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Affine higher order parallel hypersurfaces

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RÉSUMÉ. — Nous étudions des immersions équiaffines, non-dégénérées d'une variété affine (M,∇) dans $\mathbf{R}^{n+1}$ satisfaisant $\nabla^2 h = 0$. Nous prouvons que ou bien M est une quadrique ou bien M est plat. Dans le second cas, nous obtenons que M est affinement équivalente à un graphe d'une fonction polynomiale de degré 3 ou moins.

Si la connection sur M est celle qui est induite canoniquement par l'immersion, nous généralisons pour des dimensions plus grandes des résultats de K. NOMIZU et de U. PINKALL.

ABSTRACT. — We study non-degenerate equiaffine immersions of an affine manifold (M,∇) into $\mathbf{R}^{n+1}$ which satisfy $\nabla^2 h = 0$. We prove that either M is a quadric or M is flat. In the second case, we obtain that M is affinely equivalent to a graph immersion of a polynomial of degree at most 3.

If the connection on M is the canonically induced connection, we extend a result by K. NOMIZU and U. PINKALL to higher dimensions and to higher order derivatives of h.

1. Introduction

In [N – P]$_2$ K. NOMIZU and U. PINKALL classify the affine surfaces in $\mathbf{R}^3$ satisfying $\nabla^2 h = 0$. More precisely, they prove the following theorem.

THEOREM .— *Let M^2 be a nondegenerate surface in $\mathbf{R}^3$. Let ∇ be the induced affine connection and let h be the second fundamental form (affine metric). If $\nabla^2 h = 0$ but $\nabla h \neq 0$, then M^2 is congruent to an open subset of the Cayley surface $z = xy + y^3$ by an equiaffine transformation of $\mathbf{R}^3$.*

Here, we give some generalizations of this theorem to higher dimensions. Also we will not restrict ourselves to the induced (canonical) affine normal,

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but we will also consider general nondegenerate equiaffine immersions. Definitions and elementary properties are described in section 2. For more details, we refer the reader to [N] and $[\mathrm{N}-\mathrm{P}]_1$. Especially, we will prove the following theorems.

THEOREM 1.— *Let* $(M^n, \nabla, \theta) \to (\mathbf{R}^{n+1}, D, \omega)$ *be a nondegenerate equiaffine immersion. Then* $(\nabla^2 h) = 0$ *if and only if one of the two following statements holds.*

> *(i)* M *is an open part of a nondegenerate quadric and* ∇ *is the induced connection.*
>
> *(ii)* M *is flat and* M *is congruent under an equiaffine transformation of* $\mathbf{R}^{n+1}$ *to the graph immersion* $x_{n+1} = F(x_1, x_2, \dots x_n)$, *where* F *is a polynomial in* $x_1, x_2, \dots, x_n$ *of degree at most 3 and with nonzero Hessian.*

THEOREM 2.— *Let* M^3 *be a nondegenerate hypersurface in* $\mathbf{R}^4$. *Let* ∇ *be the induced affine connection and let* h *be the second fundamental form. If* $\nabla^2 h = 0$ *but* $\nabla h \neq 0$, *then* M^3 *is congruent under an affine transformation to either one of the following graph immersions.*

> *(i)* $x_4 = x_1 x_2 + x_3^2 + x_1^3$,
>
> *(ii)* $x_4 = x_1 x_2 + x_3^2 + x_1^2 x_3$.

THEOREM 3.— *Let* $f : (M^n, \nabla) \to (\mathbf{R}^{n+1}, D)$ *be a convex equiaffine immersion with second fundamental form* h. *Then* $\nabla^k h = 0$ *for some* $k \in \{1, \dots, 4\}$ *implies that* M *is part of an ellipsoid and* ∇ *coincides with the induced connection or* M *is affinely equivalent to the graph immersion of a convex function.*

THEOREM 4.— *Let* M^n *be a nondegenerate convex hypersurface in* $\mathbf{R}^{n+1}$ *with induced second fundamental form* h. *Then* $\nabla^k h = 0$ *for* $k \in \{1, \dots, 4\}$ *implies that* M *is part of a nondegenerate ellipsoid or of a nondegenerate paraboloid.*

2. Preliminaries

Let M be an n-dimensional manifold with an affine connection ∇. Furthermore, let D denote the standard connection on $\mathbf{R}^{n+1}$. By an affine immersion $f : (M, \nabla) \to (\mathbf{R}^{n+1}, D)$ we mean an immersion for which there exists locally (i.e. in a neighbourhood of each point) a transversal vector field ξ such that for all tangent vector fields X and Y to M the following formula holds

$$D_X Y = \nabla_X Y + h(X, Y)\xi. \tag{2.1}$$

Clearly h is a symmetric bilinear form. If h is nondegenerate, we say that the immersion is *nondegenerate* and if h is positive or negative definite, we say that the immersion is *convex*. It is clear that these definitions do not depend on the choice of the transversal vector field ξ.

Let ω be the parallel volumeform on $\mathbf{R}^{n+1}$ given by the determinant, $f : (M, \nabla) \to (\mathbf{R}^{n+1}, D)$ an affine immersion and ξ an associated transversal field, then we can define a volumeform θ on M by

$$\theta(X_1, X_2, \ldots, X_n) = \omega(X_1, X_2, \ldots, X_n, \xi). \tag{2.2}$$

However, it is clear that, in general, this volumeform need not be parallel. If this volumeform is parallel, we say that the immersion f is an *equiaffine* immersion. It is proven in [N – P]$_1$ that if M admits a parallel volumeform, the associated transversal vector field ξ can be chosen in such a way that this parallel volumeform coincides with the induced volumeform (i.e. ξ can be chosen such that the immersion becomes an equiaffine immersion). Furthermore, it is immediately clear that if ξ_1 and ξ_2 are both associated transversal vector fields such that the immersion is equiaffine then $\xi_1 = c\xi_2$, where c is a constant on M. Such vector fields are called affine normal vector fields. From now on, we will always work with affine normal vector fields. The associated bilinear form is then called the affine second fundamental form. From [N – P]$_1$ we also recall the following formulas for equiaffine immersions.

$$D_X \xi = -SX, \tag{2.3}$$

where S is a (1.1) tensor field on M. We call S the affine Shape operator.

The equations of Gauss, Codazzi and Ricci are then given by

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \tag{2.4}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{2.5}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \tag{2.6}$$

and

$$h(X, SY) = h(SX, Y), \tag{2.7}$$

where (∇h) is defined by

$$(\nabla h)(X, Y, Z) = Xh(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Finally, we proof the following proposition from $[\mathrm{N} - \mathrm{P}]_1$, which we will use later on.

PROPOSITION 1.— *Suppose that $f : (M^n, \nabla\theta) \to \mathbf{R}^{n+1}$ is an equiaffine immersion with $S = 0$. Then f is affinely equivalent to the graph immersion of a certain function $F : M^n \to \mathbf{R}$.*

Proof.— By assuming a transversal vector field ξ to be equiaffine, $S = 0$ implies that $D_X\xi = 0$, i.e. ξ is a constant parallel vector field. Let $H = \mathbf{R}^n$ be a hyperplane in $\mathbf{R}^{n+1}$ which is transversal to ξ. Let $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be the projection along the direction of ξ so that $\pi \circ f : M^n \to \mathbf{R}^n$ is an affine immersion with image W, an open subset of $\mathbf{R}^n$. Then, we can find a differentiable function $F : M^n \to \mathbf{R}$ such that $f(x) = \pi(f(x)) + F(x)\xi$. Thus f is affinely equivalent to a graph immersion.

Now, we will define the induced connection and the canonical affine normal. For more details, see [N] . Let M^n be a hypersurface in $\mathbf{R}^{n+1}$. For any choice of transversal vector field ξ, we can define an affine connection ∇ and a bilinear form h by (2.1). Whether h is nondegenerate or not does not depend on the choice of the transversal vector field. Therefore we call the immersion nondegenerate if there exists a transversal vector field for which the corresponding h is nondegenerate. In that case, h induces a volumeform, given by its determinant which we will denote by ω_h. Then, in [N], it is proved that there exists a unique choice of ξ such that

(i) $\omega_h = \theta$, where θ is the volumeform defined by (2.2),

and

(ii) the volumeform $\omega_h = \theta$ is parallel with respect to ∇.

It is immediately clear that the immersion defined this way is indeed an equiaffine immersion and ∇ is called the induced connection and ξ is called the canonical affine normal vector field.

3. Main results

Let T be a $(0,k)$ tensor field on M and let R denote the curvature tensor on M. Then, we define a $(0,k+2)$ tensor field $R\cdot T$ on M by
$R\cdot T(X_1,X_2,\ldots,X_{k+2}) =$
$-T(R(X_1,X_2)X_3,X_4,\ldots,X_{k+2}) - T(X_3,R(X_1,X_2)X_4,X_5,\ldots,X_{k+2})\ldots$
$-T(X_3,X_4,\ldots,X_{k+1},R(X_1,X_2)X_{k+2})$.
Then, we can also define $R^m\cdot T, m\geq 2$ by
$(R^m\cdot T)(X_1,\ldots,X_{k+2m}) = R\cdot(R^{m-1}\cdot T))(X_1,\ldots,X_{k+2m})$.

LEMMA 1.— *Let M be an equiaffine immersion with second fundamental form h. Then*

$$(\nabla^k h)(X_1,X_2,\ldots,X_{k+2}) =$$
$$(\nabla^k h)(X_2,X_1,X_3,\ldots,X_{k+2}) + R\cdot(\nabla^{k-2}h)(X_1,X_2,\ldots,X_{k+2})$$

Proof.— Let $x_1,x_2,\ldots,x_{k+2}$ be tangent vectors at a point p of M. We can extend these vectors locally to the vector fields $X_1,X_2,\ldots,X_{k+2}$ which are parallel at the point p. Put

$$A = (\nabla^k h)(X_1,X_2,\ldots,X_{k+2}) - (\nabla^k h)(X_2,X_1,X_3,\ldots,X_{k+2}).$$

Then, we find, using several times that $\nabla_{X_i}X_j$ is zero at the point p that

$$\begin{aligned}
A &= X_1(\nabla^{k-1}h)(X_2,\ldots,X_{k+2}) - X_2(\nabla^{k-1}h)(X_1,X_3,\ldots,X_{k+2})\\
&= -X_1(\nabla^{k-2}h)(\nabla_{X_2}X_3,X_4,\ldots,X_{k+2})\\
&\quad + X_2(\nabla^{k-2}h)(\nabla_{X_1}X_3,X_4,\ldots,X_{k+2}) + \ldots\\
&= -(\nabla^{k-2}h)(\nabla_{X_1}\nabla_{X_2}X_3,X_4,\ldots,X_{k+2})\\
&\quad + (\nabla^{k-2}h)(\nabla_{X_2}\nabla_{X_1}X_3,X_4,\ldots,X_{k+2}) + \ldots\\
&= -(\nabla^{k-2}h)(R(X_1,X_2)X_3,X_4,\ldots,X_{k+2}) - \ldots\\
&= (R\cdot(\nabla^{k-2}h)(X_1,X_2,\ldots,X_{k+2}).
\end{aligned}$$

LEMMA 2.— *Let M be an equiaffine submanifold of $\mathbf{R}^{n+1}$ with associated second fundamental form h. Then $(\nabla^{2k}h)(x_1,x_2,\ldots,x_{2k+2}) = 0$ for all tangent vectors $x_1,\ldots,x_{2k+2}$ to M implies that*

$$(R^k\cdot h)(x_1,x_2,\ldots,x_{2k+2}) = 0$$

and that

$$(R^k \cdot (\nabla h))(x_1, x_2, \dots, x_{2k+3}) = 0.$$

Proof.— First, by applying Lemma 1, we obtain that

$$(R \cdot (\nabla^{2k-2} h))(x_1, x_2, x_3, x_4, \dots, x_{2k}) = 0$$

and

$$(R \cdot \nabla^{2k-2} h))(x_1, x_2, x_4, x_3, x_5, \dots, x_{2k}) = 0.$$

Hence, by subtracting these two equations and by applying Lemma 1 once more, we find that

$$(R^2 \cdot (\nabla^{2k-4} h))(x_1, x_2, x_3, \dots, x_{2k}) = 0.$$

Now let us assume that $(R^m \cdot (\nabla^{2k-2m} h)) = 0$. Then, we know that for all tangent vectors $x_1, x_2, \dots, x_{2k}$ the following equations hold :

$$(R^m \cdot (\nabla^{2k-2m} h))(x_1, x_2, \dots x_{2m+1}, x_{2m+2}, \dots, x_{2k}) = 0$$

and

$$(R^m \cdot (\nabla^{2k-2m} h))(x_1, x_2, \dots, x_{2m+2}, x_{2m+1}, x_{2m+3}, \dots, x_{2k}) = 0.$$

Hence, by subtracting these two equations and by applying Lemma 1 once more, we find that

$$(R^{m+1} \cdot (\nabla^{2k-2m-2} h))(x_1, x_2, x_3, \dots, x_{2k}) = 0.$$

This proves the first part of this lemma. The proof of the second part is similar, starting from the fact that $(\nabla^{2k} h) = 0$ implies that also $(\nabla^{2k+1} h) = 0$.

The proof of the following lemma is completely similar to the proof of Lemma 2.

LEMMA 3.— *Let M be an equiaffine submanifold of $\mathbf{R}^{n+1}$ with associated second fundamental form h. Then $(\nabla^{2k+1} h)(x_1, x_2 \dots, x_{2k+3}) = 0$ for all tangent vectors $x_1, \dots, x_{2k+3}$ to M implies that*

$$(R^{k+1} \cdot h)(x_1, x_2, \dots, x_{2k+4}) = 0$$

and that

$$(R^{k+1}\cdot(\nabla h))(x_1,x_2,\dots,x_{2k+3})=0.$$

Now, we can start the proof of the main theorems.

Proof of theorem 1 . — Let $p\in M$. Since h is nondegenerate, there exists a basis $\{e_1,e_2,\dots,e_n\}$ of T_pM such that $h(e_i,e_j)=\epsilon_i\delta_{ij}$, where $\epsilon_i\in\{-1,1\}$. By Lemma 2 $(\nabla^2 h)=0$ implies that $R\cdot h=0$. Hence, we obtain for $i\neq j$ that

$$\begin{aligned}0&=(R\cdot h)(e_i,e_j,e_i,e_i)\\&=-2h(R(e_i,e_j)e_i,e_i)\\&=2\epsilon_i h(Se_j,e_i).\end{aligned}$$

Therefore $e_1,e_2,\dots,e_n$ are eigenvectors of S. Thus $Se_i=\lambda_i e_i,\lambda_i\in\{1,2,\dots,n\}$. But then

$$\begin{aligned}0&=(R\cdot h)(e_i,e_j,e_i,e_j)\\&=-h(R(e_i,e_j)e_i,e_j)-h(e_i,R(e_i,e_j)e_j)\\&=\epsilon_i\epsilon_j(\lambda_j-\lambda_i)\end{aligned}$$

implies that $S=\lambda I$. Now, (2.6) becomes

$$X(\lambda)Y-Y(\lambda)X=0$$

From this, using the connectedness of M and the fact that the dimension of M is at least two, we obtain that λ is a constant on M.

Let us now assume that $\lambda\neq 0$. Then, we have to prove that M is a part of a quadric. Therefore, due to a generalisation of the classical Berwald theorem ([N – P]$_3$, [D – V]) , it is sufficient to prove that $(\nabla h)=0$ on M. In order to do so, we know from Lemma 2 that $(R\cdot(\nabla h))=0$. This implies for mutually different indices i,j and k that

$$\begin{aligned}0&=(R\cdot(\nabla h))(e_i,e_j,e_j,e_j,e_k)\\&=-2\epsilon_j\lambda(\nabla h)(e_i,e_j,e_k).\end{aligned}$$

Also, for different indices i and j, we obtain that

$$\begin{aligned}0&=(R\cdot(\nabla h))(e_i,e_j,e_j,e_j,e_j)\\&=-3\epsilon_j\lambda(\nabla h)(e_i,e_j,e_j).\end{aligned}$$

and that

$$\begin{aligned}0 &= (R\cdot(\nabla h))(e_i,e_j,e_i,e_i,e_j)\\ &= -\epsilon_j\lambda(\nabla h)(e_i,e_i,e_i)+2\epsilon_i\lambda(\nabla h)(e_i,e_j,e_j)\\ &= -\epsilon_j\lambda(\nabla h)(e_i,e_i,e_i).\end{aligned}$$

Using linearity and the symmetry of (∇h), we then obtain that $(\nabla h)=0$. Hence M is an open part of a nondegenerate quadric and the connection coincides with the induced connection.

Let us now assume that $\lambda=0$. By using the proposition, we may assume that the immersion f is affinely equivalent to an open part of a graph immersion. Furthermore, from the proof of the proposition, we see that we may assume that

$$\begin{aligned}\xi &= (0,\dots,0,1),\\ x_{n+1} &= F(x_1,x_2,\dots,x_n),\\ o &= (0,\dots,0)\in \text{dom } F \text{ and } F(o)=0,\end{aligned}$$

$$T_oM = \text{span }\{(1,0,\dots,0,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1,0)\},$$

where the third condition can be realized by a translation and the fourth condition immediately follows from taking as the transversal hyperplane in the proposition the tangent plane at the origin. Then by a straightforward computation we find that

$$h(\partial_i,\partial_j)=F_{ij}$$

and

$$\nabla_{\partial_i}\partial_i=0,$$

where $\partial_i=\frac{\partial}{\partial x_i}, F_{ij}=\frac{\partial^2 F}{\partial x_i\partial x_j}$ and $i,j\in\{1,2,\ n\}$. Using these equations, it immediately follows from $(\nabla^3 h)=0$ that the fourth derivative of F must be zero. Hence F is a polynomial of degree less then or equal to 3. Since the immersion is nondegenerate, F must also be nondegenerate. From the imposed conditions, we also see that F has no constant and no linear term. This completes the proof of this theorem.

Proof ot theorem 2 .— We will use the same notations as in the proof of Theorem 1. Since (∇h) is not identically zero, we deduce from Theorem 1 that f is affinely equivalent with the following graph.

$$x_4=F^2(x_1,x_2,x_3)+F^3(x_1,x_2,x_3)$$

where $F^i,\ i\in\{2,3\}$ is a homogeneous polynomial in x_1,x_2,x_3 of degree $i,\ o\in$ dom (F^2+F^3) and $\xi=(0,0,0,1)$. Since we assume the immersion to be nondegenerate, we know that $\det(F^2_{ij}(o)) \neq 0$. Therefore, by a well known algebraic property, we may assume that $F^2_{ij}(0) = \epsilon_i \delta_{ij}$ where $\epsilon_i \in \{-1, 1\}$. Hence by, if necessary, taking $-\xi$ as affine normal, we may assume that either

$$F^2(x_1, x_2, x_3) = (1/2)\{x_1^2 + x_2^2 + x_3^2\}$$

or

$$F^2(x_1, x_2, x_3) = (1/2)\{x_1^2 + x_2^2 - x_3^2\}.$$

It is then clear, that we can always apply a rotation like

$$\begin{cases} y_1 &= p_1 x_1 + p_2 x_2 \\ y_2 &= -p_2 x_1 + p_1 x_2 \\ y_3 &= x_3 \end{cases}$$

where $p_1^2 + p_2^2 = 1$, without changing the form of F^2. By applying such a rotation in the (x_1, x_2)-plane and another one in the (x_2, x_3)-plane, we may assume that $F = F^2 + F^3$ is given by $F(x_1, x_2, x_3) = x_1^2 + \alpha x_2^2 + \beta x_2 x_3 + \gamma x_3^2 + a_1 x_1^2 x_2 + a_2 x_1 x_2 x_3 + a_3 x_1 x_2^2 + a_4 x_1 x_3^2 + a_5 x_2^3 + a_6 x_2^2 x_3 + a_7 x_2 x_3^2 + a_8 x_3^3$, where since M is not a quadric not all the a_i are zero.

Notice also, that the last rotation did change the form of F^2. Then the condition that ξ is the canonical affine normal implies that $\det(F_{ij})$ is a non-zero constant. Let $A = \det(F_{ij})$. Looking at the term in x_1^3 of A we immediately obtain that $a_1 \cdot a_4 = 0$.

Let us first assume that $a_1 = 0$. Then, we can apply another rotation in the (x_2, x_3)-plane such that also $a_2 = 0$. But then by looking at the terms in x_2^3 and x_3^3 we obtain that $a_4 \cdot a_6 = 0$ and $a_3 \cdot a_7 = 0$. By, if necessary, interchanging x_2 and x_3 this gives three possible subcases.

subcase 1 $a_3 = a_4 = 0$. In this case, the problem becomes essentially 2-dimensional and in the same way as in [N – P]$_2$ we obtain the desired result.

subcase 2 $a_3 = a_6 = 0$. Then it follows from the term in x_1 and the term in $x_1 x_2$ that either $a_4 = 0$ or $\alpha = a_5 = 0$. If $a_4 = 0$, we can apply subcase 1 to obtain the solution. Hence, we may assume that $\alpha = a_5 = 0$. From the term in x_3^2 we then deduce that $a_7 = 0$. Therefore, we know that F can be written as

$$F(x_1, x_2, x_3) = x_1^2 + \beta x_2 x_3 + \gamma x_3^2 + a_4 x_1 x_3^2 + a_8 x_3^3,$$

where since F is nondegenerate $\beta \neq 0$. If $a_4 = 0$ or $a_8 = 0$, then we can apply the affine transformation given by

$$\begin{cases} y_1 &= x_1 \\ y_2 &= \beta x_2 + \gamma x_3, \\ y_3 &= x_3 \end{cases}$$

to obtain the desired form. However, if both a_4 and a_8 are different from zero, we first apply the affine transformation given by

$$\begin{cases} y_1 &= a_4 x_1 + a_8 x_3 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{cases}$$

such that

$$F(y_1, y_2, y_3) = b_1 y_1^2 + b_2 y_1 y_3 + \beta y_2 y_3 + b_3 y_3^2 + y_1 y_3^2.$$

Applying the affine transformation given by

$$\begin{cases} z_1 &= y_1 \\ z_2 &= b_2 y_1 + \beta y_2 + b_3 y_3, \\ z_3 &= y_3 \end{cases}$$

then completes the proof in this case.

subcase 3 $a_6 = a_7 = 0$. Looking at the term in x_1^2 gives $a_3 \cdot a_4 = 0$. This reduces this case to the previous case.

Now, we may assume that $a_1 \neq 0$. Then we know that $a_4 = 0$. From the term in x_3^3 we find that $a_2 \cdot a_8 = 0$. Again, we have to consider several subcases.

subcase 1 $a_2 = 0$. By looking at the terms in $x_1^2, x_1^2 x_2$ and $x_1^2 x_3$, we obtain that $\gamma = a_7 = a_8 = 0$. But then, since A is a constant it immediately follows that $a_1 = 0$. Hence, in this case, we obtain a contradiction.

subcase 2 $a_8 = 0 = a_4$ and $a_1 \neq 0 \neq a_2$. Then, we find by explicitly calculating the determinant A that F must have the following form.

$$F(x_1, x_2, x_3) = (1/2)(x_1^2 - x_3^2 + \alpha x_2^2) + \beta x_2 x_3 + (1/2) a_1 x_2 (x_1 + x_3 - \beta x_2)^2.$$

By applying one after another the following affine transformations, we get also in this case the desired form. First, we apply the following affine transformation.

$$\begin{cases} y_1 &= x_1 - x_3 \\ y_2 &= x_2 \\ y_3 &= x_1 + x_3 \end{cases}.$$

Up to a scaling factor, F has then the following form.

$$F(y_1, y_2, y_3) = y_1 y_3 + b_1 y_2^2 + b_2 y_2 y_3 + b_3 y_3^2 + y_2 y_3^2.$$

Then applying the affine transformation given by

$$\begin{cases} z_1 &= y_1 + b_2 y_2 + b_3 y_3 \\ z_2 &= y_2 \\ z_3 &= y_3 \end{cases}$$

and a rescaling completes the proof.

Proof of theorem 3. .— Let $p \in M$. Since M is convex, we know that the second fundamental form is positive or negative definite. Since (2.7) holds, we then know that h and S are simultaneously diagonalisable, i.e. there exists a basis $\{e_1, e_2, \dots, e_n\}$ of T_pM such that $h(e_i, e_j) = \epsilon\delta_{ij}$ and $Se_i = \lambda_i e_i$, where $\epsilon \in \{-1, 1\}$, $\lambda_i \in \mathbf{R}$ and $i, j \in \{1, 2, \dots, n\}$. Let us now assume that $(\nabla^k h) = 0, k \leq 4$. In all these cases, we know that $(\nabla^4 h) = 0$. But then, from Lemma 2 it follows that $R^2 \cdot h = 0$. Therefore, we obtain that

$$\begin{aligned} 0 &= (R^2 \cdot h)(e_i, e_j, e_i, e_j, e_i, e_i) \\ &= -(R \cdot h)(R(e_i, e_j)e_i, e_j, e_i, e_i) - (R \cdot h)(e_i, R(e_i, e_j)e_j, e_i, e_i) \\ &\quad 2(R \cdot h)(e_i, e_j, R(e_i, e_j)e_i, e_i) \\ &= 2\epsilon\lambda_j (R \cdot h)(e_i, e_j, e_j, e_i) \\ &= 2\epsilon\lambda_j \{-h(R(e_i, e_j)e_j, e_i) - h(e_j, R(e_i, e_j)e_i)\} \\ &= 2\epsilon\lambda_j(\lambda_j - \lambda_i). \end{aligned}$$

From this, it follows that $S = \lambda I$, $\lambda \in \mathbf{R}$. In the same way as in the proof of Theorem 1, it follows from (2.6) that λ is a constant on M. If $\lambda = 0$ on M, the theorem is proved.

Therefore, we may assume that $\lambda \neq 0$. In this case, we must prove that M is a part of a nondegenerate quadric. In order to do so, by the generalized BERWALD theorem ([N – P]$_3$, [D – V]), it is sufficient to proof that $(\nabla h) = 0$.

If $(\nabla^3 h) = 0$, we know from Lemma 3 that $R \cdot (\nabla h) = 0$. From this, we deduce just as in the proof of Theorem 1 that $(\nabla h) = 0$.

If $(\nabla^4 h) = 0$, we know from Lemma 2 that $R^2 \cdot (\nabla h) = 0$. Now, we take $v, u \in T_pM$ such that $h(v, v) = h(u, u) = \epsilon$ and $h(u, v) = 0$. Then, we have that

$$\begin{aligned} 0 &= R^2 \cdot (\nabla h)(u, v, u, v, v, v, v) \\ &= -3\lambda\epsilon(R \cdot (\nabla h)(u, v, u, v, v) \\ &= 3\lambda^2\{2(\nabla h)(u, u, v) - (\nabla h)(v, v, v)\}, \end{aligned}$$

and

$$\begin{aligned} 0 &= R^2 \cdot (\nabla h)(u, v, u, v, u, u, v) \\ &= -\lambda\epsilon\{R \cdot (\nabla h)(u, v, u, u, u) - 2(R \cdot (\nabla h))(u, v, u, v, v) \\ &= \lambda^2\{-3(\nabla h)(u, u, v) + 2(\nabla h)(v, v, v) - 4(\nabla h)(u, u, v)\} \\ &= \lambda^2\{-7(\nabla h)(u, u, v) + 2(\nabla h)(v, v, v)\} \end{aligned}$$

Hence for all v, $(\nabla h)(v, v, v) = 0$. Since (∇h) is symmetric, linearization then implies that $(\nabla h) = 0$. This completes the proof of Theorem 3.

Proof ot theorem 4 .— By Theorem 3 we know that either M is part of a quadric, in which case, the proof of the theorem is obvious or $S = 0$ and M is affinely equivalent to the graph immersion

$$x_{n+1} = P(x_1, x_2, \ldots, x_n)$$

where P is a convex polynomial in $x_1, x_2, \ldots, x_n$ of degree at most 5. Since $S = 0$, P satisfies in a neighbourhood of o the following differential equation

$$|\det\,(P_{ij})| = 1$$

where P_{ij} is defined as in Theorem 1. Since P is a polynomial function, it is then clear that we may assume that P is convex on the whole of $\mathbf{R}^n$ and satisfies this differential equation on the whole of $\mathbf{R}^n$. Therefore, we obtain by a theorem of Pogorelov ([P], [C - Y]) , which is a generalisation of the theorem of Calabi [C] that M is part of a nondegenerate paraboloid.

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