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RÉSUMÉ. — On donne un résultat du type B.Y. Chen et M. Okumura (voir [3]) sur la courbure scalaire d'une sous-variété M d'une variété de Vaisman (c'est-à-dire une variété localement conformément Kaehlérienne ayant la forme de Lee parallèle et les métriques locales Kaehlériennes plates). Si M est une sous-variété de Cauchy-Riemann Levi-plate (d'une variété de Vaisman), alors on calcule les courbures sectionnelles complexes de M.

ABSTRACT. — We give a B.Y. Chen and M. Okumura (see [3]) type result on the scalar curvature of a submanifold M of a Vaisman manifold (i.e. a locally conformal Kaehler manifold having a parallel Lee form and flat local Kaehler metrics). If M is a Levi-flat Cauchy-Riemann submanifold (of a Vaisman manifold), the complex sectional curvatures of M are estimated.

1. Introduction and statement of results

Let (M, g, J) be a Hermitian manifold of complex dimension n, with the complex structure J and the Hermitian metric g. It is *locally conformal Kaehler* (l.c.K.) if there exists an open covering $(U_i)_{i\in I}$ of M and a family $(f_i)_{i\in I}$ of real valued smooth functions $f_i \in C^\infty(U_i)$ such that each $g_i = \exp(-f_i)g$ is a Kaehler metric on U_i, $i \in I$.

The local 1-forms df_i of a l.c.K. manifold M are known to glue up to a globally defined (closed) 1-form ω on M, namely the *Lee form*.

A l.c.K. manifold is a *generalized Hopf* (g.H.) manifold if its Lee form is parallel with respect to the Riemannian connection of (M, g). Typical

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examples of g.H. manifolds are products $S \times I\!R$ between a Sasaki manifold S and the real line, see [10], p. 614.

Let M be a g.H. manifold. It is said to be a *Vaisman manifold* if the local Kaehler metrics $g_i, i \in I$, of M are flat. Each complex Hopf manifold $CH^n = W/G_d$, $W = C^n - \{0\}$, $G_d = \{d^m I : m \in z\}$, $d \in C - \{0\}$, $|d| \neq 1$, is a Vaisman manifold in a natural way. Indeed, let $g_0 = |z|^{-2}\delta_{ij}dz^i \otimes d\overline{z}^j$, where $|z|^2 = \delta_{ij}z^i\overline{z}^j$ and $(z^1, \ldots, z^n)$ are the natural complex analytic coordinates on W. Note that g_0 is G_d-invariant, thus giving rise to a (globally defined) l.c.K. metric on CH^n.

Let M be a Vaisman manifold. Since the Lee form ω is parallel, its norm is constant; set $\|\omega\| = 2c$, $c \in I\!R - \{0\}$. The local structure of Vaisman manifolds is completely understood due to a deep result of I. Vaisman, (thus justifying our terminology), i.e. the theorem 3.8. in [12], p. 277, asserting that the universal covering of M is W with the metric $\rho^2 g_0$, $\rho = \dfrac{1}{c}$.

The curvature form of a Vaisman manifold is expressed by

$$(1.1)\qquad \begin{aligned} R(X,Y)Z = \frac{1}{4}\{&[\omega(X)Y - \omega(Y)X]\omega(Z) + \\ &+ [g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]B\} + \\ &+ \frac{1}{4}\|\omega\|^2\{g(Y,Z)X - g(X,Z)Y\} \end{aligned}$$

for any tangent vector fields X, Y, Z on M, see (2.1) of [13], p. 441. Here $B = \omega^{\#}$ is the *Lee field* of M, while $\#$ denotes raising of indices with respect to g. As a consequence of (1.1) one obtains the following results :

THEOREM 1.— *Let M be an n-dimensional submanifold of a Vaisman manifold. If the scalar curvature ρ of M is subject to :*

$$(1.2)\qquad \rho \geq (n-2)\|h\|^2 + (n-2)(n-1)c^2 + 2(n-1)A$$

at a point $x \in M$ for some $A \in I\!R$ then the sectional curvatures of M are $\geq A$ at the point x.

If $\overline{M}$ is a Vaisman manifold and $j : M \to \overline{M}$ the given immersion of M in $\overline{M}$, then h denotes the second fundamental form of j. Let ω_0 be the Lee form of $\overline{M}$ and $\omega = j^*\omega_0$. Since ω is closed, the distribution Ker(ω) is integrable thus defining a canonical foliation $\mathcal{F}$ on M, see also [4].

THEOREM 2.— *Let M be a Levi flat Cauchy-Riemann submanifold of a Vaisman manifold. Let $p \in G_2(M)$, $p \subseteq D_{\pi(p)}$, $J(p) = p$. Then the complex* *sectional curvature* k_C *of* M *verifies :*

$$k_C(p) \leq c^2 - \omega_0(h(X,X)) \tag{1.3}$$

for any $X \in p, \|X\| = 1$. *The equality holds if and only if* p *is tangent to some leaf of* $\mathcal{F}$ *passing through* x *and* $h_x = 0$ *on* $p \times p$.

Here $\pi : G_2(M) \to M$ denotes the Grassman bundle of all 2-planes tangent to M. Also D stands for the Levi distribution of the C.R. submanifold M, (i.e. D_x is the maximal holomorphic subspace of $T_x(M)$, $x \in M$).

For other results concerning the geometry of (the second fundamental form of) submanifolds in l.c.K. manifolds see [4], [5], [6], [7], [8].

2. Scalar curvature of submanifolds in Vaisman manifolds

Let M be an n-dimensional submanifold of a Vaisman manifold $(\overline{M}, \overline{g}, J)$. By (2.8) in [4], p. 203, the Gauss equation of M in $\overline{M}$ is given by :

$$\begin{aligned} R(X,Y)Z = {} & A_{h(Y,Z)}X - A_{h(X,Z)}Y + \\ & + \frac{1}{4}\{[\omega(x)Y - \omega(Y)X]\omega(Z) + \\ & + [g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]B\} + \\ & + \frac{1}{4}\|\omega_0\|^2\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{2.1}$$

for any tangent vector fields X, Y, Z on M. Here $g = j^*\overline{g}$. Moreover A_ξ is the Weingarten operator (associated with the normal section ξ). Suitable contraction of indices in (2.1) leads to the expression of the Ricci tensor of (M, g), i.e.

$$\begin{aligned} R_{jk} = {} & h^a_{jk} \text{ Trace } (A_a) - g^{is}h^a_{ik}h^b_{js}\delta_{ab} + \\ & + c^2(n-2)g_{jk} - \frac{n-2}{4}\omega_j\omega_k \end{aligned} \tag{2.2}$$

Indices $i, j, k, \ldots$ run from 1 to n, while $a, b, c, \ldots$ from 1 to codim$(M) = 2m - n$. Further contraction of indices in (2.2) gives :

$$\rho = n^2\|H\|^2 - \|h\|^2 + c^2(n-1)(n-2) \tag{2.3}$$

Here ρ, H denote respectively the scalar curvature of (M, g) and the mean curvature vector (i.e. $H = \frac{1}{n}$ Trace (h)) of the given immersion j.

Let $k : G_2(M) \to IR$ be the sectional curvature of (M, g). Let $p \in G_2(M)$ and $\{X, Y\}$ an orthonormal basis in p. By (2.1) one obtains :

$$k(p) = \overline{g}(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \\ + c^2 - \frac{1}{4}\{\omega(X)^2 + \omega(Y)^2\} \tag{2.4}$$

At this point we may prove our Theorem 1. To this end, let $x \in M$ and (U, x^i) be normal coordinates at x.

Substitution from (2.3) into(1.2) furnishes :

$$n^2\|H\|^2 \geq (n-1)\|h\|^2 + 2(n-1)A \tag{2.5}$$

Let ξ_a, $1 \leq a \leq 2m - n$, $\dim(\overline{M}) = 2m$, be an orthonormal frame in the normal bundle $T(M)^\perp$ of the given immersion. For simplicity, we may choose ξ_1 to be collinear with H at x (if $H_x \neq 0$, and arbitrary if $H_x = 0$ occurs). Let $X_i = \dfrac{\partial}{\partial x^i}$, $1 \leq i \leq n$. We set $h(X_i, X_j) = h^a_{ij}\xi_a$. Also $h^i_{ja} = g^{ik}h^a_{jk}, h^{ij}_a = g^{jk}g^{is}h^a_{ks}$. Clearly $h^a_{ij} = h^a_{ji}$. All computations are carried out at x (where $g_{ij} = \delta_{ij}$) so that $h^a_{ji} = h^i_{ja} = h^{ji}_a$ at x. Let us put $h_{ij} = h^1_{ij}$. Then :

$$n^2\|H\|^2 = \left(\sum_{i=1}^n h_{ii}\right)^2 \tag{2.6}$$

Substitution from (2.6) into (2.5) gives :

$$\left(\sum_{i=1}^n h_{ii}\right)^2 \geq (n-1)\left\{\sum_{i=1}^n (h_{ii})^2 + \right. \\ \left. + \sum_{i\neq j}(h_{ij})^2 + \sum_{\substack{a\geq 2 \\ 1\leq i,j,\leq n}} (h^a_{ij})^2 \right\} + \\ + 2(n-1)A \tag{2.7}$$

since $\|h\|^2 = h^a_{ji}h^{ji}_a$. We shall need the following :

LEMMA .— *(B.Y. Chen and M. Okumura, [3])*

Let $a_1, \ldots, a_n, b$ be real numbers, $n > 1$, with the property :

$$\left(\sum_{i=1}^n a_i\right)^2 \geq (n-1)\sum_{i=1}^n (a_i)^2 + b$$

Then for any $i \neq j$ *one has* $2a_i a_j \geq \dfrac{b}{n-1}$.

At this point we may use (2.7) and the Lemma (for $a_i = h_{ii}$) such as to yield :

$$(2.8) \qquad h_{ii}h_{jj} - (h_{ij})^2 \geq \\ \geq \sum_{a=2}^{2m-n} \{|h^a_{ii}h^a_{jj}| + (h^a_{ij})^2\} + A$$

for any $i \neq j$. Set $\omega_i = \omega(X_i)$. Then $\omega_i^2 + \omega_j^2 \leq \sum_{i=1}^{n} \omega_i^2 = \|\omega\|^2 = 4c^2$. Let $p_{ij} \in G_2(M)$ be spanned by X_i, X_j, $i \neq j$. Finally, using (2.4) and (2.8) we have :

$$k(p_{ij}) = \sum_{a=1}^{2m-n} (h^a_{ii}h^a_{jj} - (h^a_{ij})^2) + \\ + c^2 - \frac{1}{4}(\omega_i^2 + \omega_j^2) \geq A$$

Q.E.D.

This extends Theorem 4.1. in [1], p. 55, to the case of submanifolds in Vaisman manifolds.

3. Cauchy-Riemann submanifolds of Vaisman manifolds

Let $(\overline{M}, \overline{g}, J)$ be a Vaisman manifold of complex dimension m and M a real n-dimensional Cauchy-Riemann (C.R.) submanifold of $\overline{M}$. That is M carries a pair of orthogonal (complementary) distributions $D, D^\perp$ such that D is holomorphic, i.e. $J_x(D_x) = D_x$, $x \in M$, while $D^\perp$ is totally-real, i.e. $J_x(D_x^\perp) \subseteq T_x(M)^\perp$, $x \in M$. See also [15], p. 83. Hereafter D is called the *Levi distribution* of M. Moreover, if D is integrable, the C.R. submanifold M is said to be *Levi flat*.

Let $B_0 = \omega_0^\#$ and $A_0 = -JB_0$ be the Lee, respectively the *anti-Lee vector* fields of $\overline{M}$. Also $\theta_0 = \omega_0 \circ J$ will denote the *anti-Lee form*.

Let X, ξ be respectively a tangent vector field on M and a normal section. We set $PX = \tan(J, X)$, $FX = \mathrm{nor}(JX)$, $t\xi = \tan(J\xi)$, $f\xi = \mathrm{nor}(J\xi)$.

Here $\tan_x$, nor_x denote the natural projections associated with the direct sum decomposition $T_x(\overline{M}) = T_x(M) \oplus T_x(M)^\perp$, for any $x \in M$. Note that

P is D-valued. Also $F = 0$ on D. Moreover the following identities hold :

$$\begin{aligned} F \circ P &= 0, f \circ F = 0 \\ P^2 + t \circ F &= -I \\ t \circ f &= 0, P \circ t = 0 \\ f^2 + F \circ t &= -I \end{aligned} \tag{3.1}$$

Let $B = \tan(B_0)$, $B^\perp = \operatorname{nor}(B_0), A = \tan(A_0)$ and $A^\perp = \operatorname{nor}(A_0)$. Note that :

$$A = -PB - tB^\perp, A^\perp = -FB - fB^\perp \tag{3.2}$$

The complex structure J is not parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$. Nevertheless $\overline{M}$ admits a significant almost complex connection $\overline{D}$, namely the *Weyl connection*, i.e.

$$\begin{aligned} \overline{D}_X Y = \overline{\nabla}_X Y - \frac{1}{2}\{\omega_0(X)Y + \omega_0(Y)X - \\ - \overline{g}(X,Y)B_0\} \end{aligned} \tag{3.3}$$

Since $\overline{D}J = 0$, (3.3) yields :

$$\begin{aligned} \overline{\nabla}_X JY = J\overline{\nabla}_X Y + \\ + \frac{1}{2}\{\theta_0(Y)X + \omega_0(Y)JX - \\ - \overline{\Omega}(X,Y)B_0 - \overline{g}(X,Y)A_0\} \end{aligned} \tag{3.4}$$

Here $\overline{\Omega}$ denotes the Kaehler 2-form of $\overline{M}$. By (3.4) and the Gauss formula (1.10) of [1], p. 38, one obtains :

$$\begin{aligned} \nabla_X JY = P\nabla_X Y + th(X,Y) + \\ + \frac{1}{2}\{\theta(Y)X + \omega(Y)JX - \\ - \Omega(X,Y)B - g(X,Y)A\} \end{aligned} \tag{3.5}$$

$$\begin{aligned} h(X,JY) = fh(X,Y) + F\nabla_X Y - \\ - \frac{1}{2}\{\Omega(X,Y)B^\perp + g(X,Y)A^\perp\} \end{aligned} \tag{3.6}$$

for any $X, Y \in D$. Here ∇ denotes the Levi-Civita connection of (M, g) and $\theta = j^*\theta_0$, $\Omega = j^*\overline{\Omega}$.

Let us denote by k_C the restriction of the sectional curvature k of M to the holomorphic 2-planes $p \in G_2(M)$, $J(p) = p$, with the property $p \subseteq D_x$, $x = \pi(p)$, $x \in M$. Then k_C is called the *complex sectional curvature* of the C.R. submanifold M.

At this point we may prove our Theorem 2. As the Levi distribution D is integrable, one has $F\nabla_X Y = F\nabla_Y X$, for any $X, Y \in D$. By (3.5)-(3.6) one obtains :

$$h(JX, JY) = -h(X, Y) - g(X, Y)B^{\perp} \tag{3.7}$$

for any $X, Y \in D$. Let us apply (2.4) for the 2-plane $p \in G_2(M)$ spanned by $\{X, JX\}$, $X \in D_x, \|X\| = 1$, $x = \pi(p)$. It follows :

$$\begin{aligned} k_C(p) = \overline{g}(h(X,X), h(JX,JX)) - \|h(X,JX)\|^2 + \\ + c^2 - \frac{1}{4}\{\omega(X)^2 + \theta(X)^2\} \end{aligned} \tag{3.8}$$

Finally (3.7)-(3.8) lead to (1.3). If equality holds, then $h(X, X) = 0$, $\omega(X) = 0$, $\omega(JX) = 0$, (and actually (1.3) reads $k_C(p) = c^2$). The converse is obvious, Q.E.D.

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