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Dynamical connections and non-autonomous lagrangian systems


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Dynamical connections and non-autonomous Lagrangian systems\(^{(1)}\)

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RÉSUMÉ. — On montre que si \(\xi\) est une équation différentielle du deuxième ordre (semigerbe) sur le fibré des jets \(J^1(R, M)\) telle que les courbes intégrales sont des solutions de l'équation de Lagrange non-autonome alors il existe une connexion \(\Gamma\) sur \(J^1(R, M)\) dont les courbes intégrales sont aussi des solutions de la même équation. En plus, \(\Gamma\) est une connexion ayant comme semigerbe \(\xi\). L'étude est une extension à la dynamique Lagrangienne non-autonome de quelques résultats de Grifone pour le cas autonome.

ABSTRACT. — We show that if \(\xi\) is a second-order differential equation (semispray) on the jet bundle \(J^1(R, M)\) whose paths are solutions of the non-autonomous Lagrange equations then there is a connection \(\Gamma\) on \(J^1(R, M)\) whose paths are also solutions of the same equations. Moreover, \(\Gamma\) is a connection whose associated semispray is precisely \(\xi\). This is an extension to non-autonomous Lagrangian dynamics of a previous result due to Grifone for autonomous Lagrangians.

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1. Introduction

The geometrical description of autonomous Lagrangian systems, started with GALLISOT [G], was elucidated by KLEIN [K1], [K2] (see also GODBILLON [GB]). He showed that the differential geometry of Lagrangian dynamics is intrinsically related to a (1.1) tensor field \(J\), called almost tangent structure, defined on the tangent bundle of a manifold.

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Since the early works of KLEIN some articles have been published showing that almost tangent geometry provides a natural framework in which interesting generalizations of autonomous Lagrangian systems may be developed (see for instance, CRAMPIN [C], CRAMPIN et al. [CSC], de LEON and RODRIGUES [DLR1], [DLA2], GOTAY and NESTER [GN], SARLET et al. [SCC]). In particular, an extensive study about the theory of connections on tangent bundles in terms of the almost tangent geometry, including some aspects of autonomous Lagrangians, was proposed by GRIFONE [GR] around 1972.

As far as we know the non-autonomous case has been practically unknown in the literature (an exception, for example, is the recent paper of CRAMPIN and co-workers [CPT]). It is the purpose of this paper to establish some intrinsic properties about almost tangent theory of connections and its relation with non-autonomous Lagrangian dynamics. We will see that the theory of connections on the jet manifold $J^1(R,M)$ is surprisingly more simpler than the theory of connections on $TM$, where $M$ is a given manifold (the reader is invited to compare our results with GRIFONE).

2. Preliminaries

Throughout the text we shall keep in mind all results, definitions and notations previously introduced in [DLR 1] (see also [DLR 2]). All structures, functions, etc, are assumed to be smooth ($C^\infty$). Let $M$ be a manifold of dimension $m$ (called configuration manifold) and $\Gamma$ a connection on the tangent bundle $TM$ of $M$. We recall here that a connection $\Gamma$ on $TM$ generates two projectors $h : T(TM) \rightarrow Hor(TM), v : T(TM) \rightarrow Ver(TM)$ such that $T(TM) = Hor(TM) \oplus Ver(TM)$, where $Hor(TM)$ (resp. $Ver(TM)$) is the horizontal (resp. vertical) bundle over $TM$. If $\tilde{\xi}$ is an arbitrary semispray (second-order differential equation) on $TM$ then $\xi = h(\tilde{\xi})$ is a semispray on $TM$ which does not depend on the choice of $\xi$. We call $\xi$ the associated semispray of $\Gamma$. A connection $\Gamma$ and its associated semispray have same paths.

If $\xi$ is a semispray on $TM$ then it can be shown that $\Gamma = -\mathcal{L}_\xi J$ is a connection on $TM$ (here $\mathcal{L}_\xi$ is the Lie derivative and $\mathcal{L}_\xi J$ is defined by

$$(\mathcal{L}_\xi J)(Y) = ([\xi, JY] - J[\xi, Y]).$$

When $\xi$ is a spray (homogeneous second-order differential equation) then $\Gamma = -\mathcal{L}_\xi J$ is a connection on $M$ such that its associated semispray is precisely $\xi$. For a semispray $\xi$ there is a family of connections $\Gamma = -\mathcal{L}_\xi J + T$, where $T$ is a given linear connection on $M$. The reader is invited to compare our results with GRIFONE.
where $T$ is a semibasic tensor field of type (1.1) on $TM$ in equilibrium with $\xi$ (in fact $T$ is the strong torsion of $\Gamma$) (see [GR]). In the non-autonomous situation the relation between connections and semisprays becomes much more simpler, as we will show below.

The jet manifold $J^1(\mathbb{R}, M)$ is fibred over $\mathbb{R} \times M$, $\mathbb{R}$ and $M$ with projection maps $\pi$, $\pi_1$, and $\pi_2$. We notice that $J^1(\mathbb{R}, M)$ can be identified with $\mathbb{R} \times TM$ in a very natural way. Therefore we transport the geometric structures defined on $TM$ to $J^1(\mathbb{R}, M)$ like the almost tangent structure $J$ and the Liouville vector field $C$ on $TM$. We may define a new tensor field $\tilde{J}$ of type (1.1) on $J^1(\mathbb{R}, M)$ by

$$\tilde{J} = J - C \otimes dt,$$

which is locally characterized by

$$\tilde{J}(\partial/\partial t) = -C; \quad \tilde{J}(\partial/\partial x^i) = \partial/\partial x^i; \quad \tilde{J}(\partial/\partial y^i) = 0$$

where $(t, x, y)$ are local coordinates for $J^1(\mathbb{R}, M)$.

Hence $\tilde{J}$ has rank $m$ and satisfies $(\tilde{J})^2 = 0$. We define the adjoint of $\tilde{J}$, $\tilde{J}^*$, as the endormorphism of the exterior algebra $\Lambda(J^1(\mathbb{R}, M))$ of $J^1(\mathbb{R}, M)$ locally given by

$$\tilde{J}^*(dt) = 0, \quad \tilde{J}^*(dx^i) = 0, \quad \tilde{J}^*(dy^i) = dx^i - y^i \, dt. \quad \text{(3)}$$

Like in the autonomous situation we associate to $\tilde{J}$ operators $i_{\tilde{J}}$ and $d_{\tilde{J}}$ on the algebra $\Lambda(J^1(\mathbb{R}, M))$ by

$$i_{\tilde{J}} \omega(X_1, \cdots, X_r) = \sum_{t=1}^r \omega(X_1, \cdots, \tilde{J}X_t, \cdots, X_r), \quad \text{ and so we have} \quad \text{ } \quad \text{ (4)}$$

$$d_{\tilde{J}} = i_{\tilde{J}} d - di_{\tilde{J}},$$

and so we have

$$i_{\tilde{J}}(df) = \tilde{J}^*(df), \text{ for all } f \text{ on } J^1(\mathbb{R}, M) \quad \text{ and so we have} \quad \text{ (5)}$$

$$i_{\tilde{J}}(dt) = i_{\tilde{J}}(dx^i) = 0; \quad i_{\tilde{J}}(dy^i) = dx^i - y^i \, dt \quad \text{ and so we have} \quad \text{ (6)}$$

$$d_{\tilde{J}} f = \frac{\partial f}{\partial y^i} (dx^i - y^i \, dt)$$

$$d_{\tilde{J}}(dt) = d_{\tilde{J}}(dx^i) = 0; \quad d_{\tilde{J}}(dy^i) = -d(dx^i - y^i \, dt) = dy^i \wedge dt. \quad \text{ (6)}$$

In the following we will set

$$\theta^i = dx^i - y^i \, dt, \quad 1 \leq i \leq m. \quad \text{ (7)}$$
Also, it is not hard to see that a vector field $\xi$ on $J^1(\mathbb{R}, M)$ is a semispray iff $\theta^i(\xi) = 0$ and $dt(\xi) = 1$, $1 \leq i \leq m$. In such a case $\xi$ is locally given by

$$\xi = \partial/\partial t + y^i \partial/\partial x^i + \xi^i \partial/\partial y^i.$$  \hspace{1cm} (8)

Furthermore, a vector field $\xi$ on $J^1(\mathbb{R}, M)$ is a semispray iff $J\xi = C$ and $\tilde{J}\xi = 0$.

Let $\xi$ be a semispray on $J^1(\mathbb{R}, M)$. A curve $s$ in $M$ is called a path of $\xi$ if its canonical prolongation is an integral curve of $\xi$.

Let $s$ be a curve in $M$ locally given by $(x^i(t))$. Then $\xi(t) = (t, x^i(t), \dot{x}^i(t))$ and so $s$ is a path of $\xi$ if and only if satisfies the following non-autonomous system of differential equations

$$\frac{d^2 x^i}{dt^2} = \xi^i \left(t, x, \frac{dx}{dt}\right), \quad 1 \leq i \leq m$$

where $\xi$ is given by (8).

3. Semisprays and dynamical connections

The tensor fields $J$ and $\tilde{J}$ on $J^1(\mathbb{R}, M)$ permit us to give a characterization of a kind of connections for the fibration $\pi: J^1(\mathbb{R}, M) \to \mathbb{R} \times M$.

**Definition (1).**—By a dynamical connection on $J^1(\mathbb{R}, M)$ we mean a tensor field $\Gamma$ of type $(1,1)$ on $J^1(\mathbb{R}, M)$ satisfying

$$J\Gamma = \tilde{J}\Gamma = \tilde{J}, \quad \Gamma\tilde{J} = -\tilde{J}, \quad \Gamma J = -J.$$ \hspace{1cm} (9)

By a straightforward computation from (9) we deduce that the local expressions of $\Gamma$ are

$$\begin{align*}
\Gamma(\partial/\partial t) &= -y^i \partial/\partial x^i + \Gamma^i_1 \partial/\partial y^i, \\
\Gamma(\partial/\partial x^i) &= \partial/\partial x^i + \Gamma^i_1 \partial/\partial y^i, \\
\Gamma(\partial/\partial y^i) &= -\partial/\partial y^i.
\end{align*}$$ \hspace{1cm} (10)

The functions $\Gamma^i = \Gamma^i(t, x, y)$, $\Gamma^i_1 = \Gamma^i_1(t, x, y)$ will be called the components of the connection $\Gamma$. From (10) we easily deduce that

$$\Gamma^3 - \Gamma = 0 \text{ and } \text{rank } (\Gamma) = 2m.$$
This type of polynomial structure is called $f(3, -1)$-structure in the literature (see [YI]). Now, we can associate to $\Gamma$ two canonical operators $\ell$ and $m$ given by

$$\ell = \Gamma^2, \quad m = -\Gamma^2 + I.$$ 

Then we have

$$\ell^2 = \ell, m^2 = m, \ell m = m\ell = 0, \quad \ell + m = I, \quad (11)$$

and $\ell$ and $m$ are complementary projectors. From (11) we deduce that $\ell$ and $m$ are locally given by

$$\ell(\partial/\partial t) = -y^i \partial/\partial x^i - (\Gamma^i + y^j \Gamma^i_j)\partial/\partial y^i; \ell(\partial/\partial x^i) =$$

$$= \partial/\partial x^i; \ell(\partial/\partial y^i) = \partial/\partial y^i; m(\partial/\partial t) =$$

$$= \partial/\partial t + y^i \partial/\partial x^i + (\Gamma^i + y^j \Gamma^i_j)\partial/\partial y^i; m(\partial/\partial x^i) =$$

$$= m(\partial/\partial y^i) = 0. \quad (12)$$

If we put $L = \text{Im} \ell$, $M = \text{Im} m$, then we have that $L$ and $M$ are complementary distributions on $J^1(\mathbb{R}, M)$, that is,

$$T(J^1(\mathbb{R}, M)) = M \oplus L.$$ 

From (12) we deduce that $L$ is $2m$-dimensional and is locally spanned by $\{\partial/\partial x^i, \partial/\partial y^i\}$. $M$ is one-dimensional, globally spanned by the vector field $\xi = m(\partial/\partial t)$. Taking into account the local expression of $\xi$, we deduce that $\xi$ is a semispray which will be called the canonical semispray associated to the dynamical connection $\Gamma$.

Furthermore, we have $\Gamma^2 \ell = \ell$ and $\Gamma m = 0$. Thus $\Gamma$ acts on $L$ as an almost product structure and trivially on $M$. Since $M = \ker \Gamma$, $\Gamma$ is said to be an $f(3, -1)$-structure on $J^1(\mathbb{R}, M)$ of rank $2m$ and parallelizable kernel. Moreover, $\Gamma/L$ has eigenvalues $+1$ and $-1$. From (10) the eigenspaces corresponding to the eigenvalue $-1$ are the vertical subspaces $V_z$, $z \in J^1(\mathbb{R}, M)$. Recall that for each $z \in J^1(\mathbb{R}, M)$, $V_z$ is the set of all tangent vectors to $J^1(\mathbb{R}, M)$ at $z$ which are projected to $0$ by $T\pi$. Thus $V$ is a distribution given by $z \mapsto V_z$. The eigenspace at $z \in J^1(\mathbb{R}, M)$ corresponding to the eigenvalue $+1$ will be denoted by $H_z$ and called the strong-horizontal subspace at $z$. We have a canonical decomposition

$$T_z(J^1(\mathbb{R}, M)) = M_z \oplus H_z \oplus V_z,$$
and obviously, 
\[ T(J^1(R,M)) = \mathcal{M} \oplus H \oplus V, \] (13)
where \( H \) is the distribution \( z \mapsto H_z \).

Let us put \( H'_z = M_z \oplus H_z \); \( H'_z \) will be called the weak-horizontal subspace at \( z \). Then we have the following decompositions
\[ T_z(J^1(R,M)) = H'_z \oplus V_z, \quad z \in J^1(R,M) \]
and
\[ T(J^1(R,M)) = H' \oplus V, \] (14)
where \( H' : x \rightarrow H'_z \) is the corresponding distribution.

We notice that \( L, \mathcal{M}, H \) and \( H' \) may be considered as vector bundles over \( J^1(R,M) \); the bundles \( H \) and \( H' \) will be called strong and weak-horizontal bundles, respectively. Thus, from (14) \( \Gamma \) defines a connection on the fibration \( \pi : J^1(R,M) \rightarrow R \times M \) with horizontal bundle \( H' \) (see Roux [R] and de Leon & Rodrigues [DLR 1]). But not every connection on the fibration \( \pi : J^1(R,M) \rightarrow R \times M \) arises in this way.

A vector field \( X \) on \( J^1(R,M) \) which belongs to \( H \) (resp. \( H' \)) will be called a strong (resp. weak) horizontal vector field. From (14), we have that the canonical projection \( \pi : J^1(R,M) \rightarrow R \times M \) induces an isomorphism
\[ \pi_* : H'_z \rightarrow T_{\pi(z)}(R \times M), \quad z \in J^1(R,M). \]
Then, if \( X \) is a vector field on \( R \times M \), there exists a unique vector field \( X^{H'} \) on \( J^1(R,M) \) which is weak-horizontal and projects to \( X \). The projection of \( X^{H'} \) to \( H \) will be denoted by \( X^H \).

From (10) and by a straightforward computation, we obtain
\[ (\partial/\partial t)^{H'} = \partial/\partial t + (\Gamma^j + \frac{1}{2} y^j \Gamma^j_i) \partial/\partial y^j \]
\[ (\partial/\partial x^i)^{H'} = \partial/\partial x^i + \frac{1}{2} \Gamma^j_i \partial/\partial y^j. \] (15)

Then, if we put \( H_i = (\partial/\partial x^i)^{H'} \) and \( V_i = \partial/\partial y^i \), one deduces that \( \{\xi, H_i, V_i\} \) is a local basis of vector fields on \( J^1(R,M) \). In fact, \( \mathcal{M} = \langle \xi \rangle \), \( H = \langle H_i \rangle \), and \( V = \langle V_i \rangle \); \( \{\xi, H_i, V_i\} \) is called an adapted basis to the \( f(3,-1) \)-structure \( \Gamma \). In terms of \( \{\xi, H_i, V_i\} \) (15) becomes
\[ (\partial/\partial t)^{H'} = \xi - y^i H_i, \quad (\partial/\partial x^i)^{H'} = H_i. \]
Therefore, we obtain

\[(\partial/\partial t)^H = -y^i H_i, \quad (\partial/\partial x^i)^H = H_i.\]

If \(X = \tau \partial/\partial t + X^i \partial/\partial x^i\) is a vector field on \(R \times M\), we have

\[X^H = (X^i - \tau y^i)H_i\]  \hspace{1cm} (16)

(compare with CRAMPIN, PRINCE and THOMPSON [CPT]). Finally, we notice that the dual local basis of 1-forms of the adapted basis \(\{\xi, H_i, V_i\}\) is given by \(\{dt, \theta^i, \psi^i\}\), where \(\theta^i = dx^i - y^i dt\), and \(\psi^i = -\left(\Gamma^i + \frac{1}{2} y^j \Gamma^j_1 dx^j + dy^i\right)\). This fact can be shown by a straightforward computation.

Let \(\xi\) be a semispray on \(J^1(R, M)\) and suppose that \(\xi\) is locally expressed by

\[\xi = \partial/\partial t + y^i \partial/\partial x^i + \xi^i \partial/\partial y^i.\]  \hspace{1cm} (17)

Then a simple computation in local coordinates shows that

\[
\begin{align*}
\{\xi, \partial/\partial t\} &= -\frac{\partial \xi^i}{\partial t} \partial/\partial y^i, \\
\{\xi, \partial/\partial x^i\} &= -\frac{\partial \xi^i}{\partial x^i} \partial/\partial y^i, \\
\{\xi, \partial/\partial y^i\} &= -\frac{\partial}{\partial x^i} - \frac{\partial \xi^i}{\partial y^i} \partial/\partial y^j.
\end{align*}
\]  \hspace{1cm} (18)

**Proposition (1).** — Let \(\Gamma = -L_{\xi} \tilde{\theta}\). Then \(\Gamma\) is a dynamical connection on \(J^1(R, M)\) whose associated semispray is, precisely, \(\xi\).

**Proof.** — In fact from (18) we have

\[
\begin{align*}
\Gamma(\partial/\partial t) &= -y^i \partial/\partial x^i - \left(y^j \frac{\partial \xi^i}{\partial y^j} - \xi^i\right) \partial/\partial y^i, \\
\Gamma(\partial/\partial x^i) &= \partial/\partial x^i + \frac{\partial \xi^i}{\partial y^i} \partial/\partial y^i, \\
\Gamma(\partial/\partial y^i) &= -\partial/\partial y^i.
\end{align*}
\]  \hspace{1cm} (19)

Now, from (19) we easily deduce that \(\Gamma\) is a dynamical connection on \(J^1(R, M)\). Furthermore, taking into account (12), we have that the associated semispray to \(\Gamma\) is, precisely, \(\xi\).
From Proposition (1) we may observe that the theory of dynamical connections on $J^1(R, M)$ is more simpler than the theory of connections on $TM$.

Let $\Gamma$ be a dynamical connection on $J^1(R, M)$.

**Definition (2).** A curve $u : R \rightarrow M$ is called a path of $\Gamma$ if the canonical prolongation $j^1u$ of $u$ to $J^1(R, M)$ is a weak-horizontal curve.

Now, we shall find the differential equations for the paths of $\Gamma$ (the dots meaning time derivatives).

If $u : R \rightarrow M$ is locally given by $t \mapsto (x^i(t))$, then we have $j^1u(t) = (t, x^i(t), \dot{x}^i(t))$. Hence,

$$\frac{\dot{j^1u(t)}}{\dot{t}} = \frac{\partial}{\partial t} + \frac{dx^i}{dt} \frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2} \frac{\partial}{\partial y^i}.$$ 

Therefore, $u$ is a path of $\Gamma$ if and only if $\psi^i(j^1u(t)) = 0, 1 \leq i \leq m$, that is $u$ satisfies the following system of differential equations:

$$\frac{d^2x^i}{dt^2} = \Gamma^i(t, x, \frac{dx}{dt}) + \Gamma^i_j(t, x, \frac{dx}{dt}) \frac{dx^j}{dt}. \quad (20)$$

Let $\xi$ be the associated semispray of $\Gamma$. Then $\xi$ is locally given by

$$\xi = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial y^i},$$

where $\xi^i = \Gamma^i + y^j \Gamma^i_j, 1 \leq i \leq m$.

From (20) it is clear that the paths of $\Gamma$ and $\xi$ satisfy the same system of differential equations. Then we have

**Proposition (2).** A dynamical connection and its associated semispray on $J^1(R, M)$ have the same paths.

4. Dynamical connections and non-autonomous regular Lagrangian equations

Suppose that a non-autonomous regular Lagrangian $L$ is given, that is, $L$ is a non-degenerate real function on $J^1(R, M) = R \times TM$. Then it is
well-known that an extremal for $L$ is a curve $s : \mathbb{R} \rightarrow M$ (or a section of $(\mathbb{R} \times M, p, \mathbb{R})$) such that

$$(\tilde{s})^*(i_X \, dL \wedge dt) = 0$$  \hspace{1cm} (21)$$

for all vertical vector fields on $\mathbb{R} \times TM$. Also, it is known that (21) is equivalent to

$$(\tilde{s}^2)^*(i_X \, d\Omega_L) = 0,$$  \hspace{1cm} (22)$$

for all $\pi_1$-vertical vector fields on $J^1(\mathbb{R}, M)$. In (22) $\Omega_L$ is the POINCARE-CARTAN canonical form on $J^1(\mathbb{R}, M)$ locally given by

$$\Omega_L = L(t, x, y) dt + \frac{\partial L}{\partial y^i} \theta^i,$$

where $\theta^i$ is defined in (7) of section 2.

In terms of the tensor field $\tilde{J}$ and $J$ and the Liouville vector field $C$ on $J^1(\mathbb{R}, M)$, the POINCARE-CARTAN form takes the following expression:

$$\Omega_L = L \, dt + \frac{\partial L}{\partial y^i} \theta^i = L \, dt + d\tilde{J}L,$$

or equivalently,

$$\Omega_L = L \, dt + \frac{\partial L}{\partial y^i} \, dx^i - y^i \frac{\partial L}{\partial y^i} \, dt = \left( L - y^i \frac{\partial L}{\partial y^i} \right) \, dt + dJL$$

$$= (L - CL) dt + DJL - E_L dt; \quad E_L = CL - L.$$

Thus

$$\Theta_L = d\Omega_L = ddJL + dL \wedge dt$$

or

$$\Theta_L = ddJL - dE_L \wedge dt.$$

A straightforward computation in local coordinates shows that

$$\Theta_L \Lambda \cdots \Lambda \Theta_L = \pm \det \left( \frac{\partial^2 L}{\partial y^i \partial y^j} \right) dx^1 \Lambda \cdots \Lambda dx^m \Lambda dy^1 \Lambda \cdots \Lambda dy^m$$

and if $L$ is a non-autonomous regular Lagrangian we deduce that $\Theta_L$ is a contact form on $J^1(\mathbb{R}, M)$. Consequently, the characteristic bundle of $\Theta_L$

$$R_{\Theta_L} = \{ v \in T(J^1(\mathbb{R}, M)); i_v \Theta_L = 0 \}$$

-  179 -
has one-dimensional fibers, that is, they are a line-bundle over $J^1(\mathbb{R}, M)$. Let us recall here that a vector field $X$ on $J^1(\mathbb{R}, M)$ is characteristic if $X$ is a section of $R_X \Theta_L$, that is, $i_X \Theta_L = 0$. The following result can be compared with the corresponding one for autonomous Lagrangian [see [DLR 1]].

**Proposition (3).** — Let $L$ be a non-autonomous regular Lagrangian on $J^1(\mathbb{R}, M)$ and $\xi$ a characteristic vector field which satisfies $i_\xi dt = 1$. Then $\xi$ is a semispray on $J^1(\mathbb{R}, M)$ whose paths are the solutions of the Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad 1 \leq i \leq m.$$ 

We call $\xi$ the Lagrange vector field for $L$.

**Theorem (1).** — Let $L$ be a non-autonomous regular Lagrangian on $J^1(\mathbb{R}, M)$ and let $\xi$ be a Lagrange vector field for $L$. Then there exists a dynamical connection $\Gamma$ on $J^1(\mathbb{R}, M)$ whose paths are the solutions of the Lagrange equations. This connection is given by $\Gamma = -\mathcal{L}_\xi \tilde{J}$.

**Proof.** — From Proposition (1) we deduce that $\Gamma = -\mathcal{L}_\xi \tilde{J}$ is a dynamical connection whose associated semispray is precisely $\xi$. Thus the theorem follows directly from Proposition (2) and (3).

Finally, let us remark that the results of CRAMPIN, PRINCE and THOMPSON [CPT] can be re-obtained in terms of $\Gamma$. In fact, with the notation of Section 3 we have a local basis of vector fields on $J^1(\mathbb{R}, M)$ given by $\{\xi, H_i, V_i\}$ where $H_i$ is given by

$$H_i = \frac{\partial}{\partial x^i} + \frac{1}{2} \frac{\partial \xi^j}{\partial y^j} \frac{\partial}{\partial y^i}.$$ 

Thus the corresponding dual basis is $\{dt, \theta^i, \psi^i\}$, where

$$\psi^i = -\left( \xi^i - \frac{1}{2} y^j \frac{\partial \xi^j}{\partial y^i} \right) dt - \frac{1}{2} \frac{\partial \xi^i}{\partial y^j} dx^j + dy^i.$$ 

The significance of this dual basis is that the form $\Theta_L$ can be re-written as follows

$$\Theta_L = \frac{\partial^2 L}{\partial y^i \partial y^j} \theta^i \Lambda \psi^j$$

and so the semispray $\xi$ is uniquely determined by the equations

$$i_\xi \theta^i = i_\xi \psi^i = 0, \quad i_\xi dt = 1.$$
Dynamical connections and non-autonomous Lagrangian systems

References


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