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GLOBAL BEHAVIOUR AND SYMMETRY PROPERTIES OF SINGULAR SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

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INTRODUCTION

This paper deals with the study of some local and global qualitative properties of any solution of the equation

\[-\Delta u + g(u) = 0,\]
in some exterior domain $\Omega$ of $\mathbb{R}^N$, where $g$ is a non-decreasing function defined on $\mathbb{R}$. More precisely we shall investigate the three following problems

(I) What is the asymptotic behaviour of $u(x)$ when $x$ tends to infinity?

(II) If we suppose that $\Omega = \mathbb{R}^N - \{0\}$ and that $u$ is possibly singular at 0, is $u$ spherically symmetric?

(III) When any possibly singular solution of (E) in $\mathbb{R}^N - \{0\}$ is uniquely determined?

As that type of equation appeared in the modelisation of many physical phenomena, it has been intensively studied in supposing first that $u$ is positive and radial and $g(u) = u^q$. For example the Thomas-Fermi theory of interaction among atoms leads, as a first approximation, to the following differential equation (see [12], and [9])

$$\frac{d^2u}{dr^2} + \frac{2}{r} \frac{du}{dr} - u^{3/2} = 0.$$  \hspace{1cm} (0.1)

The singularities and the asymptotic behaviour of any solution of (0.1) are now well known (see [12] and [9]). Recently some new results concerning the asymptotic behaviour and the description of the isolated singularities of non positive solutions of (E) when $g(u) = \frac{2q}{q-1} u^q$ has been given in [14] and [16]. Those results where strongly linked to the existence of a very simple solutions of (E) in $\{0\}$ if $1 < q < \frac{N}{N-2}$:

$$u_s(x) = \left[ \left( \frac{2}{q-1} \right) \left( \frac{2q}{q-1} - N \right) \right]^{1/(q-1)} |x|^{2/(q-1)}.$$  \hspace{1cm} (0.2)

Moreover when $1 < q < \frac{N+1}{N-1}$ an infinite family of non-isotropic solutions of (E) was obtained under the following form

$$u(x) = |x|^{2/(q-1)} v_0 \left( \frac{x}{|x|} \right),$$  \hspace{1cm} (0.3)

where $v$ is any non-constant solution of

$$-\Delta_{S^{N-1}} v + |v|^{q-1} v = \left( \frac{2q}{q-1} - N \right) v \text{ on } S^{N-1},$$  \hspace{1cm} (0.4)

$\Delta_{S^{N-1}}$ being the Laplace-Beltrami operator on $S^{N-1}$.

However, as a physical law is just an approximation of a phenomena, it is natural to replace the exactitude of the definition of $g$ by a less restrictive assumption if we want to take into account some secondary effects, for example $g(r) \sim c r^q$ ($q = 3$ in the Relativistic Thomas-Fermi Theory). So we no longer have explicit solutions of the equation (E), but in using some of
the methods introduced in [5] and [16] we can give answers to the three problems

(1) Suppose $g$ vanishes only at $0$ and $u(x) = o(|x|)$ or $g$ vanishes at $0$ and $\lim_{|x| \to +\infty} u(x) = 0$, then $|x|^{-N-2} u(x)$ converges to some real number $\gamma$ as $x$ tends to infinity.

(II) Suppose $g$ satisfies

$$\liminf_{|r| \to +\infty} |g(r)|/|r|^{(N+1)/(N-1)} = +\infty,$$

or

$$g(r) - g(s) \geq c |r-s|^{2N/(N-1)} - d |r-s|^2,$$

for $c, d > 0$.

(III) Suppose $g$ vanishes only at $0$ and

$$\lim_{r \to +\infty} (g(r) - c r^q) r^{-(q-1)(N+1)/2} = 0, \text{ for some } 1 < q < \frac{N+1}{N-1},$$

then any solution $u$ of (E) in $\mathbb{R}^N - \{0\}$ is spherically symmetric.

(III) Suppose $g$ vanishes only at $0$ and

$$\lim_{r \to +\infty} (g(r) - c r^q) r^{-(q-1)N/2} = 0, \text{ for some } 1 < q < \frac{N}{N-2},$$

then any solution $u$ of (E) in $\mathbb{R}^N - \{0\}$ is uniquely determined by its isotropic singularity at $0$.

If we replace $\Delta$ by $L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ a strongly elliptic operator with constant coefficient all our results remain true provided $|x|$ is replaced by some $\left(\sum a_{ij} x_i x_j\right)^{1/2}$, the coefficients $a_{ij}$ being obtained after the diagonalisation of the matrix $\left(\frac{1}{2}(a_{ij} + a_{ji})\right)$.

Results concerning symmetry and singularities of positive solutions of equations of type (E) when $g(r) = -r^d$ have been given in [7] and in [8]. For general $g$, symmetry of positive regular solutions vanishing for $|x| = R$ is also given in [8].

The contents of our work is the following:

1. Behaviour at infinity.
2. Spherically symmetric solutions.
1. BEHAVIOUR AT INFINITY

In this paragraph $\Omega$ is an exterior domain (that is $\Omega$ is compact) of $\mathbb{R}^N$, $N > 3$, and $g$ is a nondecreasing function defined on $\mathbb{R}$ and vanishing at 0. The equation we consider is the following

$$ -\Delta u + g(u) = 0. \quad (1.1) $$

For the sake of simplicity we prefer to deal with $C^2$ solutions of (1.1) in $\Omega$, so we shall suppose that $g$ is Holder continuous although our results remain true when $g$ is discontinuous and $u$ is a $C^1$ solution of (1.1) in $D'(\Omega)$.

When $\int_{-1}^{1} (j(t))^{-1/2} dt < +\infty$, where $j(t) = \int_{0}^{t} g(s) ds$, any solution of (1.1) vanishing in some weak sense at infinity has a compact support (see [1]).

When $g(u) = |u|^{q-1} u$, $q > 1$, the behaviour of any solution $u$ of (1.1) has been given by Veron in [14]:

(i) if $q = 1$, $|x|^{(N-1)/2} \exp(|x|) u(x)$ converges to some non isotropic limit,

(ii) if $1 < q < \frac{N}{N-2}$ and $u \geq 0$, $|x|^{2/(q-1)} u(x)$ converges to 0 or

$$ \left( \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right) \right)^{1/(q-1)} = \varphi_{q,n}, $$

(iii) if $\frac{N+1}{N-1} \leq q < \frac{N}{N-2}$, $|x|^{2/(q-1)} u(x)$ converges to 0 or $\pm \varphi_{q,n}$,

(iv) if $q = \frac{N}{N-2}$, $|x|^{N-2} \left( \log |x| \right)^{(N-2)/2} u(x)$ converges to 0 or $\pm \left( \frac{N-2}{\sqrt{2}} \right)^{N-2}

(v) if $q > \frac{N}{N-2}$, $|x|^{N-2} u(x)$ converges to some arbitrary real number.

Moreover, when $q > 1$ and when $u$ vanishes at infinity, the hypothesis on $g$ can be weakened and replaced by $\lim_{r \to 0} g(r)/|r|^{q-1} r = c > 0$.

Our main result which generalises strongly the last one of [14] is

THEOREM 1.1. Suppose $u$ is a $C^2$ solution of (1.1) in $\Omega$ and

(i) either $\lim_{|x| \to +\infty} u(x)/|x| = 0$ and $g$ vanishes only at 0,

(ii) or $\lim_{|x| \to +\infty} u(x) = 0$. 

Then \( |x| \to N^{-2} u(x) \) converges to some real number when \( x \) tends to infinity.

We call \((r, \sigma)\) the spherical coordinates in \( \mathbb{R}^N = \mathbb{R}^+ \times S^{N-1} \) and \( \bar{u}(r) \) the average of \( u(r, \sigma) \) on \( S^{N-1} \) and we suppose that \( \{ x \mid |x| > R \} \subset \Omega \). The following estimate is fundamental.

**PROPOSITION 1.1.** There exists a constant \( C(N) \) such that if the hypotheses of Theorem 7.1 are fulfilled the following estimate holds

\[
\| u(r, \sigma) - \bar{u}(r) \|_{L^\infty(S^{N-1})} \leq \ldots
\]

\[
\leq C(N) \left( 1 + \frac{1}{r} \right)^{(N-1)/2} \frac{r}{\rho} \left( \frac{r - 1 - N}{\rho} \right) \| u(R, \sigma) - \bar{u}(R) \|_{L^2(S^{N-1})}
\]

for any \( R \leq \rho < r \).

We first need the \( L^2 \) version of (1.2)

**LEMMA 1.1.** Suppose \( u \in C^2(\Omega) \) is a solution of (1.1) such that \( \lim_{|x| \to +\infty} u(x) / |x| = 0 \), then

\[
\| u(r, \sigma) - \bar{u}(r) \|_{L^2(S^{N-1})} \leq \left( \frac{r}{R} \right)^{1-N} \| u(R, \sigma) - \bar{u}(R) \|_{L^2(S^{N-1})}
\]

for any \( R \leq r \).

**Proof.** If \( \Delta_{S^{N-1}} \) is the Laplace-Beltrami operator on \( S^{N-1} \), the function \( u \) satisfies

\[
\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_{S^{N-1}} u = g(u),
\]

in \([R, +\infty) \times S^{N-1}\). In averaging (1.4) we obtain

\[
\int_{S^{N-1}} \frac{\partial}{\partial r^2} (u - u)(u - u) \, d\sigma + \frac{N-1}{r} \int_{S^{N-1}} \frac{\partial}{\partial r} (u - u)(u - u) \, d\sigma - \frac{1}{r^2} \int_{S^{N-1}} -\Delta(u - u)(u - u) \, d\sigma \geq 0,
\]

as

\[
\int_{S^{N-1}} (g(u) - g(\bar{u}))(u - \bar{u}) \, d\sigma = \int_{S^{N-1}} (g(\bar{u}) - g(u))(u - u) \, d\sigma + \int_{S^{N-1}} (g(\bar{u}) - g(u))(u - \bar{u}) \, d\sigma \ldots
\]

Thus

\[
\ldots = \int (g(u) - g(\bar{u}))(u - \bar{u}) \, d\sigma \geq 0.
\]

Moreover

\[
\int_{S^{N-1}} -\Delta_{S^{N-1}} (u - \bar{u})(u - \bar{u}) \, d\sigma \geq (N-1) \int_{S^{N-1}} (u - \bar{u})^2 \, d\sigma \] as \( \bar{u} \) is the projection of \( u \).
on the first eigenspace of $-\Delta_{S^{N-1}}$ and $N-1$ is the second eigenvalue of $-\Delta_{S^{N-1}}$ (see [3]), so we deduce

$$\int_{S^{N-1}} \frac{\partial}{\partial r^2} (u-\bar{u})(u-\bar{u})d\sigma - \frac{N-1}{r} \int_{S^{N-1}} \frac{\partial}{\partial r} (u-\bar{u})(u-\bar{u})d\sigma - \frac{N-1}{r^2} \int_{S^{N-1}} (u-\bar{u})^2d\sigma \geq 0.$$  

We set $w(r) = \left( \int_{S^{N-1}} (u-\bar{u})^2(r,\sigma)d\sigma \right)^{1/2}$ and we have when $w \neq 0$:

$$w \frac{dw}{dr} = \int_{S^{N-1}} \frac{\partial}{\partial r} (u-\bar{u})(u-\bar{u})d\sigma, \quad \left| \frac{dw}{dr} \right| \leq \left( \int_{S^{N-1}} \frac{\partial}{\partial r} (u-\bar{u})^2d\sigma \right)^{1/2} \text{ and }$$

$$\int_{S^{N-1}} \frac{\partial}{\partial r^2} (u-\bar{u})(u-\bar{u})d\sigma \leq w \frac{d^2w}{dr^2}. \text{ If we set } \Gamma = \{ r > R : w(r) > 0 \} , \text{ we get}$$

$$\frac{d^2w}{dr^2} + \frac{N-1}{r} \frac{dw}{dr} - \frac{N-1}{r^2} w \geq 0,$$

on $\Gamma$. By the maximum principle $w$ cannot assume a strictly positive maximum value, so the set $\Gamma$ can only be of two types

(i) $\Gamma = (R,T)$, $T$ finite and $w(r) = 0$ on $(T, +\infty)$,

(ii) $\Gamma = (R,T) \cup (T', +\infty)$, and $w(T) = w(T') = 0$ if $T$ and $T'$ are finite.

Let us consider now the following differential equation

$$\frac{d^2y}{dr^2} + \frac{N-1}{r} \frac{dy}{dr} - \frac{N-1}{r^2} y = 0.$$  

That equation admits two linearly independent solutions

$$\phi_1(r) = r \quad \text{and} \quad \phi_2(r) = r^{1-N}.$$

Now we set $\psi_\epsilon(r) = e^\epsilon + \| u(R,\cdot) - \bar{u}(R) \|_{L^2(S^{N-1})} (r)^{1-N}, \quad \epsilon \geq 0$. As $\psi_\epsilon$ satisfies (1.8), we have

$$\frac{d^2}{dr^2} (w - \psi_\epsilon) + \frac{N-1}{r} \frac{d}{dr} (w - \psi_\epsilon) - \frac{N-1}{r^2} (w - \psi_\epsilon) \geq 0,$$

on $\Gamma$. If we are in the first case or in the second when $T < +\infty$, we take $\epsilon = 0$ and we deduce by the maximum principle that $0 \leq w(r) \leq \psi_0(r)$ on $(R,T)$, which is (1.3). In the second case with
T = + \infty, or on (T', \infty) we take \epsilon > 0. As \lim_{r \to + \infty} w(r)/r = 0, w - \psi_\epsilon is non positive at the end points of the interval, so \psi - \psi_\epsilon remains non positive. Making \epsilon \to 0 we deduce \psi \leq \psi_0 which ends the proof.

**Remark 1.1.** In Lemma 1.1 we need not assume \( g(0) = 0 \) (see Theorem 2.1 for an application of this method).

We set \( u^+ = \max(u,0), u^- = \max(-u,0) \) and we have

**Lemma 1.2.** Under the assumptions of Theorem 1.1 we have

\[
\begin{align*}
(1.11) & \quad u^+(x) \leq \left( \frac{|x|}{R} \right)^{2-N} \| u^+(R, \cdot) \|_{L^\infty(S^{N-1})}, \\
(1.12) & \quad u^-(x) \leq \left( \frac{|x|}{R} \right)^{2-N} \| u^-(R, \cdot) \|_{L^\infty(S^{N-1})},
\end{align*}
\]

for any \( x \) such that \( |x| \geq R \).

**Proof.** Multiplying (1.4) by \( u \) and integrating over \( S^{N-1} \) yields

\[
(1.13) \quad \frac{d^2}{dr^2} \int_{S^{N-1}} u^2 d\sigma + \frac{N-1}{r} \frac{d}{dr} \int_{S^{N-1}} u^2 d\sigma \geq 0.
\]

By the maximum principle \( r \mapsto \int_{S^{N-1}} u^2(r,\sigma) d\sigma \) is asymptotically monotone so there exists \( \gamma \in \mathbb{R}^+ \cup \{ + \infty \} \) such that \( \lim_{r \to + \infty} \| u(r, \cdot) \|_{L^2(S^{N-1})} = \gamma \) and \( \lim_{r \to + \infty} \overline{u}(r) = \gamma \) or \( \lim_{r \to + \infty} \overline{u}(r) = -\gamma \) and \( \lim_{r \to + \infty} u(r, \cdot) = \lim_{r \to + \infty} u(r, \cdot) \) in \( L^2(S^{N-1}) \).

We first suppose that \( \gamma = 0 \) (which is an hypothesis if \( \lim_{|x| \to + \infty} u(x) = 0 \)) and set \( p \) a convex function vanishing on \( (-\infty, 0) \), increasing on \( (0, + \infty) \) and such that \( 0 \leq p' \leq 1 \). We set \( \theta^+(x) = \left( \frac{|x|}{R} \right)^{2-N} \| u^+(R, \cdot) \|_{L^\infty(S^{N-1})} \). \( \theta^+ \) is a positive harmonic function and we have

\[
(1.14) \quad \frac{d^2 \theta^+}{dr^2} + \frac{N-1}{r} \frac{d \theta^+}{dr} + \frac{1}{r^2} \Delta_{S^{N-1}} \theta^+ \leq g(\theta^+).
\]

As we have

\[
(1.15) \quad -\int_{S^{N-1}} \Delta_{S^{N-1}} (u - \theta^+) p'(u - \theta^+) d\sigma \geq 0,
\]
we deduce from the monotonicity of $g$ that

\[
\frac{\partial^2}{\partial r^2} p(u^-) \geq p(u^+) \frac{\partial^2}{\partial r^2} (u^-),
\]

we deduce from the monotonicity of $g$ that

\[
\frac{d^2}{dr^2} \int_{S^{N-1}} p(u^-) d\sigma + \frac{N-1}{r} \frac{d}{dr} \int_{S^{N-1}} p(u^-) d\sigma \geq 0.
\]

The function $r \mapsto \int_{S^{N-1}} p(u(r,\alpha) - \theta^+(r)) d\sigma$ vanishes at $R$ and as $p(u^-) \leq (u^-)^+$, we have :

\[
\lim_{r \to + \infty} \int_{S^{N-1}} p(u(r,\alpha) - \theta^+(r)) d\sigma = 0.
\]

By the maximum principle

\[
\int_{S^{N-1}} p(u^-) d\sigma \leq 0,
\]

which is (1.11). In considering $\theta^-(x) = -\left( \frac{x}{R} \right)^{2-N} \| u^- (r, \cdot) \|_{L^\infty(S^{N-1})}$, we obtain (1.12) in the same way.

We suppose now that $\gamma > 0$ (so $g$ vanishes only at $0$) and, for example, $\lim u(r) = \gamma$. The function $\int_{S^{N-1}} p(\theta^- u) d\sigma$ satisfies

\[
\frac{d^2}{dr^2} \int_{S^{N-1}} p(\theta^- u) d\sigma + \frac{N-1}{r} \frac{d}{dr} \int_{S^{N-1}} p(\theta^- u) d\sigma \geq 0;
\]

it vanishes at $R$ and as $p(\theta^- u) \leq p(\theta^- \bar{u}) + \| \bar{u} - u \|$, we deduce from (1.3) that

\[
\lim_{r \to + \infty} \int_{S^{N-1}} p(\theta^- (r) - u(r,\alpha)) d\sigma = 0.
\]

By the maximum principle we get (1.12) which implies that $u(x)$ is bounded below on $\{ x \mid \| x \| \geq R \}$.

As $g(r) = g^+(r) - g^-(r)$, we set $g_N^+(r) = \min(N,g^+)$, $N > 0$ and we have in averaging (1.4)

\[
\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} \geq g^+_N(u) - g^-(u).
\]

But $g^- (u) = g^- (\bar{u})$ and $\lim_{r \to + \infty} u^- (r, \cdot) = 0$ in $L^2(S^{N-1})$. As $u^-$ is bounded below,

\[
\lim_{r \to + \infty} g^- (u^-) = 0. \quad \text{On the other hand, by Lebesgue’s Theorem,} \quad \lim_{r \to + \infty} g^+_N (u) = g^+_N (\gamma) = \frac{1}{|S^{N-1}|} \min(N,g(\gamma)) = \alpha > 0. \quad \text{There exists} \quad R' > R \quad \text{such that}
\]

\[
\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} \geq \frac{\alpha}{2N} r^2 + c r^{2-N} + c',
\]

on $(R', + \infty)$. Integrating (1.20) twice yields

\[
\bar{u}(r) \geq \frac{\alpha}{2N} r^2 + c r^{2-N} + c',
\]
LEMMA 1.3. Suppose \( g \) is a continuous nondecreasing function vanishing at 0; then for any \( \rho > 0 \) and any real \( a \) there exists a unique function \( v \) twice continuously differentiable satisfying

\[
\begin{cases}
\frac{d^2 v}{dr^2} + \frac{N-1}{r} \frac{dv}{dr} = g(v) & \text{on } (\rho, +\infty), \\
v(\rho) = a, \quad \lim_{r \to +\infty} r^{N-2} |v(r)| < +\infty.
\end{cases}
\]

**Proof.** *Uniqueness*: Consider the following change of variable and unknown

\[
s = \frac{r^{N-2}}{N-2}, \quad v(r) = r^{-N} w(s).
\]

The function \( w \) satisfies

\[
s^2 \frac{d^2 w}{ds^2} = (N-2)^2 - \frac{4N}{s^{N-2}} - \frac{N}{s^{N-2}} g\left(\frac{w}{s^{N-2}}\right).
\]

Suppose \( \tilde{w} \) is another solution of (1.24) with the same initial data, then

\[
s^2 \frac{d^2 w}{ds^2} |w - \tilde{w}| > 0,
\]

so the function \( s \mapsto |w - \tilde{w}|(s) \) is nonnegative, convex, vanishes at \( \rho^{N-2} \) and \( \lim_{s \to +\infty} \frac{1}{s} |w(s) - \tilde{w}(s)| = 0 \), so it is identically zero.

*Existence*: For any \( T > \rho \) set \( v_T \) the solution of the following two points problem

\[
\begin{cases}
\frac{d^2 v_T}{dr^2} + \frac{N-1}{r} \frac{dv_T}{dr} = g(v_T) & \text{on } (\rho, T), \\
v_T(\rho) = a, \quad v_T(T) = 0.
\end{cases}
\]

The function \( v_T \) exists and is unique; moreover \( |v_T| \) decreases. Thanks to the uniqueness of the solution of (1.26), the function \( T \mapsto |v_T(r)| \) is nondecreasing for any \( r > \rho \). As \( |v_T(r)| \leq |a| \) and \( g \) is continuous, we deduce in integrating (1.26) that \( \frac{dv_T}{dr} \) and \( \frac{d^2 v_T}{dr^2} \) remain bounded on every
compact interval of \([\rho, T)\); so \(v_T(r)\) converges uniformly on every compact interval to some \(C^2\)
function \(v\), as \(T\) tends to +\(\infty\). Moreover \(|v_T|\) is majorized by the function \(\psi\) defined on \([\rho, +\infty)\)
by \(\psi(r) = \left(\frac{\rho}{r}\right)^{N-2} |a|\) (which satisfies (1.22) with \(g \equiv 0\)). So \(r^{N-2} v(r)\) remains bounded on
\([\rho, +\infty)\) and (1.22) is satisfied.

**Lemma 1.4.** For any \(\rho > 0\) and \(a \in L^2(S^{N-1})\) there exists a unique function \(\omega \in L^\infty((\rho, +\infty); \)
\(L^2(S^{N-1})) \cap C^0((\rho, +\infty); L^2(S^{N-1})) \cap C^2((\rho, +\infty) \times S^{N-1})\) satisfying

\[
\begin{aligned}
&\frac{s^2}{\Delta} \frac{\partial^2 \omega}{\partial s^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} \omega = 0 \quad \text{on } (\rho, +\infty) \times S^{N-1}, \\
&\omega(\rho,.) = a(.) \quad \text{on } S^{N-1}.
\end{aligned}
\]

Moreover there exists a constant \(C = C(N)\) such that the following estimate holds

\[
\|\omega(s,.)\|_{L^\infty(S^{N-1})} \leq C \left(1 + \frac{1}{\log\frac{s}{\rho}}\right)^{(N-1)/2} \|a(.)\|_{L^2(S^{N-1})}
\]

for any \(s > \rho\).

**Proof.** For the uniqueness set \(\tilde{\omega}\) a solution of (1.27) taking the value \(\tilde{\alpha}\) for \(s = \rho\). We have :

\[
\int_{S^{N-1}} \frac{\partial^2}{\partial s^2} (\omega - \tilde{\omega})(\omega - \tilde{\omega}) d\sigma \geq 0.
\]

Hence \(s \mapsto \int_{S^{N-1}} (\omega - \tilde{\omega})^2(s,\sigma) d\sigma\) is a convex function. As it is bounded it is nonincreasing.

For the existence we set \(t = \log s\) and \(\phi(t,\sigma) = \omega(s,\sigma)\). The function \(\phi\) satisfies on
\((\log \rho, +\infty)\)

\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial t} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} \phi = 0.
\]

If \((T(t))_{t \geq 0}\) is the semigroup of contractions of \(L^2(S^{N-1})\) generated by

\[-\left(-\frac{1}{(N-2)^2} \Delta_{S^{N-1}} + \frac{1}{4}\right)^{1/2},\]

it is easy to check that \(\exp((t - \log \rho)/2) T(t - \log \rho)\alpha\) satisfies the equation (1.29) with initial data \(\alpha\) and is bounded; so it is \(\phi\).

Set \(H_0\) the subspace of \(L^2(S^{N-1})\) of constant functions and \(H' = (H_0)^\perp\). We have
the following hilbertian direct sum : \(L^2(S^{N-1}) = H' \oplus H_0\), and both \(H_0\) and \(H'\) are invariant
under \((T(t))_{t \geq 0}\).
the restriction $T'(t)$ of $T(t)$ to $H'$ satisfies (see [4])

\[(1.30) \quad \| T'(t)u \|_{L^2(S^{N-1})} \leq \exp(-tN/(2N-4)) \| u \|_{L^2(S^{N-1})}, \]

for any $u \in H'$: Moreover we have the following regularizing effect (see [15])

\[(1.31) \quad \| T'(t)u \|_{L^\infty(S^{N-1})} \leq C(1 + \frac{1}{t})^{(N-1)/2} \| u \|_{L^2(S^{N-1})}, \]

for any $u \in L^2(S^{N-1})$ and any $t > 0$. In combining (1.30) and (1.31), and using the semigroup property, we have for any $u \in H'$, any $t > 0$ and any $\epsilon > 0$:

\[(1.32) \quad \| T'(t)u \|_{L^\infty(S^{N-1})} \leq C(1 + \frac{1}{et})^{(N-1)/2} \exp(-t(1-\epsilon)N/(2N-4)) \| u \|_{L^2(S^{N-1})}. \]

Now we write $\alpha = \alpha_0 + \alpha'$ with $\alpha_0 \in H_0$ and $\alpha' \in H'$ (and in fact $\alpha_0 = \frac{1}{|S^{N-1}|} \int \alpha(s) \, ds$). We have

\[(1.33) \quad T(t)\alpha = T(t)\alpha_0 + T(t)\alpha'; \]

but $T(t)\alpha_0 = \exp(-t/2)\alpha_0$. In taking $\epsilon = \frac{2}{N}$ in (1.32) we get

\[(1.34) \quad \| T(t)\alpha \|_{L^\infty(S^{N-1})} \leq C(1 + \frac{1}{t})^{(N-1)/2} \exp(-t/2) \| \alpha \|_{L^2(S^{N-1})}. \]

In replacing $t$ by $\log s - \log \rho$, we obtain (1.28).

**Proof of Proposition 1.1.** Consider the change of variable and unknown

\[(1.35) \quad s = r^{N-2} \frac{N-2}{N-2}, \quad u(r, \sigma) = r^{2-N} v(s, \sigma). \]

The function $v$ satisfies

\[(1.36) \quad s^2 \frac{\partial^2 v}{\partial s^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} v = (N-2)^{N-2} \frac{4-N}{s^{N-2}} \frac{N}{s(N-2)} g\left( \frac{v}{s(N-2)} \right), \]

in $\left[ \frac{R^{N-2}}{N-2}, + \infty \right) \times S^{N-1}$. Let $y$ be the solution (from Lemma 1.3) of

\[(1.37) \quad \begin{cases} 
  s^2 \frac{d^2 y}{ds^2} = (N-2)^{N-2} \frac{4-N}{s^{N-2}} \frac{N}{s(N-2)} g\left( \frac{y}{s(N-2)} \right) & \text{on } (\rho', + \infty), \quad \rho' > R, \\
  y(\rho') = a, & \text{y bounded.} 
\end{cases} \]
We set $w = v - y$ and we have for $s \geq p'$

$$s^2 \frac{\partial^2 w}{\partial s^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} w = (N-2)^{\frac{N}{N-2}} \frac{N}{s^{\frac{N}{N-2}}} \text{hw},$$

where

$$h = \begin{cases} 
\frac{v}{s(N-2)} - \frac{y}{s(N-2)} \big/ (v - y) & \text{if } v \neq y, \\
0 & \text{if } v = y.
\end{cases}$$

The function $h$ is nonnegative as $g$ is nondecreasing. If $\omega^+$ is the solution of (1.27) taking the value $(v(p',\cdot) - a)^+$ for $s = p'$, $\omega^+$ is nonnegative and satisfies

$$s^2 \frac{\partial^2 \omega^+}{\partial s^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} \omega^+ \leq (N-2)^{\frac{N}{N-2}} \frac{N}{s^{\frac{N}{N-2}}} \text{h} \omega^+.$$ 

Introducing the nondecreasing convex function $p$ as we have done it in Lemma 1.2, we get $s^2 \frac{d^2}{ds^2} \int_{S^{N-1}} p(w - \omega^+) \text{d} \sigma \geq 0$; hence $w \leq \omega^+$. In the same way $w$ is minorized on $(p',+\infty)$ by the solution $\omega^-$ of (1.27) taking the value $-(v(p',\cdot) - a)^-$ for $s = p'$. Combining those estimates with (1.28) we get

$$\|v(s,\cdot) - v(s)\|_{L^\infty(S^{N-1})} \leq C \left(1 + \frac{1}{\log \frac{s}{\rho'}}\right)^{(N-1)/2} \|v(p',\cdot) - a\|_{L^2(S^{N-1})}.$$ 

In averaging (1.40) we deduce

$$\|v(s,\cdot) - \overline{v}(s)\|_{L^\infty(S^{N-1})} \leq 2C \left(1 + \frac{1}{\log \frac{s}{\rho'}}\right)^{(N-1)/2} \|v(p',\cdot) - a\|_{L^2(S^{N-1})}.$$ 

We take now $a = \overline{v}(\rho')$, $s = \frac{N}{\rho'}$ and apply (1.3) between $R$ and $\rho$, we get (1.2).

**Remark 1.2.** We can deduce from Lemma 1.1 a first property of symmetry of the solutions of (1.1): suppose $g$ is a monotone nondecreasing function and $u$ is a $C^2$ solution of (1.1) satisfying

$$\lim_{x \to +\infty} u(x) = 0. \text{ If } u \text{ is spherically symmetric on } \{x \mid |x| = R\} \text{ then it remains spherically symmetric on } \{x \mid |x| > R\}.$$
Proof of Theorem 1.1. In Proposition 1.1 we take $r = 2\rho$ and make $r \to +\infty$. In taking the notations of the transformation (1.35) we get

\begin{equation}
(1.42) \quad \lim_{s \to +\infty} \|v(s,.) - \overline{v}(s)\|_{L^\infty(S^{N-1})} = 0.
\end{equation}

As \(\{\overline{v}(s)\}\) is bounded, there exists a sequence \(s_n \to +\infty\) such that \(v(s_n,.)\) converges to some number \(c\) when \(n \to +\infty\).

If \(C > 0\) (or \(C < 0\) in the same way) there exists some \(n_0\) such that \(v(s_n,.) > C\) for \(n > n_0\) (it is a consequence of (1.42)). If we apply the maximum principle to the function \(v\) in the spherical shell \((s_{n_0}, s_n) \times S^{N-1}\), we deduce that \(v(s,.) \geq 0\) in that shell and therefore in \((s_{n_0}, +\infty) \times S^{N-1}\). In averaging (1.36) on \(S^{N-1}\) we deduce \(s^2 \frac{d^2 v}{ds^2} \geq 0\) for \(s \geq s_{n_0}\). Hence \(v\) is convex and, as it is bounded, it converges when \(s\) goes to \(+\infty\). The only admissible limit is \(C\) and finally \(\lim_{s \to +\infty} v(s,.) = C\) in \(L^\infty(S^{N-1})\).

If \(C < 0\) then
\[
\lim_{s \to +\infty} v(s,.) \|_{L^\infty(S^{N-1})} = 0, \text{ otherwise there would exist a sequence } s'_{n} \to +\infty \text{ and } \epsilon > 0 \text{ such that } \|v(s'_{n},.)\|_{L^\infty(S^{N-1})} > \epsilon \text{ for } s'_{n} \geq s'_{n_0} \text{ and there would exist a sequence } s''_{n} \text{ extracted from } s'_{n} \text{ and a number } \lambda, \|\lambda\| > \frac{\epsilon}{2}, \text{ such that } \lim_{s''_{n} \to +\infty} \overline{v}(s''_{n}) = \lambda. \text{ Applying what have been done when } C \neq 0, \text{ we would have } \lim_{s \to +\infty} v(s,.) = \lambda \text{ in } L^\infty(S^{N-1}), \text{ which contradicts } \lim_{s \to +\infty} v(s_{n},.) = 0.
\]

2. SPHERICALLY SYMMETRIC SOLUTIONS

In this paragraph \(g\) is a continuous nondecreasing function defined on \(\mathbb{R}\) (not necessarily vanishing at 0) and we still consider the equation

\begin{equation}
(2.1) \quad -\Delta u + g(u) = 0;
\end{equation}

but the equation is taken in \(D'(\mathbb{R}^N - \{0\})\) and \(u\) may have a singularity at 0. The following result is fundamental and its proof is very similar to the one of Lemma 1.1 (comparison of \(w\) with \(e^{\phi_1} + e^{\phi_2}, \epsilon, \epsilon' > 0\)).

THEOREM 2.1. Suppose \(u \in C^2(\mathbb{R}^N - \{0\})\) is a solution of (2.1) in \(D'(\mathbb{R}^N - \{0\})\) such that
where \((r,a) \in \mathbb{R}^+ \times S^{N-1}\) are the spherical coordinates in \(\mathbb{R}^N\) and \(\bar{u}(r) = \frac{1}{\mu S^{N-1}} \int_{S^{N-1}} u(r,a) da\), then \(u\) is spherically symmetric.

The following «universal» estimate on \(u\) when \(g\) has an asymptotic growth corresponding to a power greater than 1 is originated in [5].

Lemma 2.1. Suppose \(g\) satisfies, for some \(q > 1\),

\[
\begin{align*}
\lim_{r \to +\infty} \inf g(r)/r^q &> 0, \\
\lim_{r \to +\infty} \sup g(r)/|r|^q &< 0,
\end{align*}
\]

and \(u \in C^2(\mathbb{R}^N - \{0\})\) is a solution of (2.1) in \(D'(\mathbb{R}^N - \{0\})\); then

\[
|u(x)| \leq C |x|^{-2/(q-1)} + D,
\]

for \(x \neq 0\), where \(C\) and \(D\) depend on \(g\) and \(N\).

Proof. From the hypothesis (2.2) there exist two constants \(A\) and \(B > 0\) such that

\[
\begin{align*}
g(r) &\geq Ar^q - B & \text{on } r > 0, \\
g(r) &\leq -A |r|^q + B & \text{on } r < 0,
\end{align*}
\]

which yields

\[
-\Delta u + Au^q \leq B \quad \text{a.e. on } \{ x | u(x) > 0 \}.
\]

For \(x_0 \neq 0\) set \(G = \{ x \in \mathbb{R}^N, \ |x - x_0| < \frac{1}{2} |x_0| \} \) and consider the function

\[
v(x) = \lambda \left( \frac{1}{4} |x_0|^2 - |x - x_0|^2 \right)^{-2/(q-1)} + \mu,
\]
where $\lambda$ and $\mu$ are to be determined in order that

\begin{equation}
-\Delta v + Av^q \geq B,
\end{equation}

in $G$. For simplification set $v(r) = \lambda(R^2 - r^2)^{-2/(q-1)} + \mu$. We have in $G$

\[-\Delta v + Av^q \geq \lambda(R^2 - r^2)^{-2q/(q-1)(A\lambda^{q-1} - \frac{2NR^2}{q-1} + \frac{2}{q-1}(N - 2 \frac{q+1}{q-1} r^2) + \mu^q}.
\]

Set $\beta = \max\left(\frac{2N}{q-1}, 4 \frac{q+1}{(q-1)^2}\right)$ and we take $\lambda = \left(\frac{\beta}{A}\right)^{1/(q-1)} R^{2/(q-1)}$ and $\mu = \left(\frac{B}{A}\right)^{1/q}$, so we get (2.6).

By Kato’s inequality (see [10]) we have as in [5]

\begin{equation}
\Delta(u - v)^+ \geq \text{sign}^+(u - v) \Delta(u - v) \geq 0 \quad \text{in } D'(G),
\end{equation}

in $D'(G)$. Moreover $(u - v)^+$ vanishes in some neighbourhood of $\partial G$, so $(u - v)^+ = 0$ in $G$ and

\begin{equation}
u(x_0) \leq v(x_0) = \left(\frac{16\beta}{A}\right)^{1/(q-1)} |x_0|^{-2/(q-1)} + \left(\frac{B}{A}\right)^{1/q}.
\end{equation}

In the same way $u(x_0) \geq -v(x_0)$.

From that result we get

**THEOREM 2.2.** Suppose $g$ satisfies

\[
\left\{
\begin{array}{l}
\lim_{r \to +\infty} \inf g(r)/r^{(N+1)/(N-1)} = +\infty, \\
\lim_{r \to -\infty} \sup g(r)/r^{(N+1)/(N-1)} = -\infty,
\end{array}
\right.
\]

and $u \in C^2(\mathbb{R}^N - \{0\})$ satisfies (2.1) in $D'(\mathbb{R}^N - \{0\})$; then $u$ is spherically symmetric.

**Proof.** From (2.9), for any $n > 0$, there exists $B_n > 0$ such that

\[
\left\{
\begin{array}{l}
g(r) \geq nr^{(N+1)/(N-1)} - B_n \quad \text{for } r \geq 0, \\
g(r) \leq -n r^{(N+1)/(N-1)} + B_n \quad \text{for } r \leq 0.
\end{array}
\right.
\]
From (2.8) we get $|u(x)| \leq \left( \frac{16 \beta}{n} \right)^{1/(q-1)} |x|^{1-N} + \left( \frac{B_n}{n} \right)^{(N-1)/(N+1)}$, for $x \neq 0$, which implies $\limsup_{r \to 0} r^{N-1} \|u(r, \cdot) - \overline{u}(r)\|_{L^2(S^{N-1})} \leq 2 \left( \frac{16 \beta}{n} \right)^{1/(q-1)}$. Letting $n \to +\infty$ we obtain the condition ii) of Theorem 2.1; as for the condition i) it is an immediate consequence of (2.3).

When the rate of growth of $g$ at infinity is of order $\frac{N+1}{N-1}$, it is not enough to make a hypothesis on $g$ but we have to make it on $g'$ and we get:

**THEOREM 2.3.** Suppose $g$ satisfies

$$\tag{2.11} (g(r) - g(s))(r - s) \geq C |r - s|^{2N/(N-1)} - D(r - s)^2,$$

for some $C > 0$, $D > 0$ and all $r$ and $s$ real. If $u \in C^2(\mathbb{R}^N - \{0\})$ is a solution of (2.1) in $D'(\mathbb{R}^N - \{0\})$, then $u$ is spherically symmetric.

We first need the following result

**LEMMA 2.2.** Under the hypotheses of Theorem 2.3, we have

$$\tag{2.12} \|u(r, \cdot) - \overline{u}(r)\|_{L^2(S^{N-1})} \leq \frac{r}{R} \|u(R, \cdot) - \overline{u}(R)\|_{L^2(S^{N-1})}$$

for $0 < r \leq R$.

**Proof.** The function $u$ satisfies

$$\tag{2.13} \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_{S^{N-1}} u = g(u),$$

in $(0, +\infty) \times S^{N-1}$. We set $y(r, \sigma) = r^{N-1} u(r, \sigma)$. From Lemma 2.1 $y$ is bounded on every compact or $(0, +\infty) \times S^{N-1}$ and it satisfies

$$\tag{2.14} \frac{\partial^2 y}{\partial r^2} + \frac{1-N}{r^2} \frac{\partial y}{\partial r} + \frac{N-1}{r^2} \Delta_{S^{N-1}} y = r^{N-1} g(r^{1-N} y) .$$

Now we set

$$\tag{2.15} s = \frac{r^N}{N}, \quad v(s, \sigma) = y(r, \sigma) .$$

The function $v$ satisfies
in \((0, +\infty) \times S^{N-1}\). If \(\overline{v}\) is the average of \(v\) on \(S^{N-1}\) we get, as in Lemma 1.1,

\[
\text{hence } s \mapsto \|v(s, \cdot) - \overline{v}(s)\|_{L^2(S^{N-1})} \text{ is convex. As it is bounded, it admits a limit when } s \to 0.
\]

From (2.11) we get

\[
(\text{2.17}) \quad (N_s)^2 \int_{S^{N-1}} \frac{\partial^2}{\partial s^2} (v - \overline{v})(v - \overline{v}) d\sigma \geq 0;
\]

hence \(s \mapsto \|v(s, \cdot) - \overline{v}(s)\|_{L^2(S^{N-1})}^2\) is convex. As it is bounded, it admits a limit when \(s \to 0\).

From (2.17) we also deduce that the function \(s \mapsto \|v(s, \cdot) - \overline{v}(s)\|_{L^2(S^{N-1})}^2\) is convex (see the proof of Lemma 1.1). As it vanishes at \(0\) we get, for \(0 < s < \sigma\):

\[
\text{twice that the only admissible limit for } \|v - \overline{v}\|_{L^2(S^{N-1})}^2 = 0. \text{ From (2.17) we also deduce that }
\]

\[
(\text{2.19}) \quad \|v(s, \cdot) - \overline{v}(s)\|_{L^2(S^{N-1})} \leq \frac{s}{\sigma} \|v(\sigma, \cdot) - \overline{v}(\sigma)\|_{L^2(S^{N-1})}^2,
\]

which is (2.12).

**Remark 2.1.** The assumption of monotonicity on \(g\) can be avoided for obtaining estimates of the type (2.12) : if we suppose that \(g\) satisfies

\[
(\text{2.20}) \quad (g(r) - g(s)) \geq C |r - s|^{q+1} - D(r - s)^2,
\]

for some \(C\) and \(D > 0\), \(q \geq \frac{N+1}{N-1}\) and all \(r, s\) real, we first deduce from Lemma 2.1 the boundedness of \(|x|^{2/(q-1)} u(x)|\) on every compact of \(\mathbb{R}^N\). With the change of variable of Lemma 2.2 of [16] we obtain the following estimate

\[
(\text{2.21}) \quad \|u(r, \cdot) - \overline{u}(r)\|_{L^2(S^{N-1})} + dr^\alpha \leq (\frac{r}{R})^{2q/(q-1)} (\|u(R, \cdot) - \overline{u}(R)\|_{L^2(S^{N-1})} + dR^\alpha)
\]

for \(r < R\),

where \(d\) depends on \(D\) and \(\alpha > 0\). If we suppose moreover that \(g\) is differentiable and satisfies

\[
(\text{2.22}) \quad |g'(r)| \leq C' |r|^{q-1} + D',
\]
for some $C'$ and $D' > 0$ and all $r$, then we can obtain as in the Appendix of [16]

\[(2.23) \quad \lim_{r \to 0} \| u(r, \cdot) - \overline{u}(r) \|_{L^\infty(S^{N-1})} = 0. \]

Such a relation can be used for proving that the isolated singularities of the solutions of (2.1) are radial.

**Proof of the Theorem 2.3.** From Lemma 1.1, we have for any $p < r$,

\[(2.24) \quad \| u(r, \cdot) - \overline{u}(r) \|_{L^2(S^{N-1})} \leq \left( \frac{r}{p} \right)^{1-N} \| u(p, \cdot) - \overline{u}(p) \|_{L^2(S^{N-1})}; \]

and from the Lemma 2.2, \( \lim_{\rho \to 0} \| u(\rho, \cdot) - \overline{u}(\rho) \|_{L^2(S^{N-1})} = 0 \), which implies \( \| u(r, \cdot) - \overline{u}(r) \|_{L^2(S^{N-1})} = 0 \) for all $r > 0$ and ends the proof.

When $1 < q < \frac{N+1}{N-1}$, there exist non spherically symmetric solutions of

\[(2.25) \quad -\Delta u + |u|^{q-1} u = 0, \]

in $\mathbb{R}^N \setminus \{0\}$. For example if $v$ is a non constant solution of the equation

\[(2.26) \quad -\Delta_{S^{N-1}} v + |v|^{q-1} v = \left( \frac{2}{q-1} \right) \left( \frac{2q}{q-1} - N \right) v \quad \text{on } S^{N-1}, \]

(such a solution exists as \( \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right) > N-1 \) which is the second eigenvalue of \( -\Delta_{S^{N-1}} \)) then $x \mapsto |x|^{-2/(q-1)} v(x/|x|)$ is a non isotropic solution of (2.25). However such a solution cannot keep a constant sign, so we shall restrict ourself to positive solutions of (2.1). Our first result is an extension of Theorem 1.1 of [16].

**PROPOSITION 2.1.** Suppose $g$ satisfies

\[(2.27) \begin{align*}
\text{i)} & \quad \lim_{r \to +\infty} g(r) r^q = c, \\
\text{ii)} & \quad \limsup_{r \to 0^+} g(r)/r < +\infty,
\end{align*} \]

for some $c > 0$ and $1 < q < \frac{N}{N-2}$, and $\Omega$ is an open subset of $\mathbb{R}^N$ containing $0$. If $u \in C^2(\Omega \setminus 0$ ) is a non negative solution of (2.1) in $D'(\Omega \setminus 0$ ) then we have the following alternative
Proof. We shall just sketch it as it is not far from the proof of Theorem 1.1 of [16] (at least in its first part). Moreover we need not suppose that \( g \) is nondecreasing. The two assertions are distinct according to \( x \xrightarrow{N} u(x) \) is bounded or not near 0.

Part 1: \( x \xrightarrow{N} u(x) \) is bounded in some neighbourhood of 0 (and we can even suppose that \( u \) has not a constant sign if \( I g(r)/ |r|^q \) is bounded when \( r \rightarrow -\infty \)). We make the change of variable (1.35) of Proposition 1.1 and we deduce from Lemma 6.4 of [16] that \( \lim_{r \rightarrow 0} r^{N-2} \| u(r) - u(r_0) \|_{L^\infty(SN-1)} = 0 \). We end the proof as in Theorem 1.1 of [16].

Part 2: \( x \xrightarrow{N} u(x) \) is unbounded near 0. If we write (2.1) as follows

\[
-\Delta u + \frac{g(u)}{u} = 0,
\]

we deduce from (2.27) and Lemma 2.1 that \( \frac{g(u)}{u} \leq C |x|^{-2} + D \). Using Trudinger's estimates in Harnack inequalities as in the Lemma 1.4 of [16], we deduce that \( \lim_{x \rightarrow 0} x^{N-2} u(x) = +\infty \).

For any \( c' > c \) there exists \( \rho > 0 \) such that \( g(u(x)) \leq c'(u(x))^q \) on \( \{ x \mid |x| < \rho \} \), so \( -\Delta u + cu^q \geq 0 \) on such a shell. For any \( \alpha > 0 \) set \( v_\alpha \) the solution of

\[
\begin{cases}
-\Delta v_\alpha + cv_\alpha^q = 0 & \text{for } 0 < |x| < \rho, \\
\lim_{x \rightarrow 0} x^{N-2} v_\alpha(x) = \alpha, & v_\alpha(x) = \min u(x) \text{ for } |x| = \rho.
\end{cases}
\]

Such a solution exists (see Lemma 1.6 of [16]). Moreover, from the maximum principle, \( v_\alpha(x) \leq u(x) \) for any \( x \) with \( 0 < |x| < \rho \). When \( \alpha \rightarrow +\infty \), \( v_\alpha(x) \) increases and converges to \( v_\infty(x) \) and

\[
\lim_{x \rightarrow 0} x^{2/(q-1)} v_\infty(x) = \left( \frac{2}{c(q-1)} \left( \frac{2q}{q-1} - N \right) \right)^{1/(q-1)},
\]

from [16]. If we set

\[
l = \left( \frac{2}{c(q-1)} \left( \frac{2q}{q-1} - N \right) \right)^{1/(q-1)}
\]
and make $c' \neq c$, we deduce $\lim \inf |x|^{2/(q-1)}u(x) \geq \ell$. Now suppose $\lim \sup |x|^{2/(q-1)}u(x) > \ell$.

There exist a sequence $x_n \to 0$ and $\ell' > \ell$ such that $\lim |x_n|^{2/(q-1)}u(x_n) = \ell'$. Set $v_n(x) = |x_n|^{2/(q-1)}u(|x_n|x)$; $v_n$ satisfies

$$\nabla v_n(x) + |x_n|^{2q/(q-1)}g(|x_n|^{1-2/(q-1)}v_n(x)) = 0 \quad \text{in } \mathbb{R}^N - \{0\}.$$

By compactness there exists a subsequence $n_k$ and a function $v$ such that $v_{n_k}(x)$ converges to $v(x)$ uniformly on every compact of $\mathbb{R}^N - \{0\}$ and $v$ satisfies

$$-\Delta v + cv^q = 0 \quad \text{in } \mathbb{R}^N - \{0\}.$$

From Lemma 1.4 of [16] there exist two constants $K > 0$ and $\tau > 0$ such that the following inequality holds for any $R > 0$ and any $0 < |x| < R$:

$$v(x) \leq \ell |x|^{-2/(q-1)}(1 + K \left(\frac{|x|}{R}\right) \tau).$$

Making $R \to +\infty$ we deduce $v(x) \leq \ell |x|^{-2/(q-1)}$ for $x \neq 0$. For any $\varepsilon > 0$ there exists $n_{k_0}$ such that for $k > k_0$ and $|x| = 1$

$$|x|^{2/(q-1)}u(|x|v(x)) < \varepsilon.$$

If we take $x = \frac{x_n}{|x_n|}$ and make $n_k \to +\infty$ we deduce $\ell' - \ell < \varepsilon$ which contradicts $\ell' > \ell$; so

$$\ell = \ell' = \lim_{x \to 0} |x|^{2/(q-1)}u(x).$$

**THEOREM 2.4.** Suppose $g$ is defined on $\mathbb{R}^+$ and satisfies for some $c > 0$ and some $1 < q < \frac{N+1}{N-1}$

$$i) \quad \lim_{r \to +\infty} (g(r) - cr^q) r^{-(N+1)(q-1)/2} = 0,$$

$$ii) \quad \lim_{r \to 0^+} g(r) / r < +\infty.$$

If $u \in C^2(\mathbb{R}^N - \{0\})$ is a non negative solution of (2.1) in $D'(\mathbb{R}^N - \{0\})$, it is spherically symmetric.

Before proving that result we introduce the generalised Sommerfeld exponent $\tau$ (see
and \( x^2 - \left( \frac{2^{q+1}}{q-1} - N \right) x - 2 \left( \frac{2^q}{q-1} - N \right) = 0. \)

We have the following result which will also be used in Section 3,

**PROPOSITION 2.2.** Suppose \( q \) and \( p \) are two real numbers such that \( 1 < q < \frac{N}{N-2} \), \( 0 < p < \frac{q-1}{2} \) and \( g \) is defined on \( \mathbb{R}^+ \) and satisfies for some \( c > 0 \)

\[
(2.38) \quad \lim_{r \to +\infty} (g(r) - cr^q)r^{p-q} = 0.
\]

If \( u \in C^2(\mathbb{R}^N - \{0\}) \) is a positive solution of (2.1) in \( D'(\mathbb{R}^N - \{0\}) \) satisfying \( \lim_{|x| \to 0} \frac{|x|^{2/(q-1)}u(x)}{u(x) = \ell} \) (defined in (2.31), then for any \( \epsilon > 0 \) there exist \( p > 0 \) and \( k > 0 \) such that

\[
(2.39) \quad |\ell - |x|^{2/(q-1)}u(x)| \leq \epsilon |x|^{2p/(q-1)} + k |x|^{2/(q-1)},
\]

for any \( 0 < |x| < \rho \).

**Proof.** First we shall prove that for any \( \epsilon > 0 \), there exist \( \rho > 0 \) and \( k > 0 \) such that the following inequality holds for any \( 0 < |x| < \rho : \)

\[
(2.40) \quad u(x) \leq \ell |x|^{-2/(q-1)}(1 + \epsilon/\ell |x|^{2p/(q-1)}) + k.
\]

**Step 1.** We set \( \Psi(x) = \ell |x|^{-2/(q-1)} \) and we define as in the Proposition A.4 of [5] \( \phi(x) = \max(\Psi(x), u(x)) \). From Kato's inequality we get

\[
\Delta\phi = \Delta \frac{1}{2}(\Psi + u) + \frac{1}{2}(\Delta\Psi + \Delta u) + \frac{1}{2} \text{sign}(\Psi - u)\Delta(\Psi - u).
\]

As \( \Delta\Psi = c\Psi^q \), we get \( \Delta\phi \geq \frac{1}{2} (c\Psi^q + g(u) + \text{sign}(\Psi - u)(c\Psi^q - g(u))) \), or

\[
(2.41) \quad \Delta\phi \geq \min(c\phi^q, g(\phi)).
\]

Moreover there exists \( D > 0 \) such that

\[
(2.42) \quad g(\phi(x)) \geq c(\phi(x))^q - D(\phi(x))^{q-p},
\]

for \( 0 < |x| < 1 \).
Step 2. Set \( w(r,a) = r^{2/(q-1)} \phi(r,a) \) and \( \overline{w}(r) \) its average on \( S^{N-1} \). We have \( w(r,a) \geq \overline{w} \) and

\[
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \left( N - \frac{q+3}{q-1} \right) \frac{\partial w}{\partial r} + \frac{1}{r^2} \Delta_{S^{N-1}} w \geq \frac{c}{r^2} \left( w^{q-1} - \epsilon q^{-1} \right) - \frac{D r^{2p/(q-1)-2}}{q^{-p}} \overline{w}^{-p}.
\]

As \( w \) is bounded on \( \{ x \mid 0 \leq |x| < 1 \} \) we deduce in averaging (2.43) that

\[
\frac{d^2 \overline{w}}{dr^2} + \frac{1}{r} \left( N - \frac{q+3}{q-1} \right) \frac{dw}{dr} \geq -D_1 r^{2p/(q-1)-2},
\]

for \( 0 < r < 1 \), \( D_1 \) being a constant. Now we set \( s = \frac{r^{2(q+1)/(q-1)-N}}{2(q+1)/(q-1)} \) and \( \overline{v}(s) = \overline{w}(r) \). We have

\[
\frac{d^2 v}{ds^2} + D_2 s^{-2} \geq 0,
\]

on \( \{ s \mid 0 < s < \frac{q-1}{2(q+1)-N(q-1)} \} \) where \( D_2 \) is non negative and \( \theta = \frac{2p}{2(q+1)-N(q-1)} \). Hence the function \( s \mapsto \overline{v}(s) + \frac{D_2}{\theta(q-1)} \frac{s^2}{s^2} \) is convex (or \( s \mapsto v(s) + D_3(s \log s - s) \) if \( \theta = 1 \)) which implies that \( \overline{v}(s) \leq \overline{v}(0) + s D_3 \), where \( D_3 \) depends on \( \theta, q, N \) and \( \overline{u}(1) \); so there exists a constant \( A \) such that

\[
\overline{u}(r) \leq \overline{v}(r) = r^{2/(q-1)} + A r^{2q/(q-1)-N},
\]

for \( 0 < r < 1 \). Moreover that relation is true for any \( 0 < q < p \).

Step 3. We set \( \omega(r) = r^{2/(q-1)}(1 + \epsilon / \sqrt{r^{2p/(q-1)}}) \) and we claim that we can find \( \sigma \) such that

\[
- \Delta \omega + g(\omega) \geq 0,
\]

on \( \{ x \mid 0 \leq |x| < \sigma \} \). For a given \( \delta > 0 \), there exists \( \sigma' > 0 \) such that \( g(\omega(r)) \geq c(\omega(r))^{q-\delta(\omega(r))q^{-p}} \) for \( 0 < r < \sigma' \). We get

\[
\frac{d\omega}{dr} = -\frac{2}{q-1} r^{(q+1)/(q-1)} + \epsilon \frac{p-1}{q-1} r^{(2p-q-1)/(q-1)},
\]

\[
\frac{d^2 \omega}{dr^2} = \frac{2(q+1)}{(q-1)^2} r^{2q/(q-1)} + \epsilon \frac{2(p-1)(2p-q-1)}{(q-1)^2} r^{2(p-q)/(q-1)},
\]

\[
c q^q = c q^q r^{-2q/(q-1)}(1 + \epsilon / \sqrt{r^{2p/(q-1)}}) \geq c q^q r^{-2q/(q-1)}(1 + \epsilon / \sqrt{r^{2p/(q-1)}}),
\]
So we get
\[ -\Delta \omega + c\omega^q - \delta \omega^{q-p} \geq - \left( \varepsilon \frac{2(p-1)(2p-q-1)}{(q-1)^2} + 2(N-1) \varepsilon \frac{p-1}{q-1} \right)r^{2(p-q)/(q-1)} \]
\[ + c \varepsilon r^{q-1} q r^{2(p-q)/(q-1)} - \delta r^{q-p}(1 + \varepsilon r^{2p/(q-1)})q^{-p} r^{2(p-q)/(q-1)}. \]

And the right hand side of that inequality can be written as
\[ \left\{ \varepsilon \left[ 2 \left( \frac{2q}{q-1} - N \right) + \left( \frac{q+1}{q-1} - N \right) \left( \frac{2p}{q-1} - \left( \frac{2p}{q-1} \right)^2 \right) \right] - \delta \varepsilon r^{q-p}(1 + \varepsilon r^{2p/(q-1)})q^{-p} \right\} r^{2(p-q)/(q-1)}, \]

and, as \( 0 < \frac{2p}{q-1} < r \), the coefficient of \( \varepsilon \) is positive. So we first choose \( \sigma_1 \) such that
\[ (1 + \varepsilon r^{2p/(q-1)})q^{-p} \leq 2 \text{ for } 0 \leq r \leq \sigma_1. \]
We then choose \( \delta \) such that
\[ 2 \delta \varepsilon r^{q-p} \leq \varepsilon \left[ 2 \left( \frac{2q}{q-1} - N \right) + \left( \frac{q+1}{q-1} - N \right) \left( \frac{2p}{q-1} - \left( \frac{2p}{q-1} \right)^2 \right) \right], \]
and then we take \( \sigma = \min(\sigma_1, \sigma') \), which implies (2.47).

Step 4. We follow now the end of the proof of Proposition A.4 of [5]. Set
\[ k = \max x \phi(x). \] As \( g \) is nondecreasing we have
\[ -\Delta (\omega + k) + g(\omega + k) \geq 0, \quad (2.48) \]
on \( \{ x | 0 < |x| < \sigma \} \). Let \( \xi_n \) be a sequence of smooth functions such that
\[ \xi_n(x) = \begin{cases} 1 & \text{for } |x| \geq \frac{1}{n}, \\ \frac{1}{2n^2} & \text{for } |x| \leq \frac{1}{2n}, \end{cases} \]

Let \( \theta \) be a smooth nondecreasing function vanishing on \( (-\infty, 0] \), strictly positive on \( (0, +\infty) \) and such that \( \theta = 1 \) on \( [1, +\infty) \) and set \( j(t) = \int_0^t \theta(s)ds \). We have from Steps 1 and 3, in setting
\[ \Omega = \{ x | 0 < |x| < \sigma \}, \]
So we get: \( \int_{\Omega} \nabla(u - \omega - k) \cdot \nabla \xi_n \theta(u - \omega - k) \, dx + \int_{\Omega} |\nabla(u - \omega - k)|^2 \xi_n \theta'(u - \omega - k) \, dx \ldots \)

\[ \ldots + \int_{\Omega} (g(u) - g(\omega + k)) \xi_n \theta(u - \omega - k) \, dx \leq 0. \]

As \( \nabla(u - \omega - k) \cdot \nabla \xi_n \theta(u - \omega - k) = \nabla_j(u - \omega - k) \), so we get

\[
\int_{\Omega} |\nabla(u - \omega - k)|^2 \xi_n \theta'(u - \omega - k) \, dx + \int_{\Omega} (g(u) - g(\omega + k)) \xi_n \theta'(u - \omega - k) \, dx \leq \ldots
\]

\[ \ldots \int_{\Omega} j(u - \omega - k) \Delta \xi_n \, dx \leq Kn^2 \int_{\Omega} \frac{1}{2n} \leq |x| \leq \frac{1}{n} \]

But \( j(u - \omega - k) \leq j(u - \omega) \leq j(\phi - \ell \cdot x \cdot x^{2/q - 2/(q-1)}) \leq \phi - \ell \cdot x \cdot x^{2/q - 2/(q-1)} \) and from Step 2, \( 0 \leq \phi(r) - \ell \cdot r^{2/(q - 1)} \leq A r^{2q/(q - 1) - N} \) for \( 0 < r < 1 \).

So we get: \( Kn^2 \int_{\Omega} \frac{1}{2n} \leq |x| \leq \frac{1}{n} \quad j(u - \omega - k) \leq KA \frac{q + 1}{2q} n^{-2/(q - 1)} \). As \( n \to +\infty \) we get by Fatou's Lemma

\[
\int_{\Omega} |\nabla(u - \omega - k)|^2 \theta'(u - \omega - k) \, dx + \int_{\Omega} (g(u) - g(\omega + k)) \theta'(u - \omega - k) \, dx \leq 0,
\]

which implies that both terms are 0. If we make \( \theta(r) \to r^+ \) we deduce that \( \nabla(u - \omega - k)^+ = 0 \) a.e.

But \( (u - \omega - k)^+ \) vanishes on \( \partial \Omega \) so it is identically 0 and we have

\[
u(x) \leq \ell \cdot |x|^{2/(q - 1)} (1 + e/\ell \cdot |x|^{2p/(q - 1)}) + k,
\]

for \( 0 < |x| < \alpha \), which is (2.40).

For proving the reverse inequality

\[ u(x) \geq \ell \cdot |x|^{2/(q - 1)} (1 - \ell/\ell \cdot |x|^{2p/(q - 1)}) - k, \]

we do the same in introducing \( \phi_1(x) = \min(\psi(x), u(x)) \) which satisfies

\[
\Delta \phi_1 \leq \text{Mas}(c \phi_1^q, g(\phi_1)).
\]

With the same change of variable we obtain by concavity

\[
\bar{u}(r) \geq \ell \cdot r^{-2/(q - 1)} - A r^{2q/(q - 1) - N}.
\]
for $0 < r < 1$. We then construct a subsolution $\omega_1(r) = \ell r^{-2/(q-1)}(1 - e/\ell r^{2p(q-1)})$ for the equation (2.1) (the only slight change being in the estimation of $(\omega_1(r))^q$ where we have: $(\omega_1(r))^q \leq q^{q - 2q/(q-1)}(1 - q' e/\ell r^{2p/(q-1)})$ where $1 < q' < q$ but $q - q'$ can be as small as we want in restricting $r$). We end the proof as in the Step 4.

**Proof of the Theorem 2.4.** From (2.36) and Lemma 2.1 any solution of (2.1) is bounded at infinity. So, if $\lim_{x \to 0} |x|^{N-2} u(x) = \gamma$, we deduce from Theorem 2.1 that $u$ is spherically symmetric.

Now suppose that $\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell$. We have $\frac{2p}{q-1} = \frac{2q}{q-1} - (N+1) > 0$, and

\[
\frac{q+1}{q-1} - N \leq \left( \frac{2}{q-1} - N \right) \left( \frac{q+1}{q-1} - N \right) - 2 \left( \frac{2q}{q-1} - N \right) < 0,
\]

so we have (2.39) and

\[
r^{2/(q-1)} \| u(r, \cdot) - \bar{u}(r) \|_{L^\infty(S^{N-1})} \leq 2e^{(q+1)/(q-1)-N} + 2k r^{2/(q-1)}, \text{ for } 0 < r < \rho.
\]

So we deduce

\[
\limsup_{r \to 0} r^{N-1} \| u(r, \cdot) - \bar{u}(r) \|_{L^\infty(S^{N-1})} \leq 2e.
\]

Making $e \to 0$ we obtain

\[
\lim_{r \to 0} r^{N-1} \| u(r, \cdot) - \bar{u}(r) \|_{L^\infty(S^{N-1})} = 0
\]

and then we conclude with Theorem 2.1.

**Remark 2.2.** The following nonlinear Liouville-Hadamard type result is a consequence of Theorem 2.1: a $C^2$ solution $u$ of (2.1) in $\mathbb{R}^N$ such that $u(x) = o(\sqrt{|x|}(\sqrt{|x|} \to \infty))$ is a constant.

### 3. UNIQUENESS OF SOLUTIONS

In that part we shall still suppose that $g$ is a continuous nondecreasing function defined on $\mathbb{R}$ (Holder continuous as we want to deal with strong solutions) and we consider the equation

\[
-\Delta u + g(u) = 0,
\]

taken into $D'(\mathbb{R}^N - \{0\})$ and we investigate under what assumption on $g$ is a (possibly singular) solution of (3.1) uniquely determined. If $u$ is a solution of (3.1) and $\theta \in O(n)$, $u \circ \theta$ is also a solution of (3.1) ; so if $u$ is uniquely determined, $u$ must be spherically symmetric. The following easy-to-prove result is the key-stone of this section.

**Theorem 3.1.** Suppose $u_1$ and $u_2$ belonging to $C^2(\mathbb{R}^N - \{0\})$ are two solutions of (3.1) in $D'(\mathbb{R}^N - \{0\})$. If they satisfy
then $u_1 = u_2$.

**Proof.** We make the change of variable

$$s = \frac{r^{N-2}}{N-2}, \quad u_i(r, \omega) = r^{2-N} v_i(r, \omega), \quad i = 1, 2.$$  

The function $v_i$ satisfies

$$s^2 \frac{\partial^2 v_i}{\partial \omega^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} v_i = (N-2)(4-N)/(N-2) s^{N/(N-2)} g\left(\frac{v_i}{s^{N-2}}\right),$$

in $(0, +\infty) \times S^{N-1}$. If we set $w = v_1 - v_2$, then we get $s^2 \int_{S^{N-1}} \left(\frac{\partial^2 \omega}{\partial s^2}\right) \omega \, d\omega \geq 0$ which implies that the function $s \mapsto \|w(s, \cdot)\|_{L^2(S^{N-1})}$ is convex. As it vanishes at $0$ and satisfies

$$\lim_{s \to +\infty} \frac{1}{s} \|w(s, \cdot)\|_{L^2(S^{N-1})} = 0,$$

it is identically $0$.

As a consequence we have the following

**COROLLARY 3.1.** Suppose $g$ vanishes only at $0$ and satisfies

$$i) \quad \lim_{r \to +\infty} \inf \frac{g(r)}{r^{N/(N-2)}} > 0,$$

$$\lim_{r \to -\infty} \sup \frac{g(r)}{|r|^{N/(N-2)}} < 0.$$  

Then the only $u \in C^2(\mathbb{R}^N \setminus \{0\})$ satisfying (3.1) in $D'(\mathbb{R}^N \setminus \{0\})$ is the zero function.

**Proof.** From a result of Brezis and Veron [6] the function $u$ can be extended to whole $\mathbb{R}^N$ into a $C^2$ function. Moreover from Lemma 2.1 and Theorem 1.1, $\|x\|^{N-2} u(x)$ admits a limit when $|x|$ goes to $+\infty$. Applying Theorem 3.1 to $u$ and $0$, we get $u = 0$.

**Remark 3.1.** The assumption $g^{-1}(0) = 0$ can be cancelled if we consider the solutions of (3.1).
vanishing in some sense at infinity, for example such that \( \lim_{r \to +\infty} ||u(r,.)||_{L^2(S^{N-1})} = 0 \). Some other conditions are discussed in [1].

When the growth of \( g \) at infinity is comparable to some power \( q \) with \( 1 < q < \frac{N}{N-2} \), there exist two types of isotropic singularities at 0. We deduce from Proposition 2.1 and Theorems 1.1 and 3.1.

**COROLLARY 3.2.** Suppose \( g \) vanishes only at 0 and satisfies

\[
(3.6) \quad \lim_{|r| \to +\infty} |g(r)|/ |r|^q = c,
\]

for some \( c > 0 \) and \( 1 < q < \frac{N}{N-2} \). If \( u \in C^2(\mathbb{R}^N \setminus \{0\}) \) is a solution of (3.1) in \( D(\mathbb{R}^N \setminus \{0\}) \) such that \( |x|^{N-2} u(x) \) remains bounded in some neighbourhood of 0, then \( u \) is uniquely determined by the value of \( \gamma = \lim_{x \to 0} |x|^{N-2} u(x) \).

In fact in Corollary 3.2, we have not only the uniqueness with respect to the singularity at 0, but also the existence, as a consequence of

**LEMMA 3.1.** Suppose \( g \) vanishes at 0 and satisfies (3.6) for some \( c > 0 \) and some \( 1 < q < \frac{N}{N-2} \). Then for any \( \gamma \) there exists a unique \( u \in C^2(0, +\infty) \) satisfying

\[
(3.7) \quad \begin{cases}
\frac{d^2u}{dr^2} + \frac{N-1}{r} \frac{du}{dr} - g(u) = 0 & \text{on } (0, +\infty), \\
\lim_{r \to 0} r^{N-2} u(r) = \gamma, & \lim_{r \to +\infty} u(r) = 0.
\end{cases}
\]

**Proof.** If we set \( s = \frac{r^{N-2}}{N-2} \) and \( u(r) = r^{N-2} v(s) \), then (3.7) is equivalent to

\[
(3.8) \quad \begin{cases}
s^2 \frac{d^2v}{ds^2} - (N-2)(4-N)/(N-2) s^{N/(N-2)} g \left( \frac{v}{(N-2)s} \right) = 0 & \text{on } (0, +\infty), \\
\lim_{s \to 0} v(s) = \gamma, & \lim_{s \to +\infty} \frac{1}{s} v(s) = 0.
\end{cases}
\]

The uniqueness comes from the same argument of convexity as the one of Lemma 1.3. For the
existence, we consider for any $\epsilon > 0$ the solution $v_\epsilon$ (coming also from the Lemma 1.3) of the equation

$$\begin{cases}
(s+\epsilon)^2 \frac{d^2 v_\epsilon}{ds^2} - \frac{(N-2)(4-N)/(N-2)(s+\epsilon)^N/(N-2)}{g(v_\epsilon)} = 0 \text{ on } (0, + \infty),
\end{cases}$$

(3.9)

$$v_\epsilon(0) = \gamma, \quad \lim_{s \to + \infty} \frac{1}{s} v_\epsilon(s) = 0.$$  

As the function $s \mapsto |v_\epsilon(s)|$ is convex it is nonincreasing. From (3.6) we have

$$|g(r)| \leq c |r|^q + d,$$

for any $r$ and some $c, d > 0$; so we have for any $0 < s < T$

$$\frac{dv_\epsilon}{ds}(s) < \frac{dv_\epsilon}{ds}(T) + K \int_s^T ((s+\epsilon)^{N/(N-2)} - q-2 |v_\epsilon|^q + (s+\epsilon)^{N/(N-2)-2}) d\sigma.$$  

But as $|v_\epsilon| \leq \gamma$, $|g(v_\epsilon)|$ is bounded and it is the same with $\frac{d^2 v_\epsilon}{ds^2}$ and $\frac{dv_\epsilon}{ds}$ on any interval $(a, + \infty), a > 0$. Integrating again (3.11) yields

$$|v_\epsilon(t) - v_\epsilon(s)| \leq A_1(t-s) + A_2((t+\epsilon)^{N/(N-2)} - (s+\epsilon)^{N/(N-2)}) + ...$$

$$... A_3((t+\epsilon)^{N/(N-2)} - (s+\epsilon)^{N/(N-2)}),$$

for $0 < s < t < T$. As the functions $t \mapsto t^{N/(N-2)-q}$ and $t \mapsto t^{N/(N-2)}$ are uniformly continuous on $[0, T+1]$, the set of functions $(v_\epsilon | \epsilon \in (0,1])$ is equicontinuous on $[0, T]$. Using Arzela-Ascoli theorem and the diagonal process, there exists a continuous function $v$ on $[0, + \infty)$ and a sequence $\epsilon_n \to 0$ such that $v_\epsilon$ converges to $v$ on $[0, T]$ for any $T > 0$. The function $v$ satisfies the equation (3.8), is nonincreasing and $v(0) = \gamma$.

Remark 3.1. If we define $\tilde{u}$ on $\mathbb{R}^N \setminus \{0\}$ by $\tilde{u}(x) = u(1/|x|)$, where $u$ satisfies (3.7), one can see that $\tilde{u}$ is a solution of

$$-\Delta u + g(u) = (N-2) |S^{N-1}| \gamma \delta_0,$$

in $D'(\mathbb{R}^N)$, unique if $g$ vanishes only at 0.

THEOREM 3.2. Suppose $g$ vanishes only at 0 and satisfies for some $c > 0$ and some $1 < q < \frac{N}{N-2}$
Then there exists only one \( u \in \{0, \infty\} \) solution of (3.7) such that \( \lim_{x \to 0} |x|^{2/(q-1)} u(x) = \xi \).

**Proof.** *Existence:* For any \( \gamma > 0 \) set \( u_\gamma \) the solution of (3.7) on \((0, +\infty)\). From the Lemma 2.1, there exist \( A \) and \( B > 0 \) such that

\[
N-2 \quad \text{for any } r > 0 \text{ and } \gamma > 0.
\]

Setting \( s = -r^2 \) and \( u_r(r) = r^{2-N} v_\gamma(s) \), the function \( v_\gamma \) satisfies the equation (3.8) with initial data \( \gamma \) and vanishes at \(+\infty\). From the uniqueness, for any \( s > 0 \), the function \( \gamma \mapsto v_\gamma(s) \) is nondecreasing and as

\[
0 \leq v_\gamma(s) \leq \frac{A}{s^{2/q-1}} + B,
\]

it converges as \( \gamma \to +\infty \) to some function \( v_\infty \) satisfying (3.8). Setting \( u_\infty(r) = r^{2-N} v_\infty(s) \) the function \( u_\infty \) satisfies (3.7) and \( \lim_{r \to 0} r^{N-2} u_\infty(r) = +\infty \). If \( u(x) = u_\infty(|x|) \), \( u \) satisfies (3.1) and, from the Proposition 2.1, \( \lim_{x \to 0} |x|^{2/(q-1)} u(x) = \xi \).

*Uniqueness:* Set \( u_1 \) and \( u_2 \) two solutions of (3.1) such that \( \lim_{x \to 0} |x|^{2/(q-1)} u_i(x) = \xi \) for \( i = 1, 2 \). We apply the Proposition 2.2 with \( p = q - \frac{N}{2} (q-1) \) and we get from (2.39)

\[
|x|^{2/(q-1)} (u_1(x) - u_2(x)) \leq 2\xi|x|^{2/(q-1)} + 2^{2-N} + 2^{N-2} + 2^{2/(q-1)}
\]

which implies \( \lim_{x \to 0} |x|^{N-2} |u_1(x) - u_2(x)| = 0 \). As \( u_1 \) and \( u_2 \) vanishes at infinity we deduce \( u_1 = u_2 \) from the Theorem 3.1.

**Remark 3.2.** When \( g(r) = c |r|^{q-1} r \) the solution \( u \) of Theorem 3.2 is

\[
u(x) = \xi |x|^{-2/(q-1)}
\]

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