 - a_2| \leq (|\beta| c + c + 1) \exp(-C''D^6 \log^2 A \log^4 (DB \log A) \log^{-5}(D \log A)).$$

Let $K$ be a compact subset of $\mathbb{C}^2 \setminus \{0\}$ containing a neighbourhood of $a$ but no zeros of $\theta_0$; by Theorem 1, (18) is impossible if $C''$ is sufficiently large in terms of $c$ and $K$. 

$\blacksquare$
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(Manuscrit reçu le 26 juin 1981)