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THE SEGRE IMBEDDING AND ITS CONVERSE

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Résumé : En utilisant les coordonnées homogènes des espaces projectifs complexes, C. Segre a construit, en 1891, un plongement kaehlerien de $\mathbb{C}P^{\alpha_1} \times \ldots \times \mathbb{C}P^{\alpha_n}$ dans $\mathbb{C}P^{N(\alpha_1,\ldots,\alpha_n)}$, où $N(\alpha_1,\ldots,\alpha_n) = (1+\alpha_1)\ldots(1+\alpha_n) - 1$. Dans cette Note, nous considérons le problème inverse, et nous obtenons le résultat suivant :

Si $M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ est une sous-variété produit de $\mathbb{C}P^m$, et est le produit de n variétés kaehleriennes, alors $m \geqslant N(\alpha_i,\ldots,\alpha_n)$. De plus, si $m = N(\alpha_1,\ldots,\alpha_n)$, alors $M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ est un ouvert de $\mathbb{C}P^{\alpha_1} \times \ldots \times \mathbb{C}P^{\alpha_n}$ et l'immersion considérée est le plongement de Segre.

Summary : Using homogeneous coordinates of complex projective spaces, C. Segre constructed in 1891 a Kaehler imbedding of $\mathbb{C}P^{\alpha_1} \times \ldots \times \mathbb{C}P^{\alpha_n}$ in $\mathbb{C}P^{N(\alpha_1,\ldots,\alpha_n)}$ where $N(\alpha_1,\ldots,\alpha_n) = (1+\alpha_1)\ldots(1+\alpha_n) - 1$. In this paper, we consider the converse problem to the Segre imbedding and obtain the following result : If $M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ is a Kaehler submanifold of $\mathbb{C}P^m$ which is the product of n Kaehler manifolds, then $m \geqslant N(\alpha_1,\ldots,\alpha_n)$. And if $m = N(\alpha_1,\ldots,\alpha_n)$, then $M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ is an open portion of $\mathbb{C}P^{\alpha_1} \times \ldots \times \mathbb{C}P^{\alpha_n}$ and the immersion is obtained by the Segre imbedding.

0. - INTRODUCTION

Let $\mathbb{C}P^n$ be a (complex) n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. Using homogeneous coordinates, C. Segre [4] constructed in 1891 an imbedding from the product variety $\mathbb{C}P^\alpha \times \mathbb{C}P^\beta$ into $\mathbb{C}P^{N(\alpha,\beta)}$, $N(\alpha,\beta) = \alpha + \beta + \alpha\beta$, as follows

(0.1)
$$S_{\alpha,\beta} : \mathbb{C}P^\alpha \times \mathbb{C}P^\beta \to \mathbb{C}P^{N(\alpha,\beta)}$$
$$(x_i) \cdot (y_a) \mapsto (x_i y_a).$$

It is well-known that $S_{\alpha,\beta}$ is a Kaehler imbedding which is known as the Segre imbedding from $\mathbb{C}P^\alpha \times \mathbb{C}P$ into $\mathbb{C}P^{N(\alpha,\beta)}$.

In 1981, Chen [2] had considered the «converse» problem to the Segre imbedding and obtained the following.

THEOREM A. *If* $\mathbb{C}P^m$ *admits a Kaehler submanifold* $M_1^\alpha \times M_2^\beta$ *which is the product of two Kaehler manifolds of (complex) dimension* α *and* β, *respectively, then* $m \geqslant N(\alpha,\beta)$. *In particular, if* $m = N(\alpha,\beta)$, *then* (a) $M_1^\alpha \times M_2^\beta$ *is an open portion of* $\mathbb{C}P^\alpha \times \mathbb{C}P^\beta$ *and* (b) *the immersion is obtained by the Segre imbedding* $S_{\alpha,\beta}$ *up to holomorphic and isometric transformations of* $\mathbb{C}P^m$.

For the product variety $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$, using homogeneous coordinates, C. Segre defined the following imbedding

(0.2)
$$S_{\alpha_1 \dots \alpha_n} : \mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n} \to \mathbb{C}P^{N(\alpha_1,\dots,\alpha_n)}$$
$$(x_{i_1}) \dots (x_{i_n}) \mapsto (x_{i_1} \dots x_{i_n})$$

where

(0.3)
$$N(\alpha_1,\dots,\alpha_n) = s_1 + s_2 + \dots + s_n$$

where $s_1 = \sum_{i=1}^{n} \alpha_i$, $s_2 = \sum_{i<j} \alpha_i\alpha_j, \dots, s_n = \alpha_1 \dots \alpha_n$. It is clear that $S_{\alpha_1 \dots \alpha_n}$ is also a Kaehler imbedding. We call it the Segre imbedding from $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ into $\mathbb{C}P^{N(\alpha_1,\dots,\alpha_n)}$.

In view of Theorem A, it is natural and interesting to consider the following two problems :

Problem 1. Is $N(\alpha_1,...,\alpha_n)$ the smallest possible dimension of a complex projective space to admit a Kaehler submanifold $M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$ which is the product of n Kaehler manifolds ?

Problem 2. If $N(\alpha_1,...,\alpha_n)$ is the smallest possible dimension of a complex projective space to admit such a product submanifold, does this product submanifold have to the obtained from the Segre imbedding ?

In this paper we will solve these two problems completely. More precisely, we will obtain the following.

THEOREM 1. *If* $\mathbb{C}P^m$ *admits a Kaehler submanifold* $M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$ *which is the product of n Kaehler manifolds* $M_1^{\alpha_1},...,M_n^{\alpha_n}$ *of complex dimensions* $\alpha_1,...,\alpha_n$, *respectively, then we have*

(1) $m \geqslant N(\alpha_1,...,\alpha_n)$,

(2) *if* $m = N(\alpha_1,...,\alpha_n)$, *then*

(2.1) $M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$ *is an open portion of* $\mathbb{C}P^{\alpha_1} \times ... \times \mathbb{C}P^{\alpha_n}$, *and*

(2.2) *the immersion is given by the Segre imbedding* $S_{\alpha_1 \cdots \alpha_n}$ *up to holomorphic and isometric transformations of* $\mathbb{C}P^m$.

Let h denote the second fundamental form of the immersion and $\bar{\nabla}^p h$ the p-th covariant derivative of h. Then we also have the following best possible inequalities.

THEOREM 2. *Let* $M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$ *be a Kaehler submanifold of* $\mathbb{C}P^m$. *Then we have*

$$\|\bar{\nabla}^{\ell-2} h\|^2 \geqslant \ell!\, 2^\ell \sum_{i_1 < ... < i_\ell} \alpha_{i_1} \cdots \alpha_{i_\ell} \tag{0.4}$$

for $\ell = 2,3,...,n$. *The equality of* (0.4) *holds for some* ℓ *if and only if* $M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$ *is an open portion of* $\mathbb{C}P^{\alpha_1} \times ... \times \mathbb{C}P^{\alpha_n}$ *and the immersion is given by the Segre imbedding* $S_{\alpha_1 \cdots \alpha_n}$ *up to holomorphic and isometric transformations of* $\mathbb{C}P^m$. *Moreover, in this case, the equality of* (0.4) *holds for all* ℓ, $\ell = 2,3,...,n$.

It seems to be interesting to point out that $N(\alpha_1,...,\alpha_n) = (1+\alpha_1) \cdots (1+\alpha_n) - 1$ is much biger than the dimension of M in general. For example, Theorem 1 shows that if $\mathbb{C}P^m$ contains a Kaehler submanifold M which is the product of twenty 3-dimensional Kaehler manifolds, then M is only 60-dimensional, however, $\mathbb{C}P^m$ is at least 1,099,511,627,776-dimensional !!! Moreover, if m is 1,099,511,627,776, M has to be obtained by the Segre imbedding !!

1. - BASIC FORMULAS

Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with Riemannian metric $<\,,>$ and Riemannian connection ∇'. Denote by ∇ the induced connection on M. The second fundamental form h of the immersion is given by

$$(1.1) \qquad h(X,Y) = \nabla'_X Y - \nabla_X Y$$

where X and Y are vector fields tangent to M. For a vector field ξ normal to M and X tangent to M, we put

$$(1.2) \qquad \nabla'_X \xi = -A_\xi X + D_X \xi$$

where $-A_\xi X$ and $D_X\xi$ denote the tangential and normal components of $\nabla'_X\xi$, respectively. We have

$$(1.3) \qquad < h(X,Y),\xi > = < A_\xi X, Y >.$$

For the second fundamental form h, we define its first covariant derivative $\bar{\nabla}h$ to be a normal-bundle-valued tensor of type (0,3) given by

$$(1.4) \qquad (\bar{\nabla}h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

Let R', R and R^D denote the curvature tensors associated with ∇', ∇, and D, respectively. The equations of Gauss, Codazzi, and Ricci are then given respectively by

$$(1.5) \qquad R'(X,Y;Z,W) = R(X,Y;Z,W) - < h(X,W),h(Y,Z) > + < h(X,Z), h(Y,W) >,$$

$$(1.6) \qquad R'(X,Y;Z,\xi) = < (\bar{\nabla}h)(X,Y,Z) - (\bar{\nabla}h)(Y,X,Z),\xi >,$$

$$(1.7) \qquad R'(X,Y;\xi,\eta) = R^D(X,Y;\xi,\eta) - < [A_\xi,A_\eta]X,Y >$$

for vector fields X,Y,Z,W tangent to M and ξ,η normal to M.

If we define the p–th $(p \geqslant 1)$ covariant derivative of h by

$$(1.8) \qquad (\bar{\nabla}^p h)(X_1,X_2,\ldots,X_{p+2}) = D_{X_1}((\bar{\nabla}^{p-1}h)(X_2,\ldots,X_{p+2})) - \sum_{i=2}^{p+2} (\bar{\nabla}^{p-1}h)(X_2,\ldots,\nabla_{X_1}X_i,\ldots,X_{p+2}),$$

then $\overline{\nabla}^p h$ is a normal-bundle-valued tensor of type (0,p+2). Moreover, it can be proved that $\overline{\nabla}^p h$ satisfies

(1.9)
$$(\overline{\nabla}^p h)(X_1,X_2,X_3,\dots,X_{p+2}) - (\overline{\nabla}^p h)(X_2,X_1,X_3,\dots,X_{p+2})$$
$$= R^D(X_1,X_2)((\overline{\nabla}^{p-2}h)(X_3,\dots,X_{p+2}))$$
$$+\sum_{i=3}^{p+2}(\overline{\nabla}^{p-2}h)(X_3,\dots,R(X_1,X_2)X_i,\dots,X_{p+2}),\quad p \geqslant 2.$$

We put $\overline{\nabla}^0 h = h$.

Let $\widetilde{M}$ be a Kaehler manifold with the complex structure J and M be a complex submanifold of $\widetilde{M}$ with the induced Kaehler metric. Then we also have the following

(1.10)
$$h(JX,Y) = h(X,JY) = Jh(X,Y),$$

(1.11)
$$A_{J\xi} = JA_\xi,\quad JA_\xi = -A_\xi J,\quad \text{and}\quad D_X J\xi = JD_X\xi.$$

Let $\widetilde{R}$ denote the curvature tensor of $\mathbb{C}P^m$. Then it is well-known that $\widetilde{R}$ takes the following form :

(1.12)
$$\widetilde{R}(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$$
$$+ \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ.$$

In particular, if M is a complex submanifold of $\mathbb{C}P^m$, (1.6) and (1.12) imply

(1.13)
$$(\overline{\nabla}h)(X,Y,Z) = (\overline{\nabla}h)(Y,X,Z)$$
$$= (\overline{\nabla}h)(Z,X,Y).$$

In section 2, we also denote

$$(\overline{\nabla}h)(X,Y,Z) \text{ by } (\overline{\nabla}_X h)(Y,Z).$$

2. - PRODUCT OF 3 KAEHLER MANIFOLDS

Throughout this section we shall assume that $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ is the Riemannian product of three Kaehler manifolds M_1^α, M_2^β and M_3^γ of (complex) dimensions α, β, and γ, respectively. Let $x : M \to \mathbb{C}P^m$ be a Kaehler immersion from M into the m-dimensional complex projective space $\mathbb{C}P^m$.

In the following, we assume that $\{X_1,\dots,X_\alpha,JX_1,\dots,JX_\alpha\}$ (respectively, $\{Y_1,\dots,Y_\beta,JY_1,\dots,JY_\beta\}$ and $\{Z_1,\dots,Z_\gamma,JZ_1,\dots,JZ_\gamma\}$) forms an orthonormal basis for M_1^α (respectively, for M_2^β and for M_3^γ). We regard these vector fields as vector fields in M in a natural way.

We need the following results for the proof of the Main Lemma.

LEMMA 1. *Let* $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ *be a Kaehler submanifold of* $\mathbb{C}P^m$. *Then*

$$h(X_i,Y_a), Jh(X_i,Y_a), h(X_i,Z_r), Jh(X_i,Z_r), h(Y_a,Z_r), \tag{2.1}$$

$$Jh(Y_a,Z_r),\quad i = 1,\dots,\alpha\,;\ a = 1,\dots,\beta\,;\ r = 1,\dots,\gamma\,;$$

are orthonormal local vector fields in $T^\perp M$.

Proof. Let X and W be any unit vectors tangent to M_1^α and $M_2^\beta \times M_3^\gamma$, respectively. Then by (1.5) we have

$$\tilde{R}(X,W;W,X) = \langle h(X,W),h(X,W)\rangle - \langle h(X,X),h(W,W)\rangle, \tag{2.2}$$

$$\tilde{R}(X,JW;JW,X) = \langle h(X,JW),h(X,JW)\rangle - \langle h(X,X),h(JW,JW)\rangle. \tag{2.3}$$

Combining (1.10), (2.2), and (2.3) we find

$$\tilde{K}(X,W) + \tilde{K}(X,JW) = 2\,\| h(X,W)\|^2,$$

where $\tilde{K}$ denotes the sectional curvature of $\mathbb{C}P^m$. Since $X \wedge W$ is a totally real section, i.e., $\langle X,W\rangle = \langle X,JW\rangle = 0$, this implies that the length of $h(X,W)$ satisfies

$$\| h(X,W)\| = 1. \tag{2.4}$$

Therefore, by linearity, we obtain

$$\langle h(X_i,W),h(X_j,W)\rangle = 0,\quad i \neq j,\quad i,j = 1,\dots,2\alpha, \tag{2.5}$$

where we put $X_{\alpha+k} = JX_k$, $k = 1,\dots,\alpha$. Let W_1,W_2 be any two of the orthonormal vectors $Y_1,\dots,Y_\beta,Z_1,\dots,Z_\gamma$. Then we find from (2.5) that

$$<h(X_i,W_1),h(X_j,W_2)> + <h(X_i,W_2),h(X_j,W_1)> = 0. \tag{2.6}$$

On the other hand, because $R(X_i,X_j;W_1,W_2) = 0$, (1.5) and (1.12) imply

$$<h(X_i,W_1),h(X_j,W_2)> = <h(X_i,W_2),h(X_j,W_1)>. \tag{2.7}$$

Combining (2.6) and (2.7) we get $< h(X_i,W_1),h(X_j,W_2) > = 0$. From this, together with (2.4), we conclude that

$$h(X_i,Y_a),Jh(X_i,Y_a),h(X_i,Z_r),Jh(X_i,Z_r),$$

$$i = 1,\dots,\alpha;\ a = 1,\dots,\beta;\ r = 1,\dots,\gamma$$

are orthonormal. Applying the same argument to $h(Y,W)$ for unit vectors Y,W tangent to M_2^β and $M_1^\alpha \times M_3^\gamma$, respectively, we obtain Lemma 1.

LEMMA 2. *Let* $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ *be a Kaehler submanifold of* $\mathbb{C}P^m$ *and* X,Y, *and* Z *unit vector fields tangent to* M_1^α, M_2^β, *and* M_3^γ, *respectively. Then we have*

(2.8) $$<(\bar\nabla_X h)(Y,Z),h(Y,Z)> = 0$$

(2.8) *and*

$$<(\bar\nabla_X h)(Y,Z),Jh(Y,Z)> = 0.$$

Proof. The first equation of (2.8) follows from (1.4) and the identities $\nabla_X Y = \nabla_X Z = 0$ and $\| h(Y,Z) \| = 1$.

The second equation follows from the first equation and equation (1.14).

LEMMA 3. *Under the hypothesis of Lemma 2, we have*

$$A_{h(Y,Z)}X = 0. \tag{2.9}$$

Proof. Let U be any unit vector tangent to M, Lemma 1 implies

$$<A_{h(Y,Z)}X,U> = <h(Y,Z),h(X,U)> = 0.$$

This prove (2.9).

LEMMA 4. *Let* $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ *be a Kaehler submanifold of* $\mathbb{C}P^m$. *Then we have*

$$\| (\bar{\nabla}_X h)(Y,Z) \| = 1 \tag{2.10}$$

for any unit vector fields X, Y *and* Z *tangent to* M_1^α, M_2^β *and* M_3^γ *respectively.*

Proof. From the hypothesis, we have $R(X,JX)Y = R(X,JX)Z = 0$. Thus (1.9), (1.11) and (1.12) imply

$$\begin{aligned} &< (\bar{\nabla}_{JX}\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_X\bar{\nabla}_{JX} h)(Y,Z), Jh(Y,Z) > \\ &= < R^D(JX,X)(h(Y,Z)), Jh(Y,Z) > \\ &= < \tilde{R}(X,JX)Jh(Y,Z), h(Y,Z) > - < [A_{h(Y,Z)}, A_{Jh(Y,Z)}]JX, X > \\ &= 2 + 2 \| A_{h(Y,Z)}X \|^2 = 2 \end{aligned} \tag{2.11}$$

by virtue of Lemma 3. On the other hand, (1.8) and Lemma 2 give

$$\begin{aligned} &< (\bar{\nabla}_{JX}\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_X\bar{\nabla}_{JX} h)(Y,Z), Jh(Y,Z) > \\ &= < D_{JX}((\bar{\nabla}_X h)(Y,Z)) - D_X((\bar{\nabla}_{JX} h)(Y,Z)) - (\bar{\nabla}_{[JX,X]} h)(Y,Z), Jh(Y,Z) > \end{aligned} \tag{2.12}$$

Thus, by (2.11), (2.12) and Lemma 2 we find

$$\begin{aligned} 2 &= < - (\bar{\nabla}_X h)(Y,Z) + (\bar{\nabla}_{JX} h)(Y,Z), D_{JX}(Jh(Y,Z)) > \\ &= < - (\bar{\nabla}_X h)(Y,Z) + (\bar{\nabla}_{JX} h)(Y,Z), (\bar{\nabla}_{JX} h)(JY,Z) > \\ &= 2 \| (\bar{\nabla}_X h)(Y,Z) \|^2 \end{aligned}$$

by virtue of (1.13). From this we obtain (2.10).

In the following, we put

$$V = \mathrm{Span}\left\{ h(X,Y), h(X,Z), h(Y,Z) \mid X \in TM_1^\alpha,\ Y \in TM_2^\beta,\ Z \in TM_3^\gamma \right\}. \tag{2.13}$$

Then V is a complex $(\alpha\beta + \beta\gamma + \alpha\gamma)$-dimensional holomorphic subbundle of the normal bundle $T^{\perp}M$. Moreover, the vector fields given by (2.1) form an orthonormal local basis of V.

We need the following.

LEMMA 5. *Under the hypothesis of Lemma 2, we have*

$$(\bar{\nabla}_X h)(Y,Z) \text{ is perpendicular to } V. \tag{2.14}$$

Proof. Let Y and Y' (respectively, Z and Z'), be two unit vector fields tangent to M_2^{β} (respectively, M_3^{γ}). Then, for any unit vector field W tangent to M_1^{α}, Lemma 2 implies

$$<(\bar{\nabla}_X h)(Y,Z),h(Y',Z)> + <(\bar{\nabla}_X h)(Y',Z),h(Y,Z)> = 0. \tag{2.15}$$

On the other hand, from (1.4), (1.13), and Lemma 1, we get

$$\begin{aligned} <(\bar{\nabla}_X h)(Y,Z),h(Y',Z')> &= <(\bar{\nabla}_Y h)(X,Z),h(Y',Z')> \\ &= <D_Y h(X,Z),h(Y',Z')> \\ &= -<h(X,Z),D_Y h(Y',Z')> \\ &= -<h(X,Z),(\bar{\nabla}_Y h)(Y',Z')> \\ &= -<h(X,Z),(\bar{\nabla}_{Y'} h)(Y,Z')> \\ &= -<h(X,Z),D_{Y'} h(Y,Z')> \\ &= <(\bar{\nabla}_{Y'} h)(X,Z),h(Y,Z')>. \end{aligned} \tag{2.16}$$

Consequently, we have

$$\begin{aligned} <(\bar{\nabla}_X h)(Y,Z),h(Y',Z')> &= <(\bar{\nabla}_X h)(Y',Z),h(Y,Z')> \\ &= <(\bar{\nabla}_X h)(Y',Z'),h(Y,Z)>. \end{aligned} \tag{2.17}$$

Combining (2.15) and (2.17) we obtain

$$<(\bar{\nabla}_X h)(Y,Z),h(Y',Z)> = 0. \tag{2.18}$$

By linearity, (2.18) implies

$$< (\bar{\nabla}_X h)(Y,Z),h(Y',Z') > + < (\bar{\nabla}_X h)(Y,Z'),h(Y',Z) > = 0. \tag{2.19}$$

Therefore, (2.17) and (2.19) give

$$< (\bar{\nabla}_X h)(Y,Z),h(Y',Z') > = 0. \tag{2.20}$$

Since $(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) = (\bar{\nabla}_Z h)(X,Y)$, a similar argument yields $< (\bar{\nabla}_X h)(Y,Z),h(X',Y') > = < (\bar{\nabla}_X h)(Y,Z),h(X',Z') > = 0$ for any unit vectors X, X' tangent to M_1^α. These proves Lemma 5.

LEMMA 6. *Let* $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ *be a Kaehler submanifold of* $\mathbb{C}P^m$. *Then*

$$(\bar{\nabla}_{X_i} h)(Y_a,Z_r), \quad i = 1,\dots,2\alpha, \quad a = 1,\dots,\beta\,; \quad r = 1,\dots,\gamma \tag{2.21}$$

are orthonormal local vector fields in $T^\perp M$.

Proof. From (1.9), (1.12), Lemmas 1 and 3 we have

$$\begin{aligned} &< (\bar{\nabla}_{X_i}\bar{\nabla}_{X_j} h)(Y_a,Z_r) - (\bar{\nabla}_{X_j}\bar{\nabla}_{X_i} h)(Y_a,Z_r),h(Y_b,Z_t) > \\ &= < R^D(X_i,X_j)(h(Y_a,Z_r)),h(Y_b,Z_t) > \\ &= \tilde{R}(X_i,X_j;h(Y_a,Z_r),h(Y_b,Z_t)) \\ &\quad + < [A_{h(Y_a,Z_r)},A_{h(Y_b,Z_t)}]X_i,X_j > \\ &= 0 \end{aligned} \tag{2.22}$$

for $i,j = 1,\dots,2\alpha\,; \quad a,b = 1,\dots,\beta\,; \quad r,t = 1,\dots,\gamma$.

On the other hand, (1.8) and Lemma 5 imply

$$\begin{aligned} &< (\bar{\nabla}_{X_i}\bar{\nabla}_{X_j} h)(Y_a,Z_r) - (\bar{\nabla}_{X_j}\bar{\nabla}_{X_i} h)(Y_a,Z_r),h(Y_b,Z_t) > \\ &= < D_{X_i}((\bar{\nabla}_{X_j} h)(Y_a,Z_r)) - D_{X_j}((\bar{\nabla}_{X_i} h)(Y_a,Z_r)),h(Y_b,Z_t) > \\ &= < (\bar{\nabla}_{X_i} h)(Y_a,Z_r),(\bar{\nabla}_{X_j} h)(Y_b,Z_t) > \\ &\quad - < (\bar{\nabla}_{X_j} h)(Y_a,Z_r),(\bar{\nabla}_{X_i} h)(Y_b,Z_t) > . \end{aligned} \tag{2.23}$$

Hence, (2.22) and (2.23) give

$$(2.24)\qquad <(\bar\nabla_{X_i}h)(Y_a,Z_r),(\bar\nabla_{X_j}h)(Y_b,Z_t)> \\ =<(\bar\nabla_{X_j}h)(Y_a,Z_r),(\bar\nabla_{X_i}h)(Y_b,Z_t)>.$$

From Lemma 4 and linearity we also have

$$(2.25)\qquad <(\bar\nabla_{X_i}h)(Y,Z),(\bar\nabla_{X_j}h)(Y,Z)>=0,\quad i\neq j\ ;\quad i,j=1,\ldots,2\alpha.$$

Thus, by using linearity again, we find

$$(2.26)\qquad <(\bar\nabla_{X_i}h)(Y_a,Z),(\bar\nabla_{X_j}h)(Y_b,Z)> \\ +<(\bar\nabla_{X_i}h)(Y_b,Z),(\bar\nabla_{X_j}h)(Y_a,Z)>=0.$$

Combining (2.24) and (2.26) we obtain

$$(2.27)\qquad <(\bar\nabla_{X_i}h)(Y_a,Z),(\bar\nabla_{X_j}h)(Y_b,Z)>=0, \\ i\neq j\ ;\quad i,j=1,\ldots,2\alpha.$$

Thus, by applying linearity, (2.24), (2.27) and Lemma 4, we obtain Lemma 6.

Combining Lemmas 1, 5 and 6, we obtain the following.

LEMMA 7. *Let* $M=M_1^\alpha\times M_2^\beta\times M_3^\gamma$ *be a Kaehler submanifold of* $\mathbb{C}P^m$. *Then*

$$h(X_i,Y_a),\ Jh(X_i,Y_a),\ h(X_i,Z_r),\ Jh(X_i,Z_r),$$

$$h(Y_a,Z_r),\ Jh(Y_a,Z_r),\ (\bar\nabla h)(X_i,Y_a,Z_r),\ J(\bar\nabla h)(X_i,Y_a,Z_r)\ ;$$

$$i=1,\ldots,\alpha\ ;\quad a=1,\ldots,\beta\ ;\quad r=1,\ldots,\gamma\ ;$$

are orthonormal local vector fields in $T^\perp M$.

3. - MAIN LEMMA

Let $M = M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ be the Riemannian product of n Kaehler manifolds $M_1^{\alpha_1},\ldots,M_n^{\alpha_n}$ of complex dimensions $\alpha_1,\ldots,\alpha_n$ respectively. Assume that M is a Kaehler submanifold of $\mathbb{C}P^m$.

In the following, *we denote by* $X^i, Y^i, Z^i, \ldots$, etc. (with super-index i) *vector fields tangent to* $M_i^{\alpha_i}$. We shall also regard them as vector fields tangent to $M = M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ in a natural way. Moreover, *we assume that* $X_1^i,\ldots,X_{\alpha_i}^i$, $X_{\alpha_i+1}^i = JX_1^i,\ldots,X_{2\alpha_i}^i = JX_{\alpha_i}^i$ *form an orthonormal basis for* $M_i^{\alpha_i}$.

We need the following Main Lemma.

LEMMA 8. *Let* $M = M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ *be a Kaehler submanifold of* CP^m. *Then the following vectors*

$$(*)\qquad \begin{gathered} h(X_{a_1}^{i_1}, X_{a_2}^{i_2}),\ Jh(X_{a_1}^{i_1}, X_{a_2}^{i_2}),\ (\bar\nabla h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), \\ J(\bar\nabla h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}),\ldots,(\bar\nabla^{n-2}h)(X_{a_1}^{i_1},\ldots,X_{a_n}^{i_n}), \\ J(\bar\nabla^{n-2}h)(X_{a_1}^{i_1},\ldots,X_{a_n}^{i_n}), \\ i_1 < i_2 < \ldots < i_n\ ;\ 1 \leqslant j, i_1,\ldots,i_n \leqslant n\ ;\ 1 \leqslant a_j \leqslant \alpha_{i_j}\ ; \end{gathered}$$

are $2(s_2 + s_3 + \ldots + s_n)$ *orthonormal vectors normal to* M.

Proof. We will prove this lemma by induction.

If $n = 3$, this lemma is just Lemma 7. Now we assume that this lemma holds for $n \leqslant \ell - 1$, $\ell \geqslant 4$, we want to prove that it is also true for $n = \ell$.

Let $M = M_1^{\alpha_1} \times \ldots \times M_\ell^{\alpha_\ell}$ be a Kaehler submanifold of $\mathbb{C}P^m$. We put

$$\bar M_{\ell-1}^{\alpha_{\ell-1}} = M_{\ell-1}^{\alpha_{\ell-1}} \times M_\ell^{\alpha_\ell} \quad \text{and} \quad \bar M_j^{\alpha_j} = M_j^{\alpha_j},\ \ j = 1,\ldots,\ell-2.$$

We consider $\bar M_1^{\alpha_1} \times \ldots \times \bar M_{\ell-1}^{\alpha_{\ell-1}}$. Then $X_1^{\ell-1},\ldots,X_{\alpha_{\ell-1}}^{\ell-1}, X_1^\ell,\ldots,X_{\alpha_\ell}^\ell, JX_1^{\ell-1},\ldots,JX_{\alpha_{\ell-1}}^{\ell-1}, JX_1^\ell,\ldots,JX_{\alpha_\ell}^\ell$ form an orthonormal basis for $M_{\ell-1}$. Thus by induction, we know that

$$h(X^{i_1}_{a_1},X^{i_2}_{a_2}),\ Jh(X^{i_1}_{a_1},X^{i_2}_{a_2}),\ h(X^{i_1}_{a_1},X^{\ell-1}_{a_{\ell-1}}),\ Jh(X^{i_1}_{a_1},X^{\ell-1}_{a_{\ell-1}}),$$

$$h(X^{i_1}_{a_1},X^{\ell}_{a_\ell}),\ Jh(X^{i_1}_{a_1},X^{\ell}_{a_\ell}),\ (\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{i_3}_{a_3}),$$

$$J(\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{i_3}_{a_3}),\ (\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{\ell-1}_{a_{\ell-1}}),$$

$$J(\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{i_3}_{a_3}),\ (\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{\ell}_{a_\ell}),$$

$$J(\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{\ell}_{a_\ell}),\dots,(\bar\nabla^{\ell-3}h)(X^{i_1}_{a_1},\dots,X^{i_{\ell-1}}_{j_{\ell-1}}),$$

$$J(\bar\nabla^{\ell-3}h)(X^{i_1}_{a_1},\dots,X^{i_{\ell-1}}_{a_{\ell-1}}),\ (\bar\nabla^{\ell-3}h)(X^{i_1}_{a_1},\dots,X^{i_{\ell-2}}_{a_{\ell-2}},X^{\ell-1}_{a_{\ell-1}})$$

$$J(\bar\nabla^{\ell-3}h)(X^{i_1}_{a_1},\dots,X^{i_{\ell-2}}_{a_{\ell-2}},X^{\ell-1}_{a_{\ell-1}}),\ (\bar\nabla^{\ell-3}h)(X^{i_1}_{a_1},\dots,X^{i_{\ell-2}}_{a_{\ell-2}},X^{\ell}_{a_\ell})$$

$$J(\bar\nabla^{\ell-3}h)(X^{i_1}_{a_1},\dots,X^{i_{\ell-2}}_{a_{\ell-2}},X^{\ell}_{a_\ell})\ ;$$

$$i_1<\dots<i_{\ell-2}\ ;\ 1\leqslant i_1,\dots,i_{\ell-2}\leqslant \ell-2\ ;\ 1\leqslant a_j\leqslant \alpha_{i_j}\ ,$$

are orthonormal vectors normal to $M_1^{\alpha_1}\times\dots\times M_\ell^{\alpha_\ell}$. Applying the same argument to all other possible similar cases and by induction, we obtain the following.

Statement 1. The following normal vectors ;

$$h(X^{i_1}_{a_1},X^{i_2}_{a_2}),\ Jh(X^{i_1}_{a_1},X^{i_2}_{a_2}),\ (\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{i_3}_{a_3}),$$

$$J(\bar\nabla h)(X^{i_1}_{a_1},X^{i_2}_{a_2},X^{i_3}_{a_3}),\dots,$$

$$(\bar\nabla^{n-3}h)(X^{i_1}_{a_1},\dots,X^{i_{n-1}}_{a_{n-1}}),\ J(\bar\nabla^{n-3}h)(X^{i_1}_{a_1},\dots,X^{i_{n-1}}_{a_{n-1}})\ ;$$

$$i_1<i_2<\dots<i_{n-1}\ ,\ 1\leqslant j,i_1,\dots,i_{n-1}\leqslant n,\ \ 1\leqslant a_j\leqslant \alpha_{i_j}\ ;$$

are orthonormal.

We need the following.

Statement 2. For $i_1<\dots<i_{t+2}$, *and* $i\neq i_1,\dots,i_{t+2}$, *and any permutation* σ *of* $(i_1,\dots,i_{t+2})$, *we*

have

(3.1) $$A_{(\bar\nabla^t h)(X^{i_1},\dots,X^{i_{t+2}})}X^i = 0$$

and

(3.2) $$(\bar\nabla^t h)(X^{i_1},\dots,X^{i_{t+2}}) = (\bar\nabla^t h)(X^{\sigma(i_1)},\dots,X^{\sigma(i_{t+2})}).$$

Proof. First we mention that Lemma 3 implies the following

(3.3) $$A_{h(X^{i_1},X^{i_2})}X^i = 0, \quad i_1 \neq i_2.$$

If $k \neq i$, then Statement 1 yields

(3.4) $$\begin{aligned} &< A_{(\bar\nabla^t h)(X^{i_1},\dots,X^{i_{t+2}})}X^i, X^k > \\ &= < h(X^i,X^k),(\bar\nabla^t h)(X^{i_1},\dots,X^{i_{t+2}}) > \\ &= 0. \end{aligned}$$

Hence, in order to prove (3.1), it suffices to prove

(3.5) $$< A_{(\bar\nabla^t h)(X^{i_1},\dots,X^{i_{t+2}})}X^i, Y^i > = 0,$$

for any vector Y^i tangent to $M_i^{\alpha_i}$.

For $i \neq i_1, i_2, i_3$ and unit vector fields $X^{i_1}, X^{i_2}, X^{i_3}, X^i$, and Y^i, we have

$$\begin{aligned} &< A_{(\bar\nabla h)(X^{i_1},X^{i_2},X^{i_3})}X^i, Y^i > \\ &= < (\bar\nabla h)(X^{i_1},X^{i_2},X^{i_3}), h(X^i,Y^i) > \\ &= < D_{X^{i_1}} h(X^{i_2},X^{i_3}), h(X^i,Y^i) > \\ &= - < h(X^{i_2},X^{i_3}), D_{X^{i_1}} h(X^i,Y^i) > \end{aligned}$$

$$= -\langle h(X^{i_2},X^{i_3}), (\bar{\nabla}h)(X^{i_1},X^{i},Y^{i})\rangle$$

$$= -\langle h(X^{i_2},X^{i_3}), (\bar{\nabla}h)(X^{i},X^{i_1},Y^{i})\rangle$$

$$= -\langle h(X^{i_2},X^{i_3}), D_{X^i}h(X^{i_1},Y^{i})\rangle$$

$$= \langle D_{X^i}h(X^{i_2},X^{i_3}), h(X^{i_1},Y^{i})\rangle$$

$$= \langle (\bar{\nabla}h)(X^{i},X^{i_2},X^{i_3}), h(X^{i_1},Y^{i})\rangle$$

$$= 0.$$

This proves (3.1) for $t = 1$.

For (3.2), if $t = 1$, (3.2) follows from (1.13). Now, we assume that both (3.1) and (3.2) are true for $t \leqslant r-1$, $r \geqslant 2$, that is we have

$$A_{(\bar{\nabla}^t h)(X^{i_1},\dots,X^{i_{t+2}})} X^{i} = 0 \tag{3.6}$$

and

$$(\bar{\nabla}^t h)(X^{i_1},\dots,X^{i_{t+2}}) = (\bar{\nabla}^t h)(X^{\sigma(i_1)},\dots,X^{\sigma(i_{t+2})}) \tag{3.7}$$

for $t \leqslant r-1$. Then we have from Statement 1 and (3.7) that

$$\langle A_{(\bar{\nabla}^r h)(X^{i_1},\dots,X^{i_{r+2}})} X^{i}, Y^{i}\rangle \tag{3.8}$$

$$= \langle (\bar{\nabla}^r h)(X^{i_1},\dots,X^{i_{r+2}}), h(X^{i},Y^{i})\rangle$$

$$= \langle D_{X^{i_1}}((\bar{\nabla}^{r-1} h)(X^{i_2},\dots,X^{i_{r+2}})), h(X^{i},Y^{i})\rangle$$

$$= -\langle (\bar{\nabla}^{r-1} h)(X^{i_2},\dots,X^{i_{r+2}}), D_{X^{i_1}}h(X^{i},Y^{i})\rangle$$

$$= -\langle (\bar{\nabla}^{r-1} h)(X^{i_2},\dots,X^{i_{r+2}}), (\bar{\nabla}h)(X^{i_1},X^{i},Y^{i})\rangle$$

$$= -\langle (\bar{\nabla}^{r-1} h)(X^{i_2},\dots,X^{i_{r+2}}), (\bar{\nabla}h)(X^{i},X^{i_1},Y^{i})\rangle$$

$$= \langle (\bar{\nabla}^r h)(X^{i},X^{i_2},\dots,X^{i_{r+2}}), h(X^{i_1},Y^{i})\rangle.$$

If $i < i_2$, then from Statement 1, we obtain (3.5) and hence (3.6) for $t = r$.

If $i > i_2$, then (3.8) implies

$$<A_{(\bar{\nabla}^r h)(X^{i_1},\ldots,X^{i_{r+2}})}X^i,Y^i>$$

$$= <(\bar{\nabla}^r h)(X^{i_2},X^i,X^{i_3},\ldots,X^{i_{r+2}}),h(X^{i_1},Y^i)>$$

$$+ <R^D(X^i,X^{i_2})((\bar{\nabla}^{r-2}h)(X^{i_3},\ldots,X^{i_{r+2}}),h(X^{i_1},Y^i)>$$

$$= <(\bar{\nabla}^r h)(X^{i_2},X^i,X^{i_3},\ldots,X^{i_{r+2}}),h(X^{i_1},Y^i)>$$

$$+ \tilde{R}(X^i,X^{i_2};(\bar{\nabla}^{r-2}h)(X^{i_3},\ldots,X^{i_{r+2}}),h(X^{i_1},Y^i))$$

$$+ <[A_{(\bar{\nabla}^{\ell-2}h)(X^{i_3},\ldots,X^{i_{r+2}})},A_{h(X^{i_1},Y^i)}]X^i,X^{i_2}>.$$

Thus, by (1.8), (1.12), and (3.7) we find

$$<A_{(\bar{\nabla}^r h)(X^{i_1},\ldots,X^{i_{r+2}})}X^i,Y^i>$$

$$= <(\bar{\nabla}^r h)(X^{i_2},X^i,X^{i_3},\ldots,X^{i_{r+2}}),h(X^{i_1},Y^i)>$$

$$= <D_{X^{i_2}}((\bar{\nabla}^{r-1}h)(X^i,X^{i_3},\ldots,X^{i_{r+2}})),h(X^{i_1},Y^i)>$$

$$= <D_{X^{i_2}}((\bar{\nabla}^{r-1}h)(X^{i_3},\ldots,X^{i_s},X^i,X^{i_{s+1}},\ldots,X^{i_{r+2}})),h(X^{i_1},Y^i)>$$

$$= <(\bar{\nabla}^r h)(X^{i_2},\ldots,X^{i_s},X^i,X^{i_{s+1}},\ldots,X^{i_{r+2}}),h(X^{i_1},Y^i)>,$$

where $i_s < i < i_{s+1}$. Thus by Statement 1, we obtain (3.5) and hence (3.6) for $t = r$.

Now, we shall prove (3.7) for $t = r$.

Let ξ be any normal vector field normal to M. Then by (1.9), (3.7) and induction we have

$$\begin{aligned}
&<(\bar\nabla^r h)(X^{i_1},\dots,X^{i_{r+2}}),\xi>\\
&=<(\bar\nabla^r h)(X^{i_2},X^{i_1},X^{i_3},\dots,X^{i_{r+2}}),\xi>\\
&\quad+<R^D(X^{i_1},X^{i_2})((\bar\nabla^{r-2}h)(X^{i_3},\dots,X^{i_{r+2}})),\xi>\\
&=<(\bar\nabla^r h)(X^{i_2},X^{i_1},X^{i_3},\dots,X^{i_{r+2}}),\xi>\\
&\quad+\tilde R(X^{i_1},X^{i_2};(\bar\nabla^{r-2}h)(X^{i_3},\dots,X^{i_{r+2}}),\xi)\\
&\quad+<[A_{(\bar\nabla^{r-2}h)(X^{i_3},\dots,X^{i_{r+2}})},A_\xi]X^{i_1},X^{i_2}>\\
&=<(\bar\nabla^r h)(X^{i_2},X^{i_1},X^{i_3},\dots,X^{i_{r+2}}),\xi>.
\end{aligned}$$

This shows that

$$(\bar\nabla^r h)(X^{i_1},X^{i_2},\dots,X^{i_{r+2}})=(\bar\nabla^r h)(X^{i_2},X^{i_1},X^{i_3},\dots,X^{i_{r+2}}). \tag{3.9}$$

If $1<s$, then we also have from induction

$$\begin{aligned}
&(\bar\nabla^r h)(X^{i_1},\dots,X^{i_s},X^{i_{s+1}},\dots,X^{i_{r+2}}) \\
&=D_{X^{i_1}}((\bar\nabla^{r-1}h)(X^{i_2},\dots,X^{i_s},\dots,X^{i_{r+2}})\\
&=D_{X^{i_1}}((\bar\nabla^{r-1}h)(X^{i_2},\dots,X^{i_{s+1}},X^{i_s},\dots,X^{i_{r+2}})\\
&=(\bar\nabla^r h)(X^{i_1},\dots,X^{i_{s-1}},X^{i_{s+1}},X^{i_s},X^{i_{s+2}},\dots,X^{i_{r+2}}).
\end{aligned} \tag{3.10}$$

Consequently, (3.9) and (3.10) imply (3.7) for $t=r$. Thus, by induction, we obtain (3.6) and (3.7) for any t. This proves Statement 2.

Statement 3. For unit vectors $X^{i_1},\dots,X^{i_{p+2}}$, $i_1<\dots<i_{p+2}$, $1\leqslant s\leqslant p+2$, *we have*

$$(\bar\nabla^p h)(X^{i_1},\dots,JX^{i_s},\dots,X^{i_{p+2}})=J(\bar\nabla^p h)(X^{i_1},\dots,X^{i_{p+2}}). \tag{3.11}$$

Proof. If $p=0$, (3.7) follows from (1.10). If $p=1$, we obtain from (1.4) and (1.13) that

$$\begin{aligned}
&(\bar\nabla h)(JX^{i_1},X^{i_2},X^{i_3})=(\bar\nabla h)(X^{i_2},JX^{i_1},X^{i_3})\\
&=D_{X^{i_2}}(h(JX^{i_1},X^{i_3}))=D_{X^{i_2}}J(h(X^{i_1},X^{i_3}))
\end{aligned}$$

$$= JD_{X^{i_2}} h(X^{i_1}, X^{i_3}) = J(\bar{\nabla} h)(X^{i_2}, X^{i_1}, X^{i_3})$$
$$= J(\bar{\nabla} h)(X^{i_1}, X^{i_2}, X^{i_3}).$$

Similar argument also yields

$$(\bar{\nabla} h)(X^{i_1}, JX^{i_2}, X^{i_3}) = (\bar{\nabla} h)(X^{i_1}, X^{i_2}, JX^{i_3}) = J(\bar{\nabla} h)(X^{i_1}, X^{i_2}, X^{i_3}).$$

These proves (3.11) for $p = 1$.

Now, assume that (3.11) holds for $1 \leqslant p \leqslant t-1$. If $s > 1$, then, by (1.9) and induction, we have

$$(\bar{\nabla}^t h)(X^{i_1}, \ldots, JX^{i_s}, \ldots, X^{i_{t+2}}) = D_{X^{i_1}} (\bar{\nabla}^{t-1} h)(X^{i_2}, \ldots, JX^{i_s}, \ldots, X^{i_{t+2}})$$
$$= D_{X^{i_1}} J(\bar{\nabla}^{t-1} h)(X^{i_2}, \ldots, X^{i_s}, \ldots, X^{i_{t+2}})$$
$$= JD_{X^{i_1}} (\bar{\nabla}^{t-1} h)(X^{i_2}, \ldots, X^{i_{t+2}})$$
$$= J(\bar{\nabla}^t h)(X^{i_1}, \ldots, X^{i_{t+2}}).$$

If $s = 1$, then by (1.9), Statements 1 and 2, we find that for any unit normal vector field ξ, we have

$$< (\bar{\nabla}^t h)(JX^{i_1}, X^{i_2}, \ldots, X^{i_{t+2}}), \xi >$$
$$= < (\bar{\nabla}^t h)(X^{i_2}, JX^{i_1}, \ldots, X^{i_{t+2}}), \xi >$$
$$+ < R^D(JX^{i_1}, X^{i_2})((\bar{\nabla}^{t-2} h)(X^{i_3}, \ldots, X^{i_{t+2}})), \xi >$$
$$= < J(\bar{\nabla}^t h)(X^{i_2}, X^{i_1}, \ldots, X^{i_{t+2}}), \xi >$$
$$+ < \tilde{R}(JX^{i_1}, X^{i_2}; (\bar{\nabla}^{t-2} h)(X^{i_3}, \ldots, X^{i_{t+2}}), \xi >$$
$$+ < [A_{(\bar{\nabla}^{t-2} h)(X^{i_3}, \ldots, X^{i_{t+2}})}, A_\xi] JX^{i_1}, X^{i_2} >$$
$$= < J(\bar{\nabla}^t h)(X^{i_2}, X^{i_1}, \ldots, X^{i_{t+2}}), \xi >$$
$$= < J(\bar{\nabla}^t h)(X^{i_1}, X^{i_2}, \ldots, X^{i_{t+2}}), \xi >$$
$$- < R^D(X^{i_2}, X^{i_1})((\bar{\nabla}^{t-2} h)(X^{i_3}, \ldots, X^{i_{t+2}})), J\xi >$$
$$= < J(\bar{\nabla}^t h)(X^{i_1}, X^{i_2}, \ldots, X^{i_{t+2}}), \xi > .$$

Since this is true for any ξ, we obtain $(\bar{\nabla}^t h)(JX^{i_1},X^{i_2},\ldots,X^{i_{t+2}}) = J(\bar{\nabla}^t h)(X^{i_1},\ldots,X^{i_{t+2}})$. Consequently, we obtain (3.11) for $p = t$. Thus by induction, we obtain Statement 3.

Statement 4. For unit vectors $X^1,\ldots,X^n$, *we have*

$$\| (\bar{\nabla}^{n-2}h)(X^1,\ldots,X^n) \| = 1.$$

Proof. We have

$$\begin{aligned}(3.12)\qquad & <(\bar{\nabla}^{n-1}h)(JX^1,X^1,X^2,\ldots,X^n) - (\bar{\nabla}^{n-1}h)(X^1,JX^1,X^2,\ldots,X^n), \\ & \qquad J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)> \\ & = <R^D(JX^1,X^1)((\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)),J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)> \\ & = <\tilde{R}(JX^1,X^1;(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n),J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)> \\ & \quad + <[A_{(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)},A_{J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)}]JX^1,X^1> \\ & = 2 + 2\, \| A_{(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)}X^1 \|^2 = 2.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(3.13)\qquad & <(\bar{\nabla}^{n-1}h)(JX^1,X^1,X^2,\ldots,X^n) - (\bar{\nabla}^{n-1}h)(X^1,JX^1,X^2,\ldots,X^n), \\ & \qquad J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)> \\ & = <D_{JX^1}((\bar{\nabla}^{n-2}h)(X^1,\ldots,X^n)) - D_{X^1}((\bar{\nabla}^{n-2}h)(JX^1,X^2,\ldots,X^n)) \\ & \quad - (\bar{\nabla}^{n-2}h)([JX^1,X^1],X^2,\ldots,X^n),J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n)> \\ & = <(\bar{\nabla}^{n-2}h)(JX^1,X^2,\ldots,X^n),D_{X^1}(J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n))> \\ & \quad - <(\bar{\nabla}^{n-2}h)(X^1,\ldots,X^n),D_{JX^1}(J(\bar{\nabla}^{n-3}h)(X^2,\ldots,X^n))> \\ & = \| (\bar{\nabla}^{n-2})(JX^1,X^2,\ldots,X^n) \|^2 + \| (\bar{\nabla}^{n-2}h)(X^1,\ldots,X^n) \|^2 \\ & = 2\, \| (\bar{\nabla}^{n-2}h)(X^1,X^2,\ldots,X^n) \|^2 .\end{aligned}$$

Combining (3.12) and (3.13), we obtain Statement 4.

Statement 5. The following vectors

$$(\bar\nabla^{n-2}h)(X^1_{a_1},\ldots,X^n_{a_n}),\ J(\bar\nabla^{n-2}h)(X^1_{a_1},\ldots,X^n_{a_n})\ ;$$

$$a_i = 1,\ldots,\alpha_i\ ;\quad i = 1,\ldots,n\ ;$$

are orthonormal.

Proof. From (1.7), (1.9), (1.12), and Statements 1 and 2, we find

$$\begin{aligned}
&<(\bar\nabla^{n-1}h)(X^1_{a_1},X^1_{b_1},X^2_{a_2},\ldots,X^n_{a_n}) - (\bar\nabla^{n-1}h)(X^1_{b_1},X^1_{a_1},X^2_{a_2},\ldots,X^n_{a_n}),\\
&\qquad (\bar\nabla^{n-3}h)(X^2_{b_2},\ldots,X^n_{b_n})>\\
&=<R^D(X^1_{a_1},X^1_{b_1})((\bar\nabla^{n-3}h)(X^2_{a_2},\ldots,X^n_{a_n})),\ (\bar\nabla^{n-3}h)(X^2_{b_2},\ldots,X^n_{b_n})>\\
&=\tilde R(X^1_{a_1},X^1_{b_1};(\bar\nabla^{n-3}h)(X^2_{a_2},\ldots,X^n_{a_n}),(\bar\nabla^{n-3}h)(X^2_{b_2},\ldots,X^n_{b_n}))\\
&\quad+<[A_{(\bar\nabla^{n-3}h)(X^2_{a_2},\ldots,X^n_{a_n})},A_{(\bar\nabla^{n-3}h)(X^2_{b_2},\ldots,X^n_{b_n})}]X^1_{a_1},X^1_{b_1}>\\
&=0.
\end{aligned}$$

Hence, by using (1.8) and Statement 1, we may obtain

$$\begin{aligned}
(3.14)\qquad &<(\bar\nabla^{n-2}h)(X^1_{b_1},X^2_{a_2},\ldots,X^n_{a_n}),(\bar\nabla^{n-2}h)(X^1_{a_1},X^2_{b_2},\ldots,X^n_{b_n})>\\
&=<(\bar\nabla^{n-2}h)(X^1_{a_1},\ldots,X^n_{a_n}),(\bar\nabla^{n-2}h)(X^1_{b_1},\ldots,X^n_{b_n})>.
\end{aligned}$$

By continuing this process sufficient by many times, we will obtain

$$\begin{aligned}
(3.15)\qquad &<(\bar\nabla^{n-2}h)(X^1_{a_1},\ldots,X^n_{a_n}),(\bar\nabla^{n-2}h)(X^1_{b_1},\ldots,X^n_{b_n})>\\
&=<(\bar\nabla^{n-2}h)(X^1_{c_1},\ldots,X^n_{c_n}),(\bar\nabla^{n-2}h)(X^1_{e_1},\ldots,X^n_{e_n})>,
\end{aligned}$$

where $\{c_i,e_i\} = \{a_i,b_i\}$, $i = 1,\ldots,n$. Thus, by using linearity, (3.15), and Statement 4, we may conclude that

$$(\bar\nabla^{n-2}h)(X^1_{a_1},\ldots,X^n_{a_n})\ ;\ a_i = 1,\ldots,\alpha_i\ ;\quad i = 1,\ldots,n\ ;$$

are orthonormal. Therefore, it suffices to prove that

(3.16) $$<(\bar\nabla^{n-2}h)(X^1_{a_1},\ldots,X^n_{a_n}),J(\bar\nabla^{n-2}h)(X^1_{b_1},\ldots,X^n_{b_n})>=0.$$

If $a_i = b_i$ for all i, then (3.16) is trivial. Suppose that there is an i such that $a_i \neq b_i$, then we just replace $X^i_{b_i}$ by $JX^i_{b_i}$, and applying the previous case, we obtain (3.16). Consequently, we obtain Statement 5.

In the following, we put

(3.17) $$N_o = \mathrm{Span}\{h(X^{i_1}_{a_1},X^{i_2}_{a_2}),Jh(X^{i_1}_{a_1},X^{i_2}_{a_2}) \mid i_1 < i_2, \quad a_1 = 1,\ldots,\alpha_{i_1}, \quad a_2 = 1,\ldots,\alpha_{i_2}\}$$

and

(3.18) $$N_r = \mathrm{Span}\{(\bar\nabla^r h)(X^{i_1}_{a_1},\ldots,X^{i_{r+2}}_{a_{r+2}}),J(\bar\nabla^r h)(X^{i_1}_{a_1},\ldots,X^{i_{r+2}}_{a_{r+2}}) \mid i_1 < \ldots < i_{r+2}; \ a_t = 1,\ldots,\alpha_{i_t}\}$$

for $r = 1,\ldots,n-2$.

Statement 6. We have

$$N_{n-2} \perp N_t,$$

for $t = 0,1,\ldots,n-3$.

Proof. If $n = 3$, this statement is already proved in Lemma 5. Now, assume that $n \geqslant 4$.

If $t \leqslant n-4$ and $i_1 < \ldots < i_t$, then we may find one j such that $j \neq i_1,\ldots,i_t$. Using Statements 1 and 5, we have

$$<(\bar\nabla^{n-2}h)(X^1,\ldots,X^n),(\bar\nabla^t h)(Y^{i_1},\ldots,Y^{i_{t+2}})>$$

$$=<(\bar\nabla^{n-2}h)(X^j,X^1,\ldots,\hat{X}^j,\ldots,X^n),(\bar\nabla^t h)(Y^{i_1},\ldots,Y^{i_{t+2}})>$$

$$=<D_{X^j}((\bar\nabla^{n-2}h)(X^1,\ldots,\hat{X}^j,\ldots,X^n)),(\bar\nabla^t h)(Y^{i_1},\ldots,Y^{i_{t+2}})>$$

$$=-<(\bar\nabla^{n-3}h)(X^1,\ldots,\hat{X}^j,\ldots,X^n),D_{X^j}((\bar\nabla^t h)(Y^{i_1},\ldots,Y^{i_{t+2}}))>$$

$$=-<(\bar\nabla^{n-3}h)(X^1,\ldots,\hat{X}^j,\ldots,X^n),(\bar\nabla^{t+1}h)(X^j,Y^{i_1},\ldots,Y^{i_{t+2}})>$$

$$=0.$$

Consequently we have $N_{n-2} \perp N_t$ for $t=0,1,\dots,n-4$.

Now, we want to prove that $N_{n-2} \perp N_{n-3}$. Let $X^1,\dots,X^n$ be unit vector fields. Then we obtain from Statement 1 that

$$\| (\bar{\nabla}^{n-3}h)(X^2,\dots,X^n) \| = 1.$$

Thus, by (1.8), we have

$$<(\bar{\nabla}^{n-2}h)(X^1,X^2,\dots,X^n),(\bar{\nabla}^{n-3}h)(X^2,\dots,X^n)>=0.$$

Hence, by linearity, we get

$$(3.19) \qquad <(\bar{\nabla}^{n-2}h)(X^1,X^2,X^3,\dots,X^n),(\bar{\nabla}^{n-3}h)(X^2,X^3,\dots,X^n)>$$
$$+<(\bar{\nabla}^{n-2}h)(X^1,Y^2,X^3,\dots,X^n),(\bar{\nabla}^{n-3}h)(X^2,X^3,\dots,X^n)>=0.$$

On the other hand, using Statements 1 and 5, we also have

$$<(\bar{\nabla}^{n-2}h)(X^1,\dots,X^n),(\bar{\nabla}^{n-3}h)(Y^2,X^3,\dots,X^n)>$$
$$=<(\bar{\nabla}^{n-2}h)(X^2,X^1,X^3,\dots,X^n),(\bar{\nabla}^{n-3}h)(Y^2,X^3,\dots,X^n)>$$
$$=<D_{X^2}((\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n)),(\bar{\nabla}^{n-3}h)(Y^2,X^3,\dots,X^n)>$$
$$=-<(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n),(\bar{\nabla}^{n-2}h)(X^2,Y^2,X^3,\dots,X^n)>$$
$$-<(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n),(\bar{\nabla}^{n-3}h)(\nabla_{X^2}Y^2,X^3,\dots,X^n)>$$
$$=-<(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n),(\bar{\nabla}^{n-2}h)(X^2,Y^2,X^3,\dots,X^n)>$$
$$=-<(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n),(\bar{\nabla}^{n-2}h)(Y^2,X^2,X^3,\dots,X^n)$$
$$+R^D(X^2,Y^2)((\bar{\nabla}^{n-4}h)(X^3,\dots,X^n))>$$
$$=-<(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n),D_{Y^2}(\bar{\nabla}^{n-3}h)(X^2,\dots,X^n)>$$
$$-\tilde{R}(X^2,Y^2;(\bar{\nabla}^{n-4}h)(X^3,\dots,X^n),(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n))$$
$$-<[A_{(\bar{\nabla}^{n-4}h)(X^3,\dots,X^n)},A_{(\bar{\nabla}^{n-3}h)(X^1,X^3,\dots,X^n)}]X^2,Y^2>$$

$$= < D_{Y^2}((\bar\nabla^{n-3}h)(X^1,X^3,\ldots,X^n)),(\bar\nabla^{n-3}h)(X^2,\ldots,X^n) >$$

$$= < (\bar\nabla^{n-2}h)(X^1,Y^2,X^3,\ldots,X^n),(\bar\nabla^{n-3}h)(X^2,\ldots,X^n) > .$$

Combining this with (3.19), we obtain

$$(3.20) \qquad < (\bar\nabla^{n-2}h)(X^1,X^2,\ldots,X^n),(\bar\nabla^{n-3}h)(Y^2,X^3,\ldots,X^n) > = 0.$$

Continuing this process n–1 times, we will find

$$< (\bar\nabla^{n-2}h)(X^1,X^2,\ldots,X^n),(\bar\nabla^{n-2}h)(Y^2,\ldots,Y^n) > = 0.$$

Therefore, by using Statement 6, we conclude that N_{n-2} is perpendicular to N_{n-3}. Thus Statement 6 is proved.

From Statements 1, 5, and 6, we obtain Lemma 8.

(Q.E.D.)

4. - PROOF OF THEOREM 1

Let $M = M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ be a Kaehler submanifold of $\mathbb{C}P^m$ which is the product of n Kaehler manifolds. Then by the Main Lemma, we see that $m \geq N(\alpha_1,\ldots,\alpha_n)$.

Now, we assume that m is the smallest possible dimension $N(\alpha_1,\ldots,\alpha_n)$. We want to prove that each $M_i^{\alpha_i}$ is an open portion of a $\mathbb{C}P^{\alpha_i}$.

Let X^i,Y^i be any two unit vector fields tangent to $M_i^{\alpha_i}$. We need the following Lemmas.

LEMMA 9. *For any* X^i,Y^i *tangent to* $M_i^{\alpha_i}$, *we have*

$$(4.1) \qquad h(X^i,Y^i) \perp N_o ,$$

where N_o *is defined by* (3.17).

Proof. Let X^i, Y^i, and Z^i be tangent to $M_i^{\alpha_i}$ and W^j tangent to $M_j^{\alpha_j}$, $j \neq i$. Then from (1.5) of

Gauss, (1.10) and (1.12), we find

$$<h(X^i,Y^i),h(JX^i,W^j)>$$

$$=<h(X^i,W^j),h(JX^i,Y^i)>$$

$$=-<h(JX^i,W^j),h(X^i,Y^i)>$$

Hence, we have $<h(X^i,Y^i),h(X^i,W^j)>=0$. By applying linearity and using the equation of Gauss again, we obtain

$$<h(X^i,Y^i),h(Z^i,W^j)>=0, \quad i \neq j. \tag{4.2}$$

If $j,k \neq i$, then Main Lemma and equation (1.5) of Gauss also yield

$$<h(X^i,Y^i),h(W^j,V^k)>=0. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain (4.1). This proves Lemma 9.

LEMMA 10. *For* X^i *and* Y^i *tangent to* $M_i^{\alpha_i}$, *we have*

$$h(X^i,Y^i) \perp N_1. \tag{4.4}$$

Proof. Let j_1,j_2,j_3 be distinct. Then i is distinct from at least two of j_1,j_2,j_3, say $i \neq j_2,j_3$. Then from Lemmas 9 and 10, we have

$$<h(X^i,Y^i),(\bar{\nabla}h)(X^{j_1},X^{j_2},X^{j_3})> \tag{4.5}$$

$$=<h(X^i,Y^i),D_{X^{j_1}}h(X^{j_2},X^{j_3})>$$

$$=-<D_{X^{j_1}}h(X^i,Y^i),h(X^{j_2},X^{j_3})>$$

$$=-<(\bar{\nabla}h)(X^{j_1},X^i,Y^i),h(X^{j_2},X^{j_3})>$$

$$=-<D_{X^i}h(X^{j_1},Y^i),h(X^{j_2},X^{j_3})>$$

$$=<h(X^{j_1},Y^i),(\bar{\nabla}h)(X^i,X^{j_2},X^{j_3})>.$$

If $i \neq j_1$, then (4.5) and Lemma 8 imply that

$$<h(X^i,Y^i),(\bar{\nabla}h)(X^{j_1},X^{j_2},X^{j_3})>=0. \tag{4.6}$$

Assume that $i = j_1$. Then using the same method as the proof of (4.6), we may find

$$(4.7)\qquad \langle h(X^i,Y^i),(\bar\nabla h)(Z^i,X^{j_2},X^{j_3})\rangle$$
$$= \langle (\bar\nabla h)(Z^i,X^i,Y^i),h(X^{j_2},X^{j_3})\rangle.$$

On the other hand, Lemma 9 implies

$$(4.8)\qquad \langle (\bar\nabla h)(Z^i,X^i,Y^i),h(X^{j_2},X^{j_3})\rangle$$
$$+\langle h(X^i,Y^i),(\bar\nabla h)(Z^i,X^{j_2},X^{j_3})\rangle = 0.$$

Combining (4.7) and (4.8), we get

$$(4.9)\qquad \langle h(X^i,Y^i),(\bar\nabla h)(Z^i,X^{j_2},X^{j_3})\rangle = 0.$$

Therefore, by (4.6) and (4.9) we obtain (4.4). This poves Lemma 10.

LEMMA 11. *For* X^i *and* Y^i *tangent to* $M_i^{\alpha_i}$, *we have*

$$(4.10)\qquad h(X^i,Y^i) \perp N_t$$

for $t = 2,3,\dots,n-2$.

Proof. We shall prove this lemma by induction. Assume that $h(X^i,Y^i) \perp N_t$ for $t \leqslant \ell-1$. We want to prove that $h(X^i,Y^i) \perp N_\ell$ for $\ell = 2,\dots,n-2$.

Let $j_1,\dots,j_{\ell+2}$ be distinct. Then we may assume that $i \neq j_2,\dots,j_{\ell+2}$. Then by Lemma 9 and induction we have

$$\langle h(X^i,Y^i),(\bar\nabla^\ell h)(Z^{j_1},\dots,Z^{j_{\ell+2}})\rangle$$
$$= \langle h(X^i,Y^i),D_{Z^{j_1}}((\bar\nabla^{\ell-1}h)(Z^{j_2},\dots,Z^{j_{\ell+2}})\rangle$$
$$= -\langle (\bar\nabla h)(Z^{j_1},X^i,Y^i),(\bar\nabla^{\ell-1}h)(Z^{j_2},\dots,Z^{j_{\ell+2}})\rangle$$
$$= \langle h(Z^{j_1},Y^i),(\bar\nabla^\ell h)(X^i,Z^{j_2},\dots,Z^{j_{\ell+2}})\rangle.$$

If $j_1 \neq i$, this implies

$$\langle h(X^i,Y^i),(\bar\nabla^\ell h)(Z^{j_1},\dots,Z^{j_{\ell+2}})\rangle = 0.$$

If $j_1 = i$, then we have

$$< h(X^i,Y^i),(\overline{\nabla}^{\ell}h)(Z^i,Z^{i_2},...,Z^{i_{\ell+2}}) >$$

$$= < (\overline{\nabla}h)(Z^i,X^i,Y^i),(\overline{\nabla}^{\ell-1}h)(Z^{i_2},...,Z^{i_{\ell+2}}) >.$$

On the other hand, because $h(X^i,Y^i) \perp N_{\ell-1}$, we also have

$$< h(X^i,Y^i),(\overline{\nabla}^{\ell}h)(Z^i,Z^{i_2},...,Z^{i_{\ell+2}}) >$$

$$+ < (\overline{\nabla}h)(Z^i,X^i,Y^i),(\overline{\nabla}^{\ell-1}h)(Z^{i_2},...,Z^{i_{\ell+2}}) > = 0.$$

Thus, we find $h(X^i,Y^i) \perp N_{\ell}$. Consequently, by induction, we obtain Lemma 10.

From Lemmas 9,10, and 11, we conclude that $h(X^i,Y^i) \equiv 0$. Since $M_i^{\alpha_i}$ is totally geodesic in $M = M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$. We see that $M_i^{\alpha_i}$ is also totally geodesic in $\mathbb{C}P^m$, $m = N(\alpha_1,...,\alpha_n)$. Therefore, $M_i^{\alpha_i}$ is an open portion of a linear subspace $\mathbb{C}P^{\alpha_i}$. Consequently, $M = M_1^{\alpha_1} \times ... \times M_n^{\alpha_n}$ is an open portion of $\mathbb{C}P^{\alpha_1} \times ... \times \mathbb{C}P^{\alpha_n}$. This proves statement (2.1) of Theorem 1. Statement (2.2) then follows by local rigidity theorem of Kaehler submanifolds. This proves Theorem 1. (Q.E.D.)

5. - PROOF OF THEOREM 2

(0.4) follows immediately from Lemma 8. Moreover, it is clear that if we have

$$\| \overline{\nabla}^{\ell-2}h \|^2 = \ell ! \, 2^{\ell} \sum_{i_1 < ... < i_{\ell}} \alpha_{i_1} \cdots \alpha_{i_{\ell}} \tag{5.1}$$

for some ℓ, $2 \leqslant \ell \leqslant n$, then

$$(\overline{\nabla}^{\ell-2}h)(X_{a_1}^{j_1},...,X_{a_{\ell}}^{j_{\ell}}) = 0 \tag{5.2}$$

whenever two or more of $j_1,...,j_{\ell}$ are equal.

If $\ell = 2$, (5.2) implies

$$h(X^i,Y^i) = 0 \tag{5.3}$$

for any X^i, Y^i tangent to $M_i^{\alpha_i}$. Because $M_i^{\alpha_i}$ sits in $M = M_1^{\alpha_1} \times \ldots \times M_n^{\alpha_n}$ as a totally geodesic submanifold, (5.3) shows that $M_i^{\alpha_i}$ is totally geodesic in $\mathbb{C}P^m$. Thus, each $M_i^{\alpha_i}$ is an open portion of $\mathbb{C}P^{\alpha_i}$. Therefore, M is an open portion of $\mathbb{C}P^{\alpha_1} \times \ldots \times \mathbb{C}P^{\alpha_n}$. By applying Calabi's rigidity theorem, we see that the immersion is obtained by the Segre imbedding $S_{\alpha_1 \ldots \alpha_n}$.

Now, assume that (5.1) holds for some ℓ with $\ell = 3,4,\ldots$, or n. Then we have (5.2) whenever two or more of $j_1,\ldots,j_\ell$ are equal.

If two or more of $k_1,\ldots,k_{\ell-1}$ are equal, then we may choose one i with $i \neq k_1,\ldots,k_{\ell-1}$. From (1.8) and (5.2) we find

$$(5.4) \qquad \begin{aligned}&(\bar{\nabla}^{\ell-1}h)(JX^i,X^i,X_{a_1}^{k_1},\ldots,X_{a_{\ell-1}}^{k_{\ell-1}})\\ &= (\bar{\nabla}^{\ell-1}h)(X^i,JX^i,X_{a_1}^{k_1},\ldots,X_{a_{\ell-1}}^{k_{\ell-1}})\\ &= 0.\end{aligned}$$

Thus, by using (1.7), (1.9), (1.12) and (5.4), we get

$$\begin{aligned}0 &= \langle R^D(JX^i,X^i)\xi,J\xi\rangle\\ &= \tilde{R}(JX^i,X^i;\xi,J\xi) + \langle [A_\xi,A_{J\xi}]JX^i,X^i\rangle\\ &= 2\,\|\xi\|^2 + 2\,\|A_\xi X^i\|^2,\end{aligned}$$

where $\xi = (\bar{\nabla}^{\ell-3}h)(X_{a_1}^{k_1},\ldots,X_{a_{\ell-1}}^{k_{\ell-1}})$. Therefore, we find

$$(5.5) \qquad (\bar{\nabla}^{\ell-3}h)(X_{a_1}^{k_1},\ldots,X_{a_{\ell-1}}^{k_{\ell-1}}) = 0$$

whenever two or more of $k_1,\ldots,k_{\ell-1}$ are equal. Continuing this process $\ell-2$ times, we obtain (5.3) for X^i, Y^i tangent to $M_i^{\alpha_i}$. Applying the same argument as before, we conclude that M is an open portion of $\mathbb{C}P^{\alpha_1} \times \ldots \times \mathbb{C}P^{\alpha_n}$ and the immersion is obtained by the Segre imbedding. Moreover, if this is the case, we can see that the equality of (0.4) holds for all ℓ, $\ell = 2,\ldots,n$. (Q.E.D.)

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