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RÉSUMÉ. — Soit $\mu$ une mesure de probabilité sur une partie borélienne bornée non pluripolaire $E$ de $C^N$, on étudie l’allure de croissance des familles de polynômes ponctuellement bornées $\mu$-presque partout sur $E$. On définit une fonction $\mathcal{M}(t; E, \mu)$ ($0 \leq t \leq 1$) associée au couple $(E, \mu)$. Sous des hypothèses naturelles sur $E$ et $\mu$, on montre que $\mathcal{M}(1; E, \mu) = 1$ si et seulement si le couple $(E, \mu)$ satisfait à la condition polynomiale $(\mathcal{L}^*)$, généralisant la condition polynomiale de Leja dans le cas plan, si et seulement si $\mu$ est une mesure déterminante pour $E$ par rapport à la fonction $L^*_E$.

ABSTRACT. — Given a probability measure $\mu$ on a bounded nonpluripolar Borel subset $E$ of $C^N$, we study the growth behaviour of polynomial families which are pointwise bounded $\mu$-a.e. on $E$. We define a function $\mathcal{M}(t, E, \mu)$ ($0 \leq t \leq 1$) associated to the pair $(E, \mu)$. Under natural assumptions on $E$ and $\mu$ we prove that $\mathcal{M}(1; E, \mu) = 1$ if and only if the pair $(E, \mu)$ satisfies the polynomial condition $(\mathcal{L}^*)$ (a generalization of the Leja’s condition in the plane), if and only if $\mu$ is determining for $E$ with respect to the $L$-extremal function $L^*_E$.

0 - Introduction

Given a domain $\Omega$ in $C^N$, we denote by $P(\Omega)$ the class of plurisubharmonic (plsh) functions on $\Omega$. Let

$$\mathcal{L} := \{ u \in P(C^N); u(z) \leq \beta + \log(1 + |z|) \text{ in } C^N \},$$

where $\beta$ is a real constant depending on $u$. For a bounded set $E$ in $C^N$ define

$$L_E(z) := \sup \{ u(z); u \in \mathcal{L}, u \leq 0 \text{ on } E \}. \tag{0.1}$$

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The uppersemicontinuous regularization $L^*_E(z) := \limsup_{w \to z} L_E(w)$ is called the $L$-extremal function of $E$. It is known that if $E$ is a compact subset of $\mathbb{C}$ with positive logarithmic capacity then $L_E$ is identical with the Green function for $\mathbb{C} \setminus \overline{E}$ with pole at infinity.

For a bounded set $E$ in $\mathbb{C}^N$ either $L^*_E \equiv \infty$, in which case $E$ is pluripolar (plp), or $L^*_E \in \mathcal{L}$.

**Definition 0.1.** We say that a point $a$ in $\mathbb{C}^N$ is an $L$-regular point of $E \subset \mathbb{C}^N$, if $L^*_E(a) = 0$. A point $a \in \mathbb{C}^N$ such that $L_E(a) = 0$ and $L^*_E(a) > 0$ is called irregular point of $E$. It is clear that $L^*_E$ is continuous at each regular point. By Bedford-Taylor theorem on negligible sets the set of irregular points of any subset $E$ of $\mathbb{C}^N$ is plp. If $E$ is a compact set and $L_E = L^*_E$ on $E$ (i.e. if $L^*_E$ is continuous at each point of $E$) then $L_E$ is continuous in $\mathbb{C}^N$ and $L_E = L^*_E$. A compact set $E$ with $L^*_E = L_E$ is called $L$-regular. The set of $L$-regular points of a compact $L$-regular set $E$ is identical with the polynomially convex hull $\hat{E}$ of $E$.

**Definition 0.2.** A finite positive Borel measure $\mu$ on a bounded Borel set $E$ in $\mathbb{C}^N$ is called determining for $E$, if for every Borel subset $F$ of $E$ with $\mu(F) = \mu(E)$ one has $L^*_F = L^*_E$.

Observe that if $L^*_E = L_E$ and $\mu$ is determining for $E$, then for every $F \subset E$ with $\mu(F) = \mu(E)$ one has $L_F = L_E$ (because $L^*_E = L^*_E = L_E \leq L_F$).

It is known that $L^*_{E \cup A} = L^*_E$, if $A$ is plp. Therefore $L^*_F = L^*_F$ for a subset $F$ of $E$ if and only if $L^*_F = 0$ quasi-almost everywhere (q.a.e.) on $E$. We say that a property $\mathcal{P}$ holds q.a.e. on $E$, if it holds for each point of $E$ except at most of a plp subset of $E$.

We say that a property $\mathcal{P}$ holds quasi-star-almost-everywhere (q*.a.e.) on $E$, if it holds for each point of a subset $F$ of $E$ with $L^*_F \leq L^*_E$.

It is clear that if $\mu$ is determining for $E$ and $\mathcal{P}$ holds $\mu$-a.e. on $E$ then it holds q*.a.e. on $E$.

**Definition 0.3.** Let $\mu$ be a finite positive Borel measure on a bounded Borel set in $\mathbb{C}^N$. We say that the pair $(E, \mu)$ satisfies ($L^*$)-condition at a point $a$ of $\mathbb{C}^N$, if for every family $\mathcal{F}$ of polynomials of $N$-complex variables and for every number $b > 1$ the polynomial family

$$\mathcal{F}_b := \{b^{-\deg f} f; f \in \mathcal{F}\} \tag{0.2}$$

is uniformly bounded on a neighborhood $U$ of $a$. 

- 194 -
We say that the pair \((E, \mu)\) satisfies \((\mathcal{L}^*)\)-condition, if for every \(b > 1\) and for every polynomial family \(\mathcal{F}\) bounded \(\mu\)-a.e. on \(E\) the family \(\mathcal{F}_b\) is uniformly bounded on a neighborhood of \(E\).

It is clear that if \(E\) is compact then \((E, \mu)\) satisfies \((\mathcal{L}^*)\)-condition, if and only if it satisfies \((\mathcal{L}^*)\) at each point of \(E\).

All these notions are important for applications of the extremal function \(\mathcal{L}^*_E\). There are strict relations between them. Also are known important examples of pairs \((E, \mu)\) satisfying \((\mathcal{L}^*)\) and of determining measures (e.g. \([2], [5], [6], [7], [8], [14]\)).

In this paper we introduce a new function \(M(t) \equiv M(t; E, \mu)\) associated to every pair \((E, \mu)\) by the formula

\[
\log M(t) := \sup \left\{ \sup_{E} L_A; A \subset E, \mu(A) \geq t \mu(E) \right\}, \ 0 \leq t \leq 1.
\]

It is clear that \(M\) is a decreasing function and \(1 \leq M(t) \leq +\infty\). The function \(M^*(t) := \lim_{\tau \uparrow t} M(\tau) \ (0 < t \leq 1), \ M^*(0) := M(0)\), is decreasing and uppersemicontinuous on \([0, 1]\).

In the sequel we shall often assume (without loss of generality) that \(\mu\) is a probability measure (i.e. \(\mu(E) = 1\)).

The function \(M\) appears to be a useful notion strictly related to the determining measures and the \((\mathcal{L}^*)\)-condition. For example we have obtained the following results involving the function \(M\).

**Theorem A.** — If \(E \subset \mathbb{C}^N\) is compact and \(\mu\) vanishes on plp subsets of \(\mathbb{C}^N\) then the following conditions are equivalent

(i) The pair \((E, \mu)\) satisfies \((\mathcal{L}^*)\)-condition;

(ii) If \(u \in \mathcal{L} \) and \(u \leq 0 \mu\)-a.e. on \(E\), then \(u \leq 0 \) on \(E\);

(iii) \(M^*(1) = 1\);

(iv) \(M(1) = 1\);

(v) \(\mu\) is determining for \(E\) and \(E\) is \(\mathcal{L}\)-regular.

**Theorem B.** — Let \(A \subset \mathbb{C}^P, B \subset \mathbb{C}^Q\) be two bounded Borel sets and \(\mu, \nu\) two probability measures on \(A\) and \(B\), respectively. Put \(M_A(t) := M(t; A, \mu), M_B(t) := M(t; B, \nu)\) and \(M_{A \times B}(t) := M(t, A \times B, \mu \otimes \nu)\). Then

(i) \(M_{A \times B}(1) \leq M_A(1) M_B(1)\)


(ii) \( M^*_A(1) M^*_B(1) \leq M^*_{A \times B}(1) \)

**Corollary.** If \( A \subset \mathbb{C}^P, B \subset \mathbb{C}^P \) are compact sets and \( \mu, \nu \) vanish on plp sets, then if the pairs \((A, \mu), (B, \nu)\) satisfy one of the equivalent conditions of Theorem A then the pair \((A \times B, \mu \otimes \nu)\) satisfies each of the conditions.

The equivalence of the conditions (i) and (v) was earlier obtained by Levenberg [6]. Nguyen Thanh Van formulated the \((L^*)\)-condition in his paper [7]; his definition was inspired by Leja’s paper [5] containing as a special case so called “Polynomial Lemma”, which in the present language reads as follows:

Let \( \Gamma \) be a rectifiable curve in the complex plane and let \( \lambda \) be the length measure on \( \Gamma \). Then the pair \((\Gamma, \mu)\) satisfies \((L^*)\).

It is worthwhile to mention that the Leja’s paper [5] permits immediately to obtain the following estimate for the function \( M(t) \equiv M(t; [a, b], \lambda) \):

\[
M(t) \leq J \left( \frac{1 - t}{t - 9/10} \right), \quad 9/10 < t < 1,
\]

where

\[
J(\alpha) := \exp \int_0^1 \log \frac{x^2 + \alpha^2}{x^2} \, dx \leq \exp \alpha (\pi + \alpha).
\]

The exact formula for the function \( M(t; [a, b], \lambda) \), where \([a, b]\) is a bounded interval of the real line \( \mathbb{R} \) and \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \), reads as follows

\[
M(t; [a, t], \lambda) = 2t^{-1} - 1 + 2t^{-1} \sqrt{1 - t}, \quad 0 \leq t \leq 1,
\]

and may be easily derived from the following inequality due to Dudley and Randol [4]

\[
\|f\|_{[a, b]} / \|f\|_A \leq (2^{-1} - 1 + 2t^{-1} \sqrt{1 - t})^{\deg f}
\]

true for every polynomial \( f \) of a complex variable and for every compact set \( A \subset [a, b] \) with \( \lambda(A) \geq t(b - a), 0 \leq t \leq 1 \).

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1 - Determining measures for arbitrary bounded Borel subsets of \( \mathbb{C}^N \)

Let us start with the following.

**LEMMA 1.1.** Let \( F \) be a subset of a bounded set \( E \) in \( \mathbb{C}^N \). If the set

\[
G := \{ z \in E \setminus F; L_P^* (z) > 0 \}
\]

is not plp, then there exist a nonplp subset \( G_0 \) of \( G \), a number \( b > 1 \) and a polynomial family \( \mathcal{F} \) such that

1) \( \mathcal{F} \) is bounded at each point of \( F \).

2) \( \mathcal{F}_b \) given by (0.2) is unbounded at each point \( z \in G_0 \).

**Proof.** It is known that \( F_1 := \{ z \in F; L_P^* (z) > 0 \} \) is plp, so there exists a function \( w \) in the class \( \mathcal{L} \) with \( w = -\infty \) on \( F_1 \) and \( w \leq -\log 2 \) on \( E \). It is also known \([10]\) that \( w \) can be represented in the form

\[
w = \left( \limsup_{m \to \infty} \frac{1}{m} \log |P_m| \right)^*,
\]

(1.1)

where \( P_m \) is a polynomial on \( \mathbb{C}^N \) of degree \( \leq m \). We shall consider two cases: either \( F_1 = F \), or \( F_1 \neq F \).

Case \( F_1 = F \). By Bedford-Taylor theorem on negligible sets \([1]\) the set

\[
\left\{ \limsup_{m \to \infty} \sqrt[m]{|P_m|} < \left( \limsup_{m \to \infty} \sqrt[m]{|P_m|} \right)^* \right\}
\]

is plp. Hence there exists a non pluripolar subset \( G' \) of \( G \) such that

\[-\infty < w(z) = \limsup_{m \to \infty} \frac{1}{m} \log |P_m(z)| \text{ for } z \in G'.\]

There is a real number \( \epsilon \) with \( 0 < \epsilon < 1 \) such that the set \( G_0 := \{ x \in G'; w(z) \geq \log \epsilon \} \) is not plp. Take any \( b \) with \( 1 < b < 2 \). Then the family

\[
\mathcal{F} := \left\{ \left( \frac{2}{\epsilon b} \right)^m P_m; m \geq 1 \right\}
\]

has the required properties. Indeed, 1) is satisfied because

\[
\limsup_{m \to \infty} \sqrt[m]{\left( \frac{2}{\epsilon b} \right)^m |P_m(z)|} = 0 \text{ on } F.
\]
If $z \in G_o$, we have

$$\limsup_{m \to \infty} \sqrt[m]{\frac{2}{\epsilon b}} \left| P_m(z) \right| = \frac{2}{\epsilon b} \exp w(z) > \frac{2}{b} > 1,$$

which implies 2).

Case $F_1 \neq F$. Since $F_1 \neq F$, we have $L_F^* \in \mathcal{L}$ and $u_k := \frac{1}{k} w + \frac{k-1}{k} L_F^* \in \mathcal{L}$ for every $k \geq 1$. If $z \in G$ and $w(z) > 0$ the sequence $u_k(z)$ is increasing to the limit $L_F^*(z) > 0$. Therefore there exists $k$ such that the set $G_k := \{z \in G; u_k(z) > 0\}$ is not plp. For such $k$ there is $\epsilon > 0$ such that

$$G' := \{z \in G; u_k(z) \geq \log(1 + \epsilon)\}$$

is not plp. Write $u_k$ in the form

$$u_k = \left( \limsup_{j \to \infty} \frac{1}{j} \log |P_j| \right)^* \quad (\deg P_j \leq j)$$

By the theorem on negligible sets there is a non pluripolar subset $G_o$ of $G'$ with

$$u_k(z) = \frac{1}{k} w(z) + \frac{k-1}{k} L_F^*(z) = \limsup_{j \to \infty} \frac{1}{j} \log |P_j(z)|, z \in G_o.$$ 

The set $G_o$, any number $b$ with $1 < b < 1 + \epsilon$ and the polynomial family $\mathcal{F} := \{P_j; j \geq 1\}$ have the required property. Indeed $\limsup_{j \to \infty} \sqrt{j |P_j(z)|} \leq \exp u_k(z) \leq 2^{-k} < 1$ on $F$, which gives 1). On the other hand, if $z \in G_o$ then

$$\limsup_{j \to \infty} \sqrt{j |P_j(z)|} = b^{-1} \exp u_k(z) \geq \frac{1 + \epsilon}{b} > 1,$$

which implies 2).

**Lemma 1.2.**—If a polynomial family $\mathcal{F}$ is bounded q.a.e. on a subset $E$ of $\mathbb{C}^N$, then for every $b > 1$ the family $\mathcal{F}_b$ is bounded q.a.e. on $E$ and uniformly on a neighborhood of every $L$-regular point $a$ of $E$. If $E$ is compact and $L$-regular, and $\mathcal{F}$ is bounded q.a.e. on $E$ then for each $b > 1$ the family $\mathcal{F}_b$ is uniformly bounded on a neighborhood of $E$.

**Proof.**—Without loss of generality we can assume $E$ is not plp. Let $\mathcal{F}$ be a polynomial family bounded at each point of a subset $F$ of $E$ with $L_F^* = L_E^*$. Put

$$E_j := \{z \in E; |f(x)| \leq j, \forall f \in \mathcal{F}\}, j \geq 1 \quad (1.2)$$
Then $E_j \subseteq E_{j+1}$ and $F \subseteq E_0 := \bigcup_{1}^{\infty} E_j$. Hence $L_{E_j}^* \downarrow L_{E_0}^* = L_F^* = L_E^*$. By the definition of the $L$-extremal function we have

$$|f(z)| \leq j \left( \exp L_{E_j}^*(z) \right)^{\deg f}, \quad z \in \mathbb{C}^N, \ j \geq 1, \ f \in \mathcal{F} \quad (1.3)$$

which implies that for each $b > 1$ the family $\mathcal{F}_b$ is bounded at every $L$-regular point of $E$. So $\mathcal{F}_b$ is bounded q.a.e. on $E$. Moreover, if $L_E^*(a) = 0$, then $L_E^*(z) < b$ on a ball $|z-a| \leq r$. By Dini’s argument there is $j$ sufficiently large with $L_{E_j}^*(z) < b$ on the ball $|z-a| \leq r$, which implies by (1.3) that the family $\mathcal{F}_b$ is uniformly bounded on a ball $|z-a| < r$, if $a$ is any $L$-regular point of $E$. The proof of the remaining part of Lemma 1.2. is trivial.

**Theorem 1.3.** — Given a probability measure on a bounded Borel set $E$ in $\mathbb{C}^N$ the following conditions are equivalent.

I. The measure $\mu$ is determining for $E$;

II. If $u \in L$ and $u \leq 0$ $\mu$-a.e. on $E$, then $u \leq 0$ q.a.e. on $E$;

III. If $\mathcal{F}$ is a polynomial family bounded $\mu$-a.e. on $E$, then for every $b > 1$ the family $\mathcal{F}_b$ is bounded q.a.e. on $E$.

**Proof.** $I \Rightarrow II$. Let $u$ be a fixed function in the class $L$ with $u \leq 0$ $\mu$-a.e. on $E$. Put $F := \{z \in E; u(x) \leq 0\}$. Then $u(z) \leq L_F^*(z) = L_E^*(z)$. Hence $u \leq 0$ q.a.e. on $E$.

$I \Rightarrow III$. Let $\mathcal{F}$ be a polynomial family bounded $\mu$-a.e. on $E$. Let $E_j$ be given by (1.2). Then $E_j \subseteq E_{j+1}$ and $\mu(F) = \mu(E)$ for $F := \bigcup_{1}^{\infty} E_j$. By $L_{E_j}^* = L_{E_0}^*$. It is known [12] that $L_{E_j}^* \downarrow L_{E_0}^*$ as $j \to \infty$. Hence by (1.3) the family $\mathcal{F}$ is bounded q.a.e. on $E$, and by Lemma 1.2. the family $\mathcal{F}_b$ is bounded q.a.e. on $E$ for every $b > 1$.

The implication $III \Rightarrow I$ follows directly from Lemma 1.1.

It remains to show that $II \Rightarrow I$. Fix $F \subseteq E$ with $\mu(F) = \mu(E)$ and let $u$ be a function of the class $L$ such that $u \leq 0$ on $F$. Then $u \leq 0$ q.a.e. on $E$. Hence $u \leq L_{E}^*$ in $\mathbb{C}^N$. By the arbitrariness of $u$ we get $L_{E_j}^* \leq L_{E_0}^*$, which gives $L_{E_j}^* = L_{E_0}^*$, because $L_T^* \leq L_{E_j}^*$. Because $L_E^* \leq L_{E_0}^*$.

**2 - The function $\mathcal{M}(t; E, \mu)$**

Given a probability measure $\mu$ on a bounded Borel set $E$ in $\mathbb{C}^N$ the function $\mathcal{M}$ is defined by the formula

$$\log \mathcal{M}(t) := \sup \left\{ \sup_{E} L_A; A \subseteq E, \ \mu(A) \geq t \right\}, \quad 0 \leq t \leq 1 \quad (2.1)$$
It is clear that $1 \leq \mathcal{M}(t_2) \leq \mathcal{M}(t_1) \leq +\infty$ if $0 \leq t < t_2 \leq 1$. The function 
$\mathcal{M}^*(t) := \limsup_{\tau \to t} \mathcal{M}(\tau)$ is also decrasing. It follows from (0.1) that

$$\log \mathcal{M}(t) \leq \sup \left\{ \sup_{E} u; u \in \mathcal{L}, u \leq 0 \text{ on } A, A \subset E, \mu(A) \geq t \right\},$$

which implies

$$\sup_{E} u - \sup_{A} u \leq \log \mathcal{M}(\mu(A)), \text{ if } u \in \mathcal{L}, A \subset E,$$

$$L_A(z) \leq \log \mathcal{M}(\mu(A)) + L_{E}(z), \text{ } z \in C^N, \text{ } A \subset E,$$

where $A$ is any Borel subset of $E$.

Remark 2.1. If $\mu$ vanishes on plp sets then

$$\mathcal{M}(t) \equiv \mathcal{M}_1(t) := \sup \left\{ \sup_{E} \exp L_A^*; A \subset E, \mu(A) \geq t \right\}.$$  

Indeed, it is clear that $\mathcal{M} \leq \mathcal{M}_1$. In order to prove the opposite inequality observe that given $t$ with $0 \leq t \leq 1$ and $m \in \mathbb{R}$ with $m < \mathcal{M}_1(t)$ these exists $A \subset E$ such that $\mu(A) \geq t$ and $\sup_E L_A^* > \log m$. Put

$$A_0 := \{ z \in A; L_A(z) = L_A^*(z) \}.$$ 

Then $\mu(A_0) = \mu(A) \geq t$ and $L_A^* \leq L_{A_0}$. Hence $\log m < \sup_E L_A^* \leq \sup_E L_{A_0} \leq \mathcal{M}(t)$. By the arbitrariness of $m$ we get $\mathcal{M}_1(t) \leq \mathcal{M}(t)$.

Remark 2.2. If $\mathcal{M}^*(1) = 1$, then $\mathcal{M}$ is continuous at $t = 1$ and $\mathcal{M}(1) = 1$. Hence, if $\mathcal{M}^*(1) = 1$ then $L_{A_n} \to L_E$ for every sequence $A_n$ of Borel subsets of $E$ such that $\mu(A_n) \to \mu(E)$.

Remark 2.3. If $\mu$ vanishes on plp sets and $\mathcal{M}(1) = 1$ then $L_E^* = 0$ on $E$. In particular, if $E$ is compact and $\mathcal{M}(1) = 1$ then $E$ is $L$-regular. Indeed, put

$$E_0 := \{ z \in E; L_E^*(z) = 0 \}.$$ 

Then $\mu(E_0) = 1$ and $L_E^* \leq L_{E_0} \leq \log \mathcal{M}(1) = 0$ on $E$, i.e. $L_E^* = 0$ on $E$. It is clear that the pairs $(E, \mu)$ for which $\mathcal{M}(1) = 1$ or $\mathcal{M}^*(1) = 1$ are of great importance for applications of the $L$-extremal function.

PROPOSITION 2.4. Define

$$\mathcal{M}_P(t) := \sup \left\{ \frac{\|f\|_E/\|f\|_A)^{1/\deg f}}{; \deg f \geq 1, A \subset E, \mu(A) \geq t} \right\}$$

where the sup is taken over all polynomials $f$ of degree $\geq 1$ and over all compact sets $A \subset E$. If $E$ is compact, then

$$\mathcal{M}_P(t) \leq \mathcal{M}(t) \leq \mathcal{M}_P(t), \text{ } 0 \leq t \leq 1$$

- 200 -
Proof. — Fix $t$ with $0 \leq t \leq 1$. Given any number $m$ with $m < \mathcal{M}(t)$, take $A \subset E$ with $\mu(A) \geq t$ and $\sup_E L_A > \log m$. Next choose $u$ in the class $\mathcal{L}$ with $u \leq 0$ on $A$ and $\sup_E u > \log m$. By the Approximation Lemma [13] there is a sequence $u_\nu := \max \frac{1}{n_j} \log |f_j|$, where $f_j$ is a polynomial of degree $\leq n_j$, such that $u_\nu \downarrow u$ as $\nu \to \infty$. Given $\epsilon > 0$ there exists a compact set $K \subset A$ with $\mu(K) \geq t - \epsilon$. Take $\nu$ so large that $\sup_K u_\nu < \epsilon$ and choose $j$ with $\sup_E \frac{1}{n_j} \log |f_j| > \log m$. Then $m e^{-\epsilon} \leq (\|f_j\|_E/\|f_j\|_K)^{1/n_j} \leq \mathcal{M}_P(\mu(K)) \leq \mathcal{M}_P(t - \epsilon)$. Hence $m \leq \mathcal{M}_P(t)$. By the arbitrariness of $m$ we get $\mathcal{M}(t) \leq \mathcal{M}_P(t)$. The inequality $\mathcal{M}_P(t) \leq \mathcal{M}(t)$ is obvious.

**Proposition 2.5.** — If $E$ is nonplp compact set in $\mathbb{C}^N$ then $\mathcal{M}_P(t) = \lim_{n \to \infty} B_n^{1/n}(t) = \sup_{n \geq 1} B_n^{1/n}(t)$, where

$$B_n(t) := \sup \{\|f\|_E; \deg f \leq n, \|f\|_A = 1, A \subset E, \mu(A) \geq t\}.$$

Proof. — Given $m \geq 1$ and $c$ with $0 < c^m < B_m(t)$, let $A$ be a compact subset of $E$ with $\mu(A) \geq t$ and let $f_m$ be a polynomial of degree $\leq m$ such that $\|f_m\|_A = 1$ and $\|f_m\|_E > c^m$. Every natural number $n \geq m$ can be written in the form $n = km + r$ with $0 \leq r < m$. Observe that

$$c^km < \|f_m\|_E^k \leq B_n(t).$$

Hence $\liminf_{n \to \infty} B_n^{1/n}(t) \geq c$, which implies that $B_m^{1/m}(t) \leq \liminf_{n \to \infty} B_n^{1/n}(t)$, and consequently we get the required result.

**Theorem 2.6.** — Let $A \subset \mathbb{C}^p$, $B \subset \mathbb{C}^q$ be bounded Borrel sets and let $\mu, \nu$ be probability measures on $A$ and $B$, respectively. Put $\mathcal{M}_A(t) := \mathcal{M}(t; A, \mu)$, $\mathcal{M}_B(t) := \mathcal{M}(t; B, \nu)$ and $\mathcal{M}_{A \times B}(t) := \mathcal{M}(t; A \times B, \lambda)$ with $\lambda := \mu \otimes \nu$.

Then

(i) $\mathcal{M}_{A \times B}(1) \leq \mathcal{M}_A(1) \mathcal{M}_B(1)$

(ii) $\mathcal{M}_{A \times B}^*(1) \leq \mathcal{M}_A^*(1) \mathcal{M}_B^*(1)$

Proof. — (i) Let $E \subset A \times B$ with $\lambda(E) = 1$ and put $B^z := \{w \in B; (z, w) \in E\}$. Then $\nu(B^z) = 1 \mu$-a.e. on $A$. Let $u \in \mathcal{L}(\mathbb{C}^p \times \mathbb{C}^q)$, $u \leq 0$ on $E$. Then $u(z, w) \leq \log \mathcal{M}_B(1) + L_B(w)$ for all $z \in A_0$ and $w \in \mathbb{C}^q$, where $A_0 \subset A$ and $\mu(A_0) = 1$. Hence

$$u(z, w) \leq \log \mathcal{M}_B(1) + L_B(w) + \log \mathcal{M}_A(1) + L_A(z), (z, w) \in \mathbb{C}^p \times \mathbb{C}^q.$$
Hence by (2.2) one gets (i).

(ii) Let \( m \) be a fixed number with \( m < \log \mathcal{M}_{A\times B}^*(1) \). There is a sequence \( E_n \) of Borel subsets of \( A \times B \) such that \( \lambda(E_n) \geq 1 - 2^{-n} \) and \( \sup_{A \times B} L_{E_n} > m \). Define

\[
B_n^z := \{ w \in B; (z, w) \in E_n \}, \quad z \in A, \ n \geq 1
\]

\[
A_{\neq} := \{ z \in A; \nu(B_n^z) \geq 1 - \epsilon \}, \quad n \geq 1, \ 0 < \epsilon < 1
\]

We claim that \( \mu(A_{\neq}) \rightarrow 1 \) as \( n \rightarrow \infty \). Indeed,

\[
\lambda(E_n) = \int_A \nu(B_n^z) d\mu(z) = \int_{A_{\neq}} + \int_{A \setminus A_{\neq}} \leq \mu(A_{\neq}) + (1 - \mu(A_{\neq})) (1 - \epsilon) = 1 - \epsilon + \epsilon \mu(A_{\neq}).
\]

Hence \( \liminf_{n \rightarrow \infty} \mu(A_{\neq}) \geq 1 \), which implies the claim. Fix \( n \geq 1 \) and let \( u \) be a function of the class \( \mathcal{L}(C^p \times C^q) \) with \( u \leq 0 \) on \( E_n \). Then for every fixed \( z \) in \( A \) we have \( u(z, w) \leq 0 \) on \( B_n^z \). Therefore

\[
u(B_n^z) \leq \log \mathcal{M}_B(\nu(B_n^z)) + L_B(w)
\]

which implies

\[
u(B_n^z) \leq \log \mathcal{M}_B(1 - \epsilon) + L_B(w) \text{ for } z \in A_{\neq}, \ w \in C^q.
\]

Hence

\[
u(B_n^z) \leq \log \mathcal{M}_B(1 - \epsilon) + L_B(w) + \log \mathcal{M}_A(\mu(A_{\neq})) + L_A(z)
\]

for all \((z, w) \in C^p \times C^q\). By the arbitrariness of \( u \) we can replace \( u \) by \( L_{E_n}(z, w) \). Then we get

\[
m < \sup_{A \times B} L_{E_n} \leq \log \mathcal{M}_B(1 - \epsilon) + \log \mathcal{M}_A(\mu(A_{\neq})), \ n \geq 1, \ 0 < \epsilon < 1.
\]

After passing to the limits, first with \( n \) to \( \infty \) and next with \( \epsilon \) to \( 0 \), we get

\[
m \leq \log \mathcal{M}_B^*(1) + \log \mathcal{M}_A^*(1).
\]

By the arbitrariness of \( m \) we get (ii).

The following corollary is important for applications of the function \( \mathcal{M} \).

COROLLARY 2.7. — If \( \mathcal{M}_A(1) = 1, \mathcal{M}_B(1) = 1, \) (resp. \( \mathcal{M}_A^*(1) = 1, \mathcal{M}_B^*(1) = 1 \)) then \( \mathcal{M}_{A\times B}(1) = 1 \) (resp. \( \mathcal{M}_{A\times B}^*(1) = 1 \)).
Exemple 2.8. — Let $I = \{a, b\}$ be an interval of the real line $\mathbb{R}$ with end points $a, b$ such that $-\infty < a < b < +\infty$. Then

$$\mathcal{M}(t) \equiv \mathcal{M}(t; I, \lambda_1) = 2t^{-1} - 1 + 2t^{-1}\sqrt{1 - t}, \quad 0 \leq t \leq 1,$$

$\lambda_1$ denoting the Lebesgue measure on $\mathbb{R}$.

Proof. — Without loss of generality we may assume that $I = [a, b]$ is closed. By [4] for every polynomial $f$ of degree $\leq n$, $\|f\|_A/\|f\|_I \leq B_n(t)$, if $A \subset I$ and $\lambda_1(A) \geq t(b - a)$, where

$$B_n(t) := \frac{1}{2} \left[ (2t^{-1} - 1 + 2t^{-1}\sqrt{1 - t})^n + (2t^{-1} - 1 - 2t^{-1}\sqrt{1 - t})^n \right].$$

Moreover, if $A$ is a subinterval of $I$ with a common end point, this bound is best possible. Therefore by Proposition 2.5 we have

$$\mathcal{M}_p(t) = \sup_{n \geq 1} B_n^{1/n}(t) = 2t^{-1} - 1 + 2t^{-1}\sqrt{1 - t}, \quad 0 \leq t \leq 1.$$

By Proposition 2.4, $\mathcal{M}_p = \mathcal{M}$.

Remark 2.9. — If $\Omega$ is a bounded open set in $\mathbb{R}^N$ (resp. in $\mathbb{C}^N$) then for every determining measure $\mu$ for $\Omega$ one has $\mathcal{M}(1; \Omega, \mu) = 1$. Indeed, it is known [12] that $L_\Omega^* = L_\Omega$. So if $F \subset \Omega$ and $\mu(F) = 1$, then $L_F = L_\Omega$ which implies that $\mathcal{M}(1; \Omega, \mu) = 1$. As an example of such $\mu$ one can take the Lebesgue measure $\lambda_N$ in $\mathbb{R}^N$ (resp. $\lambda_2$ in $\mathbb{C}^N$).

If $\mu$ is a probability measure on $\Omega$ such that $\mathcal{M}^*(1; \Omega, \mu) = 1$, then the closure $E$ of $\Omega$, $E = \overline{\Omega}$, is an $\mathcal{L}$-regular compact. Indeed, let $K_n$ be an increasing sequence of $\mathcal{L}$-regular compact subsets of $\Omega$ such that $\mu(\Omega) = \lim_{n \to \infty} \mu(K_n)$ and $\Omega = \bigcup_{n \geq 1} K_n$. Then

$$\log \mathcal{M}(\mu(K_n)) \geq \sup_{\Omega} L_{K_n} = \sup_{E} L_{K_n} \geq \sup_{E} L_{E}, \quad n \geq 1,$$

which implies that $L_{E}^* = 0$ on $E$, i.e. $E$ is $\mathcal{L}$-regular.

Example 2.10. — We shall now construct a bounded open subset $\Omega$ of $\mathbb{C}$ with the following properties.

1) $E := \overline{\Omega}$ is $\mathcal{L}$-regular.

2) For every probability measure $\mu$ on $\Omega$, $\mathcal{M}^*(1; \Omega, \mu) > 1$.

3) There exists no finite positive Borel measure $\mu$ on $\Omega$ such that the pair $(\Omega, \mu)$ satisfies the $(\mathcal{L}^*)$-condition.
Indeed, let \( \{a_n\} \) be a discrete sequence in the upper half plane \( \{\text{Im } z > 0\} \) such that each point of \( I = [0,1] \) is a limit of a subsequence of \( \{a_n\} \) and the sequence \( \{a_n\} \) has no other limit points. There exists a sequence of positive real numbers \( \{r_n\} \) such that

\[
L_{\Omega_n}(z) \geq 5 - (2^{-1} + \ldots + 2^{-n}), \quad z \in I, \quad n \geq 1,
\]

with \( \Omega_n := \bigcup_{j=1}^{\infty} \{|z - a_j| < r_j\} \). Namely, it is clear that \( L_{\Omega_1}(z) > 5 - 2^{-1} \) on \( I \), if \( r_i > 0 \) is sufficiently small. Suppose \( r_1, \ldots, r_n \) are already chosen so that (*) is satisfied. Put \( \Omega(r) := \Omega_n \cup \{|z - a_{n+1}| < r\} \). Then \( L_{\Omega(r)} \uparrow L_{\Omega_n} \) in \( C \setminus \{a_{n+1}\} \) as \( r \uparrow 0 \). By Dini's argument the convergence is uniform on \( I \). Hence (*) is satisfied for \( n + 1 \) with \( r = r_{n+1} \) sufficiently small. The open set \( \Omega := \bigcup_{n} \Omega_n \) has the required properties. It is clear that \( E := \overline{\Omega} \) is \( L \)-regular. If \( \mu \) is a finite positive Borel measure on \( \Omega \), then \( \log M(\mu(\Omega_n)) \geq \sup_{\Omega} L_{\Omega_n} = \sup_{E} L_{\Omega_n} \geq 4 \) \( (n \geq 1) \). Hence \( M^*(1; \Omega, \mu) \geq 4 \). The set \( G := \{z \in E : L_{\Omega}^*(z) > 0\} \) contains the interval \( I \), so \( G \) is not plp. By Lemma 1.1 the pair \( (\Omega, \mu) \) does not satisfy \( (L^*) \) (see also theorem 3.1).

**Proposition 2.11.** If \( \mu \) is determining for a nonpluripolar bounded Borel set in \( \mathbb{C}^N \), \( \mu \) vanishes on plp sets and \( M^*(1; E, \mu) = 1 \) then \( (E, \mu) \) satisfies \( (L^*) \).

**Proof.** Given a polynomial family \( \mathcal{F} \) bounded \( \mu \)-a.e. on \( E \), let \( E_j \) be the sequence of subsets of \( E \) defined by (1.2). Then \( E_j \uparrow F \) with \( \mu(F) = 1 \). Therefore \( L_{E_j}^* \downarrow L_E^* = L_E \) (see Remark 2.3). Given \( b > 1 \) the set \( \Omega_b := \{L^*_F < \sqrt{b}\} \) is an open neighborhood of \( E \). By (1.2) and (2.4)

\[
|f(z)| \leq j \left( M(\mu(E_j)) \exp L_{E_j}^*(z) \right)^{\deg f}, \quad f \in \mathcal{F}, \quad j \geq 1.
\]

If \( j \) is sufficiently large the family \( \mathcal{F}_b \) is bounded by \( j \) uniformly on \( \Omega_b \).

**Problem 2.12.** Let \( \Delta = \{|z| < 1\} \) be the unit disk on the complex plane \( \mathbb{C} \). Let \( \theta \) denote the length measure on the boundary \( \partial \Delta \) of \( \Delta \) and let \( \lambda_2 \) be the Lebesgue measure on \( \mathbb{C} \equiv \mathbb{R}^2 \). Compute the functions \( M(t; \partial \Delta, \theta) \) and \( M(t; \Delta, \lambda_2) \), \( 0 \leq t \leq 1 \).

### 3 - Determining measures for bounded Borel sets with \( L \)-regular closure

The main result of this section is given by the following.
THEOREM 3.1. — Let \( \mu \) be a probability measure on a bounded Borel set \( E \) in \( \mathbb{C}^N \) such that \( \overline{E} \) is \( L \)-regular. Then the following conditions are equivalent.

(1) The pair \((E, \mu)\) satisfies \((L^*)\)-condition;

(2) If \( u \in L \) and \( u \leq 0 \) \( \mu \)-a.e. on \( E \) then \( u \leq 0 \) on \( \overline{E} \);

(3) \( M^*(1) \equiv M^*(1; E, \mu) = 1 \) and \( L_E = L_{\overline{E}} \);

(4) \( M(1) \equiv M(1; E, \mu) = 1 \) and \( L_E = L_{\overline{E}} \);

(5) If \( A \subset E \) and \( \mu(A) = 1 \), then \( L_A = L_{\overline{E}} \);

(6) For every \( b > 1 \) there exists a neighborhood \( \Omega \) of \( \overline{E} \) such that for every polynomial family \( \mathcal{F} \) bounded \( \mu \)-a.e. on \( E \) the family \( \mathcal{F}_b \) (given by \((0.2)) \) is uniformly bounded on \( \Omega \);

(7) If \( \mathcal{F} \) is a polynomial family bounded \( \mu \)-a.e. on \( E \) then for every number \( b > 1 \) the family \( \mathcal{F}_b \) is bounded q.a.e. on \( \overline{E} \).

Proof. — (1) \( \Rightarrow \) (2). Let \( u \) be a function of the class \( \mathcal{F} \) with \( u \leq 0 \) \( \mu \)-a.e. on \( E \). The function \( u \) can be written in the form

\[
u = \left( \limsup_{j \to \infty} \frac{1}{j} \log |f_j| \right)^*\]

where \( f_j \) is a polynomial of degree \( \leq j \). Given any fixed number \( b > 1 \) the polynomial family \( \mathcal{F} := \{ b^{-j} f_j; j \geq 1 \} \) is bounded \( \mu \)-a.e. on \( E \). By (1) there are a constant \( M > 0 \) and a neighborhood \( \Omega \) of \( E \) such that

\[
\|f_j\|_\infty \leq M b^{2j}, \quad j \geq 1,
\]

which implies \( \|f_j\|_\overline{E} \leq M b^{2j} (j \geq 1) \). Hence by the definition of \( L_{\overline{E}} \) we obtain

\[
\frac{1}{j} \log |f_j(z)| \leq \frac{1}{j} \log M + 2 \log b + L_E(z) \quad \text{in} \quad \mathbb{C}^N (j \geq 1).
\]

Therefore \( u(z) \leq 2 \log b \) on \( \overline{E} \). By the arbitrariness of \( b > 1 \) we get \( u \leq 0 \) on \( \overline{E} \).

(2) \( \Rightarrow \) (3). If (2) is satisfied, then \( L_E \leq L_{\overline{E}} \leq L_E \), so that \( L_E = L_{\overline{E}} \).

It remains to show that \( \lim_{t \uparrow 1} M(t) = 1 \). Suppose there exists \( b > 1 \) with \( M(t) > b \) for all \( t \) with \( 0 < t < 1 \). Let \( A_n \) be Borel subsets of \( E \) such that

\[
\mu(A_n) \geq 1 - 2^{-n} \quad \text{and} \quad \sup_{E} (\exp L_{A_n}) > b \quad (n \geq 1)
\]

(*)

Put \( E_n := A_n \cap A_{n+1} \cap \ldots \) and observe that \( E_{n+1} \supset E_n, E_n \subset A_n \) and

\[
\mu(E_n) = \mu(A_n) - \mu(A_n \setminus E_n) \geq \mu(A_n) - \mu(E \setminus E_n)
\]

\[
\geq \mu(A_n) - \sum_{j=0}^{\infty} \mu(E \setminus A_{n+j}) \geq 1 - 2^{-n} - \sum_{j=0}^{\infty} 2^{-n-j} = 1 - 3.2^{-n},
\]

- 205 -
which implies that \( \mu(E_n) \to 1 \). Put \( F := \bigcup E_n \). Then \( L_{E_n}^* \downarrow L_F^* \) and \( \mu(F) = 1 \). By (2) \( L_F \leq L_E \) and since \( L_E \leq L_F \), we get \( L_F = L_E \). By Dini’s argument \( L_{E_n} \leq L_{E_n}^* \leq \log b \) on \( \overline{E} \), if \( n > n_0 = n_0(b) \). This however contradicts the second inequality of (*)\). Therefore \( M^*(1) = \lim_{t \to 1} M(t) = 1 \).

(3) \( \Rightarrow \) (4) obvious.

(5) \( \Rightarrow \) (6) If \( b > 1 \), then the set \( \Omega_b := \{ z \in \mathbb{C}^N; L_E(z) < \sqrt{b} \} \) is by (5) an open neighborhood of \( \overline{E} \). Let \( \mathcal{F} \) be a polynomial family bounded \( \mu \)-a.e. on \( E \). Put \( E_k := \{ z \in E; |f(z)| \leq k, \forall f \in \mathcal{F} \} \). Then \( E_k \subset E_{k+1} \) and \( \mu(E_k) \uparrow 1 \). Hence by (5) \( L_{E_k}^* \downarrow L_{A}^* = L_{E} \) with \( A := \bigcup_{i=0}^{\infty} E_k \), the convergence being uniform on \( \overline{E} \). Hence \( L_{E_k} \leq \frac{1}{2} \log b \) on \( \overline{E} \) if \( k > k_0 \). It is clear that

\[
|f(z)| \leq k (\exp L_{E_k}(z))^{\deg f} \\
\leq k \cdot \left( \exp \left[ \frac{1}{2} \log b + L_{E}(z) \right] \right)^{\deg f} \\
\leq k \cdot b^{\deg f} , \text{if } z \in \Omega_b, \ f \in \mathcal{F}, \ k > k_0.
\]

(6) \( \Rightarrow \) (7) is obvious.

(7) \( \Rightarrow \) (1) follows from lemma 1.2.

4 - Determining measures for compact sets in \( \mathbb{C}^N \)

**Theorem 4.1.** — If \( \mu \) is a probability measure on a compact set \( E \) in \( \mathbb{C}^N \) vanishing on plp subsets of \( E \), then the following conditions are equivalent.

(i) The pair \((E, \mu)\) satisfies (*)\);

(ii) If \( u \in \mathcal{L} \) and \( u \leq 0 \) \( \mu \)-a.e. on \( E \), then \( u \leq 0 \) on \( E \);

(iii) \( M^*(1, E, \mu) = 1 \);

(iv) \( M(1; E, \mu) = 1 \);

(v) \( \mu \) is determining for \( E \) and \( E \) is \( \mathcal{L} \)-regular.

**Proof.** — First observe that each of the conditions (i), (ii), (iii), (iv) implies \( \mathcal{L} \)-regularity of \( E \), and next apply Theorem 3.1.

**Example 4.2.** — (most likely well known to the reader). Let \( E \) be a compact subset in the complex plane. Assume \( E \) has a positive logarithmic
capacity \( c(E) \). By the classical potential theory there exists a unique probability measure \( \lambda \) with support on \( E \) such that

\[
\log c(E) = \int_E \int_E \log |z - \zeta| d\lambda(z) d\lambda(\zeta) = \sup_{\mu} \int_E \int_E \log |z - \zeta| d\mu(z) d\mu(\zeta)
\]

the supremum being taken over all probability measures \( \mu \) on \( E \). The measure \( \lambda \) is called the equilibrium measure of \( E \). We shall show that \( \lambda \) is determining for \( E \). Indeed, if \( F \) is a Borel subset of \( E \) with \( \lambda(F) = 1 \) there is (by Choquet capacitability theorem) a sequence \( F_n \) of compact subsets of \( F \) with \( c(F_n) / c(F) \). Without loss of generality we may assume \( E \) is contained in the disk \( |z| < 1/2 \). Then

\[
\log c(E) \geq \log c(F_n) \geq \frac{1}{\lambda^2(F_n)} \int_{F_n} \int_{F_n} \log |z - \zeta| d\lambda(z) d\lambda(\zeta)
\]

\[
\geq \frac{1}{\lambda^2(F_n)} \log c(E).
\]

Therefore \( c(F_n) \uparrow c(E) = c(F) \). For all sufficiently large \( n \) the function \( u_n(z) := L^+_E(z) - L^+_E(z), u_n(\infty) := \log[c(E)/c(F_n)] \), is harmonic in \( \overline{C \setminus \hat{E}} \), \( u_{n+1} \leq u_n \) and \( u_n(\infty) \downarrow 0 \). By Harnack's theorem \( u_n \downarrow 0 \) locally uniformly in \( \overline{C \setminus \hat{E}} \). The function \( u := \lim L^+_n \) is subharmonic on \( C \), \( u \geq L^+_E \) on \( C \), and \( u = L^+_E \) in \( C \setminus \hat{E} \) as well as at each regular point of \( \partial \hat{E} \). By the generalized maximum principle for subharmonic function, \( u \leq 0 \) on \( \hat{E} \) except at most the polar set of irregular points of \( \partial \hat{E} \). On the other hand \( u \geq 0 \) on \( C \). Therefore \( u = L^+_E \). Observe that \( L_E \leq L_F \leq L_{F_n}(n \geq 1) \). Hence \( L^*_E = L^*_F \).

It follows that \( \mu \) is determining for \( E \). Hence by theorem 3.1, if \( E \) is an \( \mathcal{L} \)-regular subset of \( C \) and \( \lambda \) is the equilibrium measure of \( E \), then the pair \((E, \lambda)\) satisfies each of the equivalent conditions of theorem 4.1.

**Remark 4.3.**—Given a norm \( \mathcal{N} \) on \( \mathbb{C}^N \) the logarithmic capacity \( c(E) \equiv c(E, \mathcal{N}) \) of a bounded subset \( E \) of \( \mathbb{C}^N \) is defined by the formula

\[
-\log c(E) := \limsup_{\mathcal{N}(z) \to \infty} [L_E(z) - \log \mathcal{N}(z)].
\]

If \( E \) is a probability measure on \( E \) with \( \mathcal{M}(1; E, \mu) = 1 \), then for every \( F \subset E \) with \( \mu(F) = \mu(E) \) one has \( c(F) = c(E) \).

On the plane, if \( E \) is bounded and \( F \subset E \), then \( c(F) = c(E) \iff L^*_F = L^*_E \), which implies that \( \mu \) is determining for \( E \) iff \( F \subset E \), \( \mu(F) = 1 \Rightarrow c(F) = c(E) \) (i.e.; iff \( \mu \) is determining in the sense of Ullman [14]).
If $N \geq 2$ and $F \subset E$, it is clear that $L^*_F = L^*_E \Rightarrow \forall Nc(F, N) = c(E, N)$. But we do not know whether the inverse implication is true.

The aim of the following example is to illustrate an application of theorem 2.6.

**Example 4.4.** — Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ (resp. in $\mathbb{C}^N$). Then it is known that $\lambda_N$ (resp. $\lambda_{2N}$) is determining for $\Omega$. We can propose the following proof of this result.

It is sufficient to consider the case of $\mathbb{R}^N$ (because by (1.1) for every $u \in L(C^N)$ there is $\tilde{u} \in L(C^{2N})$ such that $\tilde{u}(x_1, y_1, \ldots, x_N, y_N) = u(x_1 + iy_1, \ldots, x_N + iy_N)$ for $(x_1 + iy_1, \ldots, x_N + iy_N) \in C^N \equiv \mathbb{R}^{2N}$. Hence, if $u \in L(C^N)$ and $u \leq 0 \lambda_{2N} - a.e.$ on $\Omega \subset C^N$ then $u \leq 0$ on $\Omega$. Let $u \in L(C^N)$ and let $u \leq 0 \lambda_N - a.e.$ on $\Omega$. Given a point $a = (a_1, \ldots, a_n)$ in $\Omega$, let $Q := \{ |x_j - a_j| \leq r \ (j = 1, \ldots, N) \}$ be a closed cube with center $a$ contained in $\Omega$. Since by Theorem 4.1 (via example 2.8) $\lambda_1$ is determining for $[a_j - r, a_j + r]$, so by theorem 2.6 the measure $\lambda_N$ is determining for the cube $Q$. Therefore $u \leq 0$ on $Q$. By the arbitrariness of $Q$ we get $u \leq 0$ on $\Omega$. Hence $L^*_\Omega = L^*_F$ for every $F \subset \Omega$ with $\lambda_N(F) = \lambda_N(\Omega)$.

Let $I^N = [0, 1]^N$ be the unit cube in $\mathbb{R}^N$. If $A$ is a nonsingular affine mapping of $\mathbb{R}^N$ onto itself, then the set $P := A(I^N)$ is called a parallelepiped.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ such that for each point $b \in \overline{\Omega}$ there exists a parallelepiped $P$ such that $P \subset \Omega \cup \{b\}$ and $b \in P$. Then $\overline{\Omega}$ is $C$-regular and the pair $(\Omega, \lambda_N)$ satisfies each of the equivalent conditions of theorem 3.1.

Indeed, it is easy to see that each parallelepiped $P$ is $C$-regular. Therefore $\overline{\Omega}$ is $C$-regular, because $L_{\overline{\Omega}} \leq L_P$. We already know that the pair $(I^N, \lambda_N)$ satisfies $(C^*)$. Hence for every parallelepiped $P$ the pair $(P, \lambda_N)$ satisfies $(C^*)$. Therefore the pair $(\Omega, \lambda_N)$ satisfies $(C^*)$ at each point of $\overline{\Omega}$, which implies that $(\Omega, \lambda_N)$ satisfies (6) of Theorem 3.1.

5 - Polynomial inequality of Bernstein-Markov type and pairs $(E, \mu)$ atisfying the $(C^*)$-condition

**Definition 5.1.** — Let $p$ be a positive number, $E$ a bounded Borel set in $\mathbb{C}^N$ and $\mu$ a probability measure on $E$. We say that the triple $(p, E, \mu)$ has Bernstein-Markov Property, if for every $b > 1$ there exist a positive constant $M$ and a neighborhood $G$ of $E$ such that for every polynomial $f$ of...
$N$ complex variables one has

$$\|f\|_G \leq M \deg f \|f\|_{\mu_p} \quad (BM)$$

with $\|f\|_{\mu_p} := (\int_E |f(x)|^p \, d\mu(z))^{1/p}$. 

It was shown in [11] that if $(E, \mu)$ satisfies $(\mathcal{L}^*)$ and $\mu$ satisfies some density condition, then the triple $(p, E, \mu)$ has BMP for every $p > 0$. Due to a remark by A. Zeriahi the density condition may be dropped and one gets the following.

**Theorem 5.1.**— Let $E$ be a Borel subset of $\mathbb{C}^N$ and let $\mu$ be a positive measure on $E$ such that $(E, \mu)$ satisfies $(\mathcal{L}^*)$. Then for every $p > 0$ the triple $(p, E, \mu)$ has the Bernstein-Markov Property (BMP).

**Proof.**— Let $s(f)$ denote the degree of $f$. It is sufficient to prove that for every $p > 0$ and for every $b > 1$ there exists a constant $M > 0$ such that for every polynomial $f$

$$\|f\| := \|f\|_E \leq M \deg f \|f\|_{\mu_p}.$$ 

Suppose the statement is not true. Then we can find $p > 0$, $b > 1$ and a sequence of polynomials $f_k$ such that

$$\|f_k\| > k^k \deg(f_k) \|f\|_{\mu_p} \text{ for } k \geq 1. \quad (5.1)$$

It follows that $\|f_k\| > 0$ and $0 < \|f_k\|_{\mu_p} < +\infty$ ($k \geq 1$). We claim that for every $q > 1$ and every $w > 1$ the sequence of polynomials $g_k := \eta^{-k} q^{-s(f_k)} f_k/\|f_k\|_{\mu_p}$ is bounded $\mu$-a.e. on $E$. Indeed, following NGUYEN THANH VAN [8], put $E_{nk} := \{z \in E; |g_k(z)| \geq n\}$, $E_n := \cup_{k=1}^\infty E_{nk}$ and observe that

$$\mu(E_n) \leq \sum_{k=1}^{\infty} n^{-p} \eta^{-kq} q^{-q s(f_k)} \leq n^{-p}/\eta^{p-1}, \quad n \geq 1,$$

whence it follow that $\{g_k\}$ is bounded $\mu$-a.e. on $E$. Now by the assumption $(E, \mu)$ satisfies $(\mathcal{L}^*)$, so that we can find $G \supset E$ and $M > 0$ such that $\|g_k\|_G \leq M q^{s(f_k)}, \quad k \geq 1$. Hence

$$\|f_k\|_G \leq M \eta^k q^{2s(f_k)} \|f\|_{\mu_p}, \quad k \geq 1 \quad (5.2)$$
Put $q = b^{1/2}$. Then (5.1) and (5.2) imply

$$k^k < M \eta^k, \; k \geq 1,$$

which is an absurd.

**Theorem 5.2.** If $M^*(1; E, \mu) = 1$ and there is $p > 0$ such that the triple $(p, E, \mu)$ has the BMP, then $(E, \mu)$ satisfies $(\mathcal{L}^*)$.

**Proof.** Take $b > 1$ and let $\mathcal{F}$ be a polynomial family bounded $\mu$-a.e. on $E$. Define $E_j$ by formula (1.2). Then $\mu(E_j) \uparrow 1$ and

$$|f(z)| \leq j \mathcal{M}(\mu(E_j))^{\deg f} \text{ for all } z \in E, \; f \in \mathcal{F}, \; j \geq 1.$$  

Hence by BMP

$$\|f\|_{\mathcal{O}} \leq j \cdot M [b \cdot \mathcal{M}(\mu(E_j))]^{\deg f} \; , \; f \in \mathcal{F}, \; j \geq 1,$$

which implies the required result.

**Corollary.** If $M^*(1; E, \mu) = 1$, then the pair $(E, \mu)$ satisfies $(\mathcal{L}^*)$ if and only if for every $p > 0$ (for some $p > 0$) the triple $(p, E, \mu)$ has the BMP.

**References**


Families of polynomials and determining measures


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