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Existence, uniqueness and regularity for Kruzkov's solutions of the Burger-Carleman's system

VINCENT CASELLES^{*(1)} AND CHARBEL KLAIANY⁽²⁾

RÉSUMÉ. — Nous montrons l'existence et l'unicité d'une solution $(u(t), v(t))$ au sens kruzkov du système de Burger-Carleman avec condition initiale $(u_0, v_0) \in L^1(\mathbf{R})_+ \times L^1(\mathbf{R})_+$. Nous montrons que pour tout $t > 0, u(t), v(t) \in L^\infty(\mathbf{R})$. Cet effet régularisant est lié à la possibilité de définir la solution au sens Kruzkov du système de Burger - Carleman.

ABSTRACT. — We prove existence and uniqueness of a Kruzkov solution $(u(t), v(t))$ of the Burger-Carleman's system with initial data $(u_0, v_0) \in L^1(\mathbf{R})_+ \times L^1(\mathbf{R})_+$. Moreover, we show that for any $t > 0, u(t), v(t) \in L^\infty(\mathbf{R})$ with precise estimates. In fact, this regularizing effect is related to the possibility of defining Kruzkov's solutions for the Burger-Carleman's system.

We consider the following first order system which will be called the Burger-Carleman's system :

$$\begin{aligned}
 & u_t + \left(\frac{u^2}{2}\right)_x + u^2 - v^2 = 0 \quad \text{on } [0, +\infty) \times \mathbf{R} \\
 (BC) \quad & v_t - \left(\frac{v^2}{2}\right)_x + v^2 - u^2 = 0 \quad \text{on } [0, +\infty) \times \mathbf{R} \\
 & u(0, x) = u_0(x), v(0, x) = v_0(x)
 \end{aligned}$$

with initial data $u_0, v_0 \in L^1(\mathbf{R})_+$. We prove the existence and uniqueness of a Kruzkov's solution of (BC) (see definition 1 below) using the theory of nonlinear semigroups generated by accretive operators. We notice that

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the possibility of defining Kruzkov's solutions for (BC) when the initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ depends on the $\mathbf{L}^1 - \mathbf{L}^\infty$ regularizing effect for homogeneous equations proved in [2]. In fact, the estimates proved in [2] imply that for any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ and any $t > 0, u(t), v(t) \in \mathbf{L}^\infty(\mathbf{R})_+$ with precise estimates given below. Before stating the precise result, let us define the notion of Kruzkov's solution for (BC) :

DEFINITION 1. — *Let $T > 0$. The pair of functions $(u, v) \in \mathbf{L}^\infty([0, T], \mathbf{L}^1(\mathbf{R})_+)^2 \cap \mathbf{L}^\infty([\tau, T] \times \mathbf{R})^2$ for any $\tau > 0$ will be called a kruzakov's solution of (BC) in $[0, T] \times \mathbf{R}$ with initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ if $(u(t), v(t)) \rightarrow (u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2$ as $t \rightarrow 0$ and*

$$\int_0^T \int_{\mathbf{R}} |u - k| \xi_t + \text{sign}_0(u - k) \left[\left(\frac{u^2}{2} - \frac{k^2}{2} \right) \xi_x + (v^2 - u^2) \xi \right] dx dt \geq 0$$

$$\int_0^T \int_{\mathbf{R}} |v - k'| \eta_t + \text{sign}_0(v - k') \left[\left(\frac{k'^2}{2} - \frac{v^2}{2} \right) \eta_x + (u^2 - v^2) \eta \right] dx dt \geq 0$$

holds for all $\xi, \eta \in C_0^\infty((0, T) \times \mathbf{R}), \xi, \eta \geq 0$ and all $k, k' \in \mathbf{R}$.

As it is costumary

$$\text{sign}_0(r) = +1 \text{ if } r > 0, 0 \text{ if } r < 0$$

$$\text{sign}(r) = +1 \text{ if } r > 0, [-1, 1] \text{ if } r = 0, -1 \text{ if } r < 0$$

$$\text{sign}^+(r) = +1 \text{ if } r > 0, [0, 1] \text{ if } r = 0, 0 \text{ if } r < 0$$

Similarly one defines $\text{sign}_0^+(r)$.

Then, our result says :

THEOREM 1. — *For any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$, there exists a unique Kruzkov's solution $(u, v) \in C([0, T], \mathbf{L}^1(\mathbf{R})_+^2)$ of (BC) in $[0, T] \times \mathbf{R}$ for any $T > 0$ with initial data (u_0, v_0) such that for any $t > 0$:*

$$(RE) \quad \|u(t)\|_{\mathbf{L}^\infty(\mathbf{R})} \leq \left(\frac{2}{t} \|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})} + \frac{2\sqrt{2}}{\sqrt{t}} (\|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})})^{3/2} \right)^{1/2}$$

The same estimate holds for $\|v(t)\|_{\mathbf{L}^\infty(\mathbf{R})}$. Moreover, if $(u, v), (\hat{u}, \hat{v})$ are two Kruzkov's solutions of (BC) in $[0, T] \times \mathbf{R}, T > 0$, corresponding to the initial data $(u_0, v_0), (\hat{u}_0, \hat{v}_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ respectively, then for all $t \in [0, T]$.

$$\begin{aligned} & \| (u(t) - \hat{u}(t))^+ \|_{\mathbf{L}^1(\mathbf{R})} + \| (v(t) - \hat{v}(t))^+ \|_{\mathbf{L}^1(\mathbf{R})} \\ & \leq \| (u_0 - \hat{u}_0)^+ \|_{\mathbf{L}^1(\mathbf{R})} + \| (v_0 - \hat{v}_0)^+ \|_{\mathbf{L}^1(\mathbf{R})} \end{aligned}$$

To begin with the proof, let us introduce the following operators A, B :

$$D(A) := \{(u, v) \in \mathbf{L}^1(\mathbf{R})_+^2 : u^2, v^2 \in AC(\mathbf{R})\}$$

$$D(B) := \{(u, v) \in \mathbf{L}^1(\mathbf{R})_+^2 : u^2, v^2 \in \mathbf{L}^1(\mathbf{R})\}$$

where $AC(\mathbf{R})$ is the set of absolutely continuous functions on \mathbf{R} ,

$$A(u, v) = \left(\left(\frac{u^2}{2} \right)_x, - \left(\frac{v^2}{2} \right)_x \right), B(u, v) = (u^2 - v^2, v^2 - u^2)$$

for $(u, v) \in D(A), (u, v) \in D(B)$ respectively. Notice that $D(A) \subset D(B)$. Thus $D(A+B) = D(A)$ and (BC) can be written in the abstract form : let $U = (u, v)$

$$(BC)_a \quad \frac{dU}{dt} + (A+B)U = 0$$

$$U(0) = (u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$$

We show that one can use the Grandall-Liggett's theorem to solve $(BC)_a$. This is the purpose of the next two lemmas. Before stating them, let us recall the definition of T -accretivity. Let E be a Banach lattice. A (in general, multivalued) operator B on E called T -accretive if

$\|(x - \hat{x})^+\|_E \leq \|(x - \hat{x} + \lambda y - \lambda \hat{y})^+\|_E$ holds for all $[x, y], [\hat{x}, \hat{y}] \in B$ and all $\lambda > 0$.

If $E = \mathbf{L}^1(\mathbf{R}) \times \mathbf{L}^1(\mathbf{R})$ endowed with the norm

$\|(u, v)\|_E = \int_{\mathbf{R}} |u| + \int_{\mathbf{R}} |v|, (u, v) \in E$, then this is equivalent to say that for all $[(x_1, x_2), (y_1, y_2)], [(\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2)] \in B$ there exists some $\alpha_1 \in \text{sign}^+(x_1 - \hat{x}_1), \alpha_2 \in \text{sign}^+(x_2 - \hat{x}_2)$ such that

$$\int_{\mathbf{R}} \alpha_1 (y_1 - \hat{y}_1) + \alpha_2 (y_2 - \hat{y}_2) dx \geq 0. \text{ Then :}$$

LEMMA 1. — $A+B$ is T -accretive in $\mathbf{L}^1(\mathbf{R})^2$. Moreover, for any $p \in W^{1,\infty}(\mathbf{R})$ such that $p' \geq 0$ has compact support :

$$(1) \quad \int_{\mathbf{R}} p(u)w + p(v)h \, dx \geq 0$$

holds for any $(u, v) \in D(A)$ where $(w, h) = (A+B)(u, v)$.

LEMMA 2. — For all $\lambda > 0$, $\text{Ran}(I + \lambda(A+B)) = \mathbf{L}^1(\mathbf{R})_+^2$.

Proof of lemma 1. — Let $U = (u, v), \hat{U} = (\hat{u}, \hat{v}) \in D(A)$. One easily checks that

$$\int_{\mathbf{R}} \left[\left(\frac{u^2}{2} \right)_x \left(\frac{\hat{u}^2}{2} \right)_x \right] \text{sign}_0^+(u - \hat{u}) dx = \int_{\mathbf{R}} \left[\left(\frac{v^2}{2} \right)_x - \left(\frac{\hat{v}^2}{2} \right)_x \right] \text{sign}_0^+(v - \hat{v}) dx = 0$$

since $u, \hat{u}, v, \hat{v} \geq 0$ and sign_0^+ is an increasing function, then

$$\begin{aligned} & \int_{\mathbf{R}} B(u, v)(\text{sign}_0^+(u - \hat{u}), \text{sign}_0^+(v - \hat{v}))dx = \\ & \int_{\mathbf{R}} [\text{sign}_0^+(u - \hat{u}) - \text{sign}_0^+(v - \hat{v})][(u^2 - \hat{u}^2) - (v^2 - \hat{v}^2)]dx = \\ & \int_{\mathbf{R}} [\text{sign}_0^+(u^2 - \hat{u}^2) - \text{sign}_0^+(v^2 - \hat{v}^2)][(u^2 - \hat{u}^2) - (v^2 - \hat{v}^2)]dx \geq 0 \end{aligned}$$

Both remarks imply that $A + B$ is T accretive in $\mathbf{L}^1(\mathbf{R})_+^2$.

Let $\beta(r) := r^{1/2}, r \geq 0$. Let $p \in W^{1,\infty}(\mathbf{R})$ be such that $p' \geq 0$ has compact support.

Let $j : \mathbf{R}_+ \rightarrow \mathbf{R}$ be $j(r) = \int_0^r (p \circ \beta)(s)ds$. Then, if $z = u^2$

$$\int_{\mathbf{R}} \left(\frac{u^2}{2}\right)_x p(u)dx = \int_{\mathbf{R}} \left(\frac{z}{2}\right)_x (p \circ \beta)(z)dx = \frac{1}{2} \int_{\mathbf{R}} j(z)_x dx = 0.$$

Similarly $\int_{\mathbf{R}} \left(\frac{v^2}{2}\right)_x p(x)dx = 0$ and

$$\int_{\mathbf{R}} (u^2 - v^2)p(u) + (v^2 - u^2)p(v)dx = \int_{\mathbf{R}} (u^2 - v^2)(p(u) - p(v))dx \geq 0$$

since p is increasing and $u, v \geq 0$. Putting this things together we get the inequality (1).

Proof of lemma 2.— Since the proof below is independent of the value of $\lambda > 0$ we take $\lambda = 1$. We have to solve the following equations : let $f, g \in \mathbf{L}^1(\mathbf{R})_+$.

$$u + \left(\frac{u^2}{2}\right)_x + u^2 - v^2 = f \tag{2.1}$$

$(SP)_{f,g}$

$$v - \left(\frac{v^2}{2}\right)_x + v^2 - u^2 = g \tag{2.2}$$

1st step : We work in a \mathbf{L}^2 -framework. Let $I_n = [-n, n]$. Let us solve the equations $(SP)_{f,g}$ for $f, g \in \mathbf{L}^2(I_n)_+$. Let β be as above. Then $(SP)_{f,g}$ is equivalent to

$$\beta(w) + \left(\frac{w}{2}\right)_x + w - h = f$$

$(SP)_{\beta,f,g}$

$$\beta(h) - \left(\frac{h}{2}\right)_x + h - w = g$$

through the change of variable $w = u^2, h = v^2$. Let $\bar{\beta}(r) = \sqrt{r}$ if $r \geq 0$, $-\sqrt{|r|}$ if $r < 0$. Let us first consider the system :

$$\bar{\beta}(w) + \left(\frac{w}{2}\right)_x + w - h = f$$

$(SP)_{\bar{\beta},f,g}$

$$\bar{\beta}(h) - \left(\frac{h}{2}\right)_x + h - w = g$$

where $f, g \in \mathbf{L}^2(I_n)$. The existence of a solution of $(SP)_{\bar{\beta},f,g}$ is a consequence of standard perturbation results for maximal monotone operators ([5]). Let $T_{\bar{\beta}} : \mathbf{L}^2(I_n)^2 \rightarrow \mathbf{L}^2(I_n)^2$ be given by $T_{\bar{\beta}}(w, h) = (\bar{\beta}(w), \bar{\beta}(h))$. Let $T : \mathbf{L}^2(I_n)^2 \rightarrow \mathbf{L}^2(I_n)^2$ with domain.

$\text{Dom}(T) = \{(w, h) \in H^1(I_n) \times H^1(I_n) : w(-n) = h(-n), w(n) = h(n)\}$ be given by $T(w, h) = \left(\frac{w_x}{2} + w - h, -\frac{h_x}{2} + h - w\right)$. Since $T_{\bar{\beta}}, T$ are maximal monotone and $\text{Dom}(T_{\bar{\beta}}) = \mathbf{L}^2(I_n)^2, T_{\bar{\beta}} + T$ is maximal monotone ([4], Corol. 2.7). Moreover, since $\bar{\beta}$ is the subgradient of a convex function, by [5], thm. 4, $\text{Int Ran}(T_{\bar{\beta}} + T) = \text{Int}(\text{Ran } T_{\bar{\beta}} + \text{Ran } T)$. But it is an exercise to see that $\text{Ran } T = \mathbf{L}^2(I_n)^2$. Therefore, $\text{Ran}(T_{\bar{\beta}} + T) = \mathbf{L}^2(I_n)^2$. Therefore, for $f, g \in \mathbf{L}^2(I_n), (SP)_{\bar{\beta},f,g}$ has a solution $(w, h) \in H^1(I_n) \times H^1(I_n)$ with $w(-n) = h(-n), w(n) = h(n)$. To go back to problem $(SP)_{\beta,f,g}$ it suffices to remark that $w, h \geq 0$ if $f, g \geq 0$. For that we multiply the first equation in $(SP)_{\bar{\beta},f,g}$ by w^- and the second by h^- . Adding both equations and integrating over \mathbf{R} , one gets :

$$\int_{\mathbf{R}} gh^- + fw^- + (w^-)^{3/2} + (h^-)^{3/2} + (w^- - h^-)^2 + 2w^+h^- dx = 0$$

Since each term in the integrand is positive, $w^- = h^- = 0, i.e., w, h \geq 0$. Thus, given $f, g \in \mathbf{L}^2(I_n)_+$, there exists $w, h \in H^1(I_n)$ with $w(-n) = h(-n), w(n) = h(n), w, h \geq 0$ which solve $(SP)_{\beta,f,g}$. Then $u = \sqrt{w}$ on $I_n, 0$ in $\mathbf{R} - I_n, v = \sqrt{h}$ on $I_n, 0$ in $\mathbf{R} - I_n$ solve $(SP)_{f,g}$.

2nd step : Let $f, g \in \mathbf{L}^1(\mathbf{R})_+$. Let $f_n, g_n \in \mathbf{L}^2(I_n)_+$ be such that $f_n \uparrow f, g_n \uparrow g$. Let (u_n, v_n) be the solutions of $(SP)_{f_n, g_n}$ found in step 1.

Notice that the accretivity of $A+B$ implies that u_n, v_n are Cauchy sequences in $\mathbf{L}^1(\mathbf{R})$. Let $u, v \in \mathbf{L}^1(\mathbf{R})_+$ be the limits of u_n, v_n in $\mathbf{L}^1(\mathbf{R})$. Now adding the corresponding equations to (2.1), (2.2) for $(SP)_{f_n, g_n}$ and using that $u_n, v_n \geq 0$ we get;

$$(3) \quad \left(\frac{u_n^2 - v_n^2}{2} \right)_x \leq f_n + g_n$$

Since $u_n(-n) = v_n(-n), u_n(n) = v_n(n)$, integrating from $-\infty$ to x and from x to ∞ we get $\|u_n^2 - v_n^2\|_\infty \leq 2\|f_n + g_n\| \mathbf{L}^1(\mathbf{R})$. Since, for $a, b \geq 0$, $|a - b| \leq |a^2 - b^2|^{1/2}$, the sequence $u_n - v_n$ is bounded in $\mathbf{L}^\infty(\mathbf{R})$. Then, $u_n^2 - v_n^2 = (u_n - v_n)(u_n + v_n)$ is bounded in $\mathbf{L}^1(\mathbf{R})$. From $(SP)_{f_n, g_n}$ it follows that $\left(\frac{u_n^2}{2} \right)_x, \left(\frac{v_n^2}{2} \right)_x$ are bounded in $\mathbf{L}^1(\mathbf{R})$. This, together with $u_n \rightarrow u, v_n \rightarrow v$ in $\mathbf{L}^1(\mathbf{R})$ implies that u_n, v_n are bounded in $\mathbf{L}^\infty(\mathbf{R})$ and $u_n^2 \rightarrow u^2, v_n^2 \rightarrow v^2$ in $\mathbf{L}^1(\mathbf{R})$. Thus

$$\left(\frac{u_n^2}{2} \right)_x \rightarrow \left(\frac{u^2}{2} \right)_x, \left(\frac{v_n^2}{2} \right)_x \rightarrow \left(\frac{v^2}{2} \right)_x \text{ in } \mathbf{L}^1(\mathbf{R}), (u, v) \in D(A)$$

and letting $n \rightarrow \infty$ in $(SP)_{f_n, g_n}$ we get a solution $(u, v) \in D(A)$ for $(SP)_{f, g}$.

Using the Crandall - Liggett's theorem in combination with lemmas 1 and 2 above, one gets :

PROPOSITION 1. — *For any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ and any $t > 0$, there exists a unique mild (or semigroup) solution $(u, v) \in C([0, T], \mathbf{L}^1(\mathbf{R})_+^2)$ of (BC) with initial data $u(0) = u_0, v(0) = v_0$. If $(u, v), (\hat{u}, \hat{v})$ are two mild solutions of (BC) with initial data $(u_0, v_0), (\hat{u}_0, \hat{v}_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ respectively, then :*

$$\|(u(t) - \hat{u}(t))^+\|_{\mathbf{L}^1(\mathbf{R})} + \|(v(t) - \hat{v}(t))^+\|_{\mathbf{L}^1(\mathbf{R})} \leq \|(u_0 \hat{u}_0)^+\|_{\mathbf{L}^1(\mathbf{R})} + \|(v_0 \hat{v}_0)^+\|_{\mathbf{L}^1(\mathbf{R})}$$

Moreover, if $u_0, v_0 \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^p(\mathbf{R})_+^2, 1 \leq p, \infty$, then $(u(t), v(t)) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^p(\mathbf{R})_+^2$ and for any $t \geq 0$

$$\|u(t)\|_{\mathbf{L}^p(\mathbf{R})} + \|v(t)\|_{\mathbf{L}^p(\mathbf{R})} \leq \|u_0\|_{\mathbf{L}^p(\mathbf{R})} + \|v_0\|_{\mathbf{L}^p(\mathbf{R})}.$$

Proof. — Just remark that the last assertion is a consequence of the inequalities (1) in Lemma 1 ([1], section 2).

Before proving the regularizing estimate (RE) let us prove that the semigroup solution (u, v) of (BC) with initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^\infty(\mathbf{R})_+^2$ obtained via the Crandal-Liggett's theorem is a Kruzkov's solution. This is a consequence of two facts : first, if $(u(t), v(t))$ is the mild solution of (BC) with initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^\infty(\mathbf{R})_+^2$ then $u(t)$ and $v(t)$ are, respectively, the mild solutions of

$$(*) \quad u_t + \left(\frac{u^2}{2} \right)_x = \Psi(t),$$

$$(**) \quad v_t - \left(\frac{v^2}{2} \right)_x = -\Psi(t),$$

where $\Psi(t) \equiv v^2(t) - u^2(t)$ ([10], Lemma 1.7) and second, the well known fact that mild or semigroup solutions of $(*)$ and $(**)$ are in fact Kruzkov's solutions of $(*)$, $(**)$ respectively ([1], Prop. 2.11). Writing what means that $u(t), v(t)$ are Kruzkov's solutions of $(*)$, $(**)$ respectively we get that $(u(t), v(t))$ is a Kruzkov solution of (BC) in the sense of definition 1. One can argue directly using only [1], Prop. 2.11. Recall that (u, v) is obtained in the following way : let $\mathcal{P}_n = \{0 = a_0^n < \dots < a_n^n = T\}$ where $a_k^n = \frac{kT}{n}$. Let $u_n(t), v_n(t)$ be the step functions given by $u_n(0) = 0, v_n(0) = 0, u_n(t) = u_k^n, v_n(t) = v_k^n$ in $]a_{k-1}^n, a_k^n]$, where (u_k^n, v_k^n) are constructed as solutions of the difference scheme :

$$\frac{u_k^n - u_{k-1}^n}{a_k^n - a_{k-1}^n} + \left(\frac{(u_k^n)^2}{2} \right)_x + (u_k^n)^2 - (v_k^n)^2 = 0$$

(DS)

$$\frac{v_k^n - v_{k-1}^n}{a_k^n - a_{k-1}^n} - \left(\frac{(v_k^n)^2}{2} \right)_x + (v_k^n)^2 - (u_k^n)^2 = 0$$

with $u_0^n = u_0, v_0^n = v_0$. Then $u_n(t), v_n(t) \rightarrow u(t), v(t)$ in $\mathbf{L}^1(\mathbf{R})$ uniformly on $[0, T]$. Let $\Psi_n(t) = (v_k^n)^2 - (u_k^n)^2$ on $]a_{k-1}^n, a_k^n]$. Since $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^\infty(\mathbf{R})_+^2$ then $\Psi_n(t) \rightarrow \Psi(t) := v(t)^2 - u(t)^2$ in $\mathbf{L}^1([0, T], \mathbf{L}^1(\mathbf{R}))$ as $n \rightarrow \infty$. Thus $u(t), v(t)$ are mild solutions of

$$\begin{cases} u_t + \left(\frac{u^2}{2} \right)_x = \Psi(t) \\ u(0) = u_0 \end{cases} \quad \begin{cases} v_t - \left(\frac{v^2}{2} \right)_x = \Psi(t) \\ v(0) = v_0 \end{cases}$$

respectively therefore $(u(t), v(t))$ is the Kruzkov's solution of (BC) in $[0, T] \times \mathbf{R}$ with initial data (u_0, v_0) in the sense of Definition 1 ([1], Prop. 2.11).

Since $(u_0, v_0) \in \mathbf{L}^\infty(\mathbf{R})^2$, then $u, v \in \mathbf{L}^\infty([0, 1] \times \mathbf{R})$.

Taking $k > \|u(\cdot, \cdot)\|_\infty, k' > \|v(\cdot, \cdot)\|_\infty$ and then $k < -\|u(\cdot, \cdot)\|_\infty, k' < -\|v(\cdot, \cdot)\|_\infty$ we see that u, v are distributional solutions of (BC) . We can now easily show the regularizing estimate (RE) of theorem 1. Let $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+$. Let $(u_{0n}, v_{0n}) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^\infty(\mathbf{R})_+^2$ be such that $u_{0n} \uparrow u_0, v_{0n} \uparrow v_0$. Let $u_n(t), v_n(t)$ be the solutions of (BC) given by proposition 1. Using [2], Theorem 2, it follows that

$$\begin{aligned} \frac{u_n(t+h) - u_n(t)}{h} &\geq -\frac{1}{t+h} u_n(t) \\ \frac{v_n(t+h) - v_n(t)}{h} &\geq -\frac{1}{t+h} v_n(t) \end{aligned}$$

for $t, h > 0$. This implies that for any $t > 0$ and any $t \in]0, T[$ $u_{nt} \geq -\frac{u_n}{t}, v_{nt} \geq -\frac{v_n}{t}$ in $\mathcal{D}'((0, T) \times \mathbf{R})$. It follows that

$$\left(\frac{u_n^2 - v_n^2}{2} \right)_x \leq \frac{u_n + v_n}{t}$$

in $\mathcal{D}'((0, T) \times \mathbf{R})$. Thus, for any $\varphi \in C_0^\infty(\mathbf{R})$ with $\|\varphi\|_\infty \leq 1, \varphi \geq 0$:

$$(4) \quad \int_{\mathbf{R}} \frac{u_n^2(t, x) - v_n^2(t, x)}{2} \varphi'(x) dx \leq \frac{\|u_n(t) + v_n(t)\|_{\mathbf{L}^1(\mathbf{R})}}{t} \leq \frac{\|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})}}{t}$$

holds *a.e.* with respect to t . Since $u_n, v_n \in C([0, T], \mathbf{L}^1(\mathbf{R})_+)$ it holds for all $t \in]0, T[$. As we remarked above, since $(u_{0n}, v_{0n}) \in \mathbf{L}^\infty(\mathbf{R})^2, (u_n(t), v_n(t)) \in \mathbf{L}^\infty(\mathbf{R})^2$. Then, $u_n(t)^2 - v_n(t)^2 \in \mathbf{L}^1(\mathbf{R})$. Now the following argument can be justified : let x_0 be a Lebesgue point of $u_n(t)^2 - v_n(t)^2$. For each $k \in \mathbf{N}$, take $\varphi_k(x) = 0$ if $x < x_0, k(x - x_0)$ if $x \in]x_0, x_0 + 1/k], 1$ if $x \geq x_0 + 1/k$. Plug φ_k into (4) to get :

$$-k \int_x^{x+1/k} \frac{u_n^2(t, x) - v_n^2(t, x)}{2} dx \leq \frac{\|u_n(t) + v_n(t)\|_{\mathbf{L}^1(\mathbf{R})}}{t}$$

Since x_0 is a Lebesgue point of $u_n^2(t) - v_n^2(t)$, letting $k \rightarrow \infty$ we get :

$$-(u_n^2(t, x_0) - v_n^2(t, x_0)) \leq \frac{2}{t} \|u_n(t) + v_n(t)\|_{\mathbf{L}^1(\mathbf{R})}$$

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Taking now $\varphi_k(x) = 1$ if $x \leq x_0$, $1 - k(x - x_0)$ if $x \in]x_0, x_0 + 1/k[$, 0 if $x \geq x_0$ and repeating the argument above, one gets :

$$(u_n^2(t, x_0) - v_n^2(t, x_0)) \leq \frac{2}{t} \|u_n(t) + v_n(t)\|_{\mathbf{L}^1(\mathbf{R})}$$

Therefore, $u_n^2(t) - v_n^2(t) \in \mathbf{L}^\infty([0, T] \times \mathbf{R})$ and

$$\|u_n^2(t) - v_n^2(t)\|_\infty \leq \frac{2}{t} \|u_n(t) + v_n(t)\|_{\mathbf{L}^1(\mathbf{R})}$$

for all $t \in]0, T]$. Since for $a, b \geq 0$, $|a - b| \leq |a^2 - b^2|^{1/2}$, it follows that

$$\|u(t) - v(t)\|_\infty \leq \frac{\sqrt{2}}{t^{1/2}} (\|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})})^{1/2}$$

and

$$\|u_n^2(t) - v_n^2(t)\|_{\mathbf{L}^1(\mathbf{R})} \leq \frac{\sqrt{2}}{t^{1/2}} (\|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})})^{3/2}$$

Since $u_{nt} + \left(\frac{u_n^2}{2}\right)_x + u_n^2 - v_n^2 = 0$ holds in $\mathcal{D}'((0, T) \times \mathbf{R})$ then :

$$\left(\frac{u_n^2}{2}\right)_x \leq v_n^2 - u_n^2 + \frac{u_n}{t}. \text{ As before, this implies that } u_n \in \mathbf{L}^\infty([0, T] \times \mathbf{R})$$

$$\text{and } \|u_n(t)\|_\infty \leq \left\{ \frac{2}{t} \|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})} + \frac{2\sqrt{2}}{t^{1/2}} (\|u_0 + v_0\|_{\mathbf{L}^1(\mathbf{R})})^{3/2} \right\}^{1/2}$$

for all $n \in \mathbf{N}$ and $t > 0$, Letting $n \rightarrow \infty$ we get (RE) for $u(t)$. Similarly, (RE) holds for $v(t)$.

Now, it is easy to show that for any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$, the semigroup solution of (BC) given by proposition 1 is in fact a Kruzkov's solution of (BC) . (RE) implies that $(u, v) \in \mathbf{L}^\infty([0, T], \mathbf{L}^1(\mathbf{R}))^2 \cap \mathbf{L}^\infty([\tau, T] \times \mathbf{R})^2$ for any $\tau > 0$. Let $u_{0n}, v_{0n} \in \mathbf{L}^1(\mathbf{R})_+ \cap \mathbf{L}^\infty(\mathbf{R})_+$ be such that $u_{0n} \uparrow u_0, v_{0n} \uparrow v_0$. As has been proved above, the semigroup solutions u_n, v_n of (BC) in $[0, T]$ with initial data u_{0n}, v_{0n} satisfy :

$$\int_0^T \int_{\mathbf{R}} |u_n - k|\zeta_t + \text{sign}_0(u_n - k) \left[\left(\frac{u_n^2}{2} - \frac{k^2}{2}\right) \zeta_x + (v_n^2 - u_n^2)\zeta \right] dx dt \geq 0$$

(5)

$$\int_0^T \int_{\mathbf{R}} |v_n - k'|\eta_t + \text{sign}_0(v_n - k') \left[\left(\frac{k'^2}{2} - \frac{v_n^2}{2}\right) \eta_x + (u_n^2 - v_n^2)\eta \right] dx dt \geq 0$$

for all $\zeta, \eta \in C_0^\infty((0, T) \times \mathbf{R})$, $\zeta, \eta \geq 0$, all $k, k' \in \mathbf{R}$ and all $n \in \mathbf{N}$.

Since u_n, v_n satisfy the estimate (RE) , $u_n^2 - v_n^2 \rightarrow u^2 - v^2$ in $L^1([\tau, T] \times \mathbf{R})$ for any $\tau \in [0, T]$ and one can let $n \rightarrow \infty$ in (5) to get

$$\int_0^T \int_{\mathbf{R}} \alpha(t, x, k) \left[(u - k)\zeta_t + \left(\frac{u^2}{2} - \frac{k^2}{2} \right) \zeta_x + (v^2 - u^2)\zeta \right] dx dt \geq 0$$

$$\int_0^T \int_{\mathbf{R}} \beta(t, x, k') \left[(v - k')\eta_t + \left(\frac{k'^2}{2} - \frac{v^2}{2} \right) \eta_x + (u^2 - v^2)\eta \right] dx dt \geq 0$$

for all $\zeta, \eta \in C_0^\infty((0, T) \times \mathbf{R})$, $\zeta, \eta \geq 0$ and all $k, k' \in \mathbf{R}$ where $\alpha(t, x, k) \in \text{sign}(u(t, x) - k)$, $\beta(t, x, k') \in \text{sign}(v(t, x) - k')$.

Using [1], Lemme 2.2, we see that (u, v) is a Kruzkov's solution of (BC) on $[0, T] \times \mathbf{R}$ with initial data (u_0, v_0) .

The uniqueness of Kruzkov's solutions of (BC) follows easily adaptating the arguments of [1], Sect. II. Firts of all we observe that if $(u, v), (\hat{u}, \hat{v})$ are Kruzkov's solutions of (BC) on $[0, T] \times \mathbf{R}$ with respective initial data $(u_0, v_0), (\hat{u}_0, \hat{v}_0) \in L^1(\mathbf{R})_+^2$ then ([1], Prop. 2.7) there exists some $\alpha(t, x) \in \text{sign}(u(t, x) - \hat{u}(t, x))$, $\beta(t, x) \in \text{sign}(v(t, x) - \hat{v}(t, x))$ such that

$$\int_0^T \int_{\mathbf{R}} |u - \hat{u}| \zeta_t + \alpha(t, x) \left[\left(\frac{u^2 - \hat{u}^2}{2} \right) \zeta_x + ((v^2 - u^2) - (\hat{v}^2 - \hat{u}^2))\zeta \right] dx dt \geq 0$$

$$\int_0^T \int_{\mathbf{R}} |v - \hat{v}| \eta_t + \beta(t, x) \left[\left(\frac{\hat{v}^2 - v^2}{2} \right) \eta_x + ((u^2 - v^2) - (\hat{u}^2 - \hat{v}^2))\eta \right] dx dt \geq 0$$

holds for all $\zeta, \eta \in C_0^\infty((0, T) \times \mathbf{R})$, $\zeta, \eta \geq 0$. Take $\zeta, \eta \in C_0^\infty((0, T) \times \mathbf{R})$, $\zeta \geq 0$ in both inequalities and add them. Then, observing that

$$[(u^2 - \hat{u}^2) - (v^2 - \hat{v}^2)](\beta(t, x) - \alpha(t, x))\zeta \leq 0 \text{ a.e. one gets :}$$

$$\int_0^T \int_{\mathbf{R}} (|u - \hat{u}| + |v - \hat{v}|)\zeta_x + \left[\left(\frac{u^2}{2} - \frac{\hat{u}^2}{2} \right) \alpha + \left(\frac{v^2}{2} - \frac{\hat{v}^2}{2} \right) \beta \right] \zeta dx dt \geq 0$$

As in [1], Lemme 2.5, one obtains : for any $\tau \in]0, T[$ fixed

$$\int_{|x| \leq R - C\tau} |u(t, x) - \hat{u}(t, x)| + |v(t, x) - \hat{v}(t, x)| dx$$

$$\leq \int_{|x| \leq R - C\tau} |u(s, x) - \hat{u}(s, x)| + |v(s, x) - \hat{v}(s, x)| dx$$

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for $0 < \tau \leq s \leq t \leq T$, where C is the Lipschitz constant of the function $r \rightarrow \frac{r^2}{2}$ on $\{|r| \leq \max(|u(t, x)|, |\hat{u}(t, x)|, |v(t, x)|, |\hat{v}(t, x)|) : t \in [\tau, T], x \in \mathbf{R}\}$ and $R > Ct$. Thus :

$$(6) \quad \int_{|x| \leq Ct} |u(t, x) - \hat{u}(t, x)| + |v(t, x) - \hat{v}(t, x)| dx \\ \leq \int_{\mathbf{R}} |u(s, x) - \hat{u}(s, x)| + |v(s, x) - \hat{v}(s, x)| dx$$

for any $0 < \tau \leq s \leq t \leq T$. Since $(u(s), v(s)) \rightarrow (u_0, v_0)$ on $\mathbf{L}^1(\mathbf{R})$ as $s \rightarrow 0$, letting $R \rightarrow \infty$ on (6) and then $\tau, s \rightarrow 0$ we get :

$$\int_{\mathbf{R}} |u(t, x) - \hat{u}(t, x)| + |v(t, x) - \hat{v}(t, x)| dx \leq \int_{\mathbf{R}} |u_0 - \hat{u}_0| + |v_0 - \hat{v}_0| dx$$

for any $t > 0$. From this estimate, the uniqueness of Kruzkov's solutions of (BC) follows. This finishes the proof of theorem 1.

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