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Asymptotic estimates in Weighted Hölder spaces for a class of elliptic scale-covariant second order operators

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**Résumé.** — On dérive des estimations a priori dans les espaces de Hölder à poids pour des opérateurs elliptiques de second ordre "covariant" sous l'action des transformations de changement d'échelle, \(r \to \lambda r\). On démontre l'existence des solutions du "problème de Dirichlet extérieur modulo un nombre fini d'harmoniques sphériques", ce qui est un résultat utile lorsqu'on s'intéresse au problème d'existence de solutions de l'équation avec source indépendamment des valeurs au bord de la solution (l'alternative habituelle de Fredholm ne garantit pas existence de solutions pour toute source). Finalement les résultats obtenus sont utilisés pour établir un développement asymptotique fin de certaines solutions de l'équation de \(p\)-Laplace pour des variétés asymptotiquement euclidiennes.

**Abstract.** — Pointwise asymptotic estimates for solutions of second order elliptic equations of a certain form are derived.

1. Introduction

It has been recently observed [Ki] [JK] that the existence of solutions of the \(p\)-Laplace equation.

\[
\text{div}(|\text{grad } f|^{p-2} \text{ grad } f) = 0 \tag{1.1}
\]

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satisfying certain conditions allows an elementary proof to the positive
energy theorem in general relativity. In order to carry out this proof one
needs certain fine asymptotic estimates on the behaviour on the solutions
of (1.1) at infinity. The proof of such estimates, presented in section 4 of
this paper, requires an asymptotic estimate on the behaviour of solutions
of linear equations of the form(*):

$$\left[ \delta_{ij} + (p - 2) \frac{x^i x^j}{r^2} \right] f,ij = \rho, \quad \rho = O(r^{\alpha-2}). \quad (1.2)$$

A sharp estimate for equations of this type does no seem to exist in the
literature unless $p = 2$ [Me]. Equations of this type have been studied by
Bagirov and Kondratev [BK] in weighted Sobolev spaces, their results can
be used to obtain pointwise estimates via weighted embeddings theorems
which, however, are not sharp as far as the pointwise decay (or growth) rates
are concerned. If $p = 2$ pointwise estimates of the appropriate type have
been obtained by N. Meyers [Me] (cf. also [CSCB] and references therein
for a restricted range of decay rates but for more general operators). The
object of this paper is to present an elementary argument which leads to a
priori estimates for solutions of equations of the form:

$$\left[ a \frac{d^2}{dr^2} + \frac{b}{r} \frac{d}{dr} + \frac{c + \Delta S}{r^2} \right] f = \rho \quad (1.3)$$

$a \in \mathbb{R}^+, b, c \in \mathbb{R}$, $\Delta S$ is the Laplace-Beltrami operator on a compact
Riemannian manifold $S$ (in the case of interest, eq. $(1.2)$, $S$ is a sphere).
In the particular case $a = 1$, $b = n - 1$, $c = 0$, $\Delta S = S_{n-1}$, $(1.3)$ is just
the Laplace equation and we recover the results of Meyers [Me], using an
apparently much simpler method. The idea of the proof given here is to
solve $(1.3)$ by explicit integrals for a finite number of terms in the harmonic
decomposition of $\rho$ and to use the comparison principle for an “energy
integral” to show that the remainder also has the required asymptotic
properties. When $(1.3)$ is the Laplace equation, it is well known that for $\rho$
decaying as $r^{-k}$, $k$-integer, $k \geq n - 2$, there will in general be logarithmic
terms in $f$; therefore it is natural to consider a logarithmic weight for $\rho$ as
well since in some applications on needs to iterate $(1.3)$. Our main result
can loosely be described as follows: if

$$|\rho| \leq C r^{\alpha}(1 + \ln r)^q,$$

(*) A comma denotes partial differentation, the summation convention is used
throughout.
there always exists a solution of (1.3) satisfying:

$$|f| \leq C'r^{\alpha+2}(1 + \ln r)^{q+2}.$$  \hfill (1.4)

If we know the spectrum of $S$, the $q + 2$ exponent in the logarithm above can be replaced by $q + 1$ or by $q$ under certain conditions. In section 2 the asymptotic estimates are derived. In section 3 existence of solutions of (1.3) satisfying (1.4) is established using the results of section 2. In section 4 the results of section 2 are used to obtain the desired asymptotic estimate for solutions of the $p$-Laplace equation.

2. Asymptotic Estimates

Let $r \in [1, \infty)$, let $S$ be a compact Riemannian manifold, let $\Omega = [1, \infty) \times S$, let $a$ a twice differentiable function $f$ satisfy:

$$L f = \left[ a \frac{d^2}{dr^2} + \frac{b}{r} \frac{d}{dr} + \frac{c + \Delta S}{r^2} \right] f = \rho$$ \hfill (2.1)

$a \in \mathbb{R}^+$, $b, c \in \mathbb{R}$. Let $\{\varphi_i\}$ be a complete orthonormal set of eigenfunctions of $\Delta_S$, $\Delta_S \varphi_i = -\lambda_i \varphi_i$, $\varphi_i$ ordered in such a way that $\lambda_i \leq \lambda_j$ for $i < j$. Let $\mathcal{H}_i$ denote an eigenspace of $-\Delta_S$, with the indices ordered with increasing eigenvalues, let $O_k = \bigcup_{i \leq k} \mathcal{H}_i$, let $V_k$ be the orthogonal complement of $O_k$ (in $L_2(S)$). Associated to the eigenvalues of $-\Delta_S$ are the "characteristic decay exponents $\mu_i^\pm$:

$$\mu_i^\pm = \frac{a - b \pm \sqrt{(b-a)^2 + 4a(\lambda_i - c)}}{2a}.$$ \hfill (2.2)

For $x = (r_1, p_1)$, $y = (r_2, p_2)$, we set $|x - y|^2 = (r_1 - r_2)^2 + d(p_1, p_2)^2$, where $d$ is the Riemannian distance on $S$. For $k \in \mathbb{N} \cup \{0\}$, $\alpha, q \in \mathbb{R}$, $\lambda \in (0, 1)$, we define:

$$\|f\|_{C_{\alpha, q}^k} = \sum_{|i| \leq k} \sup_{x \in \Omega} \left| D_i f(x) r^{-\alpha + |i|} (1 + \ln r)^{-q} \right|$$

$$[f]_{\lambda, \alpha, q} = \sup_{x \in \Omega} \sup_{y \in \{[\frac{1}{2} r(x), 2r(x)] \times S \cap \Omega}} \frac{|f(x) - f(y)| r^{-\alpha + \lambda} (1 + \ln r)^{-q}}{|x - y|^\lambda}$$

$$\|f\|_{C_{\alpha, \lambda}^k} = \|f\|_{C_{\alpha, q}^k} + \sum_{|i| = k} [r^k D_i f]_{\lambda, \alpha, q}.$$ 

(*) We assume that $S$ has no boundary. All the results presented here will be valid if $f$ is assumed to vanish on $\partial S$, when non-empty.
We shall need the following elementary result:

**Lemma 2.1.** Let \( \rho = \rho_i(r) \varphi_i \) (no summation over \( i \)), with \( \Delta_S \varphi_i = -\lambda_i \varphi_i \), let the function \( f_i \) be given by:

a) If \( \mu_i^+ \neq \mu_i^- \), let:

\[
f_i = \frac{1}{a(\mu_i^+ - \mu_i^-)} \left\{ f_i^+ r^{\mu_i^+} + f_i^- r^{\mu_i^-} + r^{\mu_i^+} \int_1^r s^{1-\mu_i^-} \rho_i(s) \, ds - r^{\mu_i^-} \int_1^r s^{1-\mu_i^+} \rho_i(s) \, ds \right\}, \quad f_i^\pm \in \mathbb{R}. \tag{2.3}
\]

b) If \( \mu_i^+ \neq \mu_i^- \) and if \( s^{1-\mu_i^+} \rho_i(s) \in L_1([1, \infty)) \) (*) let:

\[
f_i = \frac{1}{a(\mu_i^+ - \mu_i^-)} \left\{ r^{\mu_i^-} \left( f_i^- - \int_1^r s^{1-\mu_i^-} \rho_i(s) \, ds \right) - r^{\mu_i^+} \int_r^\infty s^{1-\mu_i^+} \rho_i(s) \, ds \right\}, \quad f_i^- \in \mathbb{R}. \tag{2.4}
\]

c) If \( \mu_i^+ \neq \mu_i^- \) and if \( s^{1-\mu_i^+} \rho_i(s) \in L_1([1, \infty)) \), let:

\[
f_i = \frac{1}{a(\mu_i^+ - \mu_i^-)} \left\{ r^{\mu_i^-} \left( f_i^- + \int_r^\infty s^{1-\mu_i^-} \rho_i(s) \, ds \right) - r^{\mu_i^+} \int_r^\infty s^{1-\mu_i^+} \rho_i(s) \, ds \right\}, \quad f_i^- \in \mathbb{R}. \tag{2.5}
\]

d) If \( \mu_i^+ = \mu_i^- = \mu_i \), let:

\[
f_i = \frac{r^{\mu_i}}{a} \left\{ \alpha_i + \beta_i \ln r + \frac{1}{2} \int_1^r s^{1-\mu_i} \rho_i(s) \, ds s \, ds \right\}, \quad \alpha_i, \beta_i \in \mathbb{R}. \tag{2.6}
\]

\( (*) \) (2.4) and (2.5) are clearly special cases of (2.3), it is however useful to write them explicitly for the sake of clarity of the proof of theorem 2.1. A similar remark applies to (2.6)-(2.8).
The function $f = f_i \varphi_i$ satisfies $L f = \rho$.

The following computational lemma shows that under appropriate conditions the functions $\pm (C_1 + \ln r)^q r^\alpha$ are super- and sub-solutions of the equation (2.1).

**Lemma 2.2.** Let $a > 0$, $b, e, a, q \in \mathbb{R}$. There positive constants $C_1(a, b, e, a, q)$ and $C_2(a, b, e, a, q)$ such that, if:

$$e < -\alpha [a(\alpha - 1) + b]$$

then:

$$\left[ a \frac{d^2}{dr^2} + \frac{b}{r} \frac{d}{dr} + \frac{e}{r^2} \right] (C_1 + \ln r)^q r^\alpha \leq -C_2 (1 + \ln r)^q r^{\alpha - 2}. \quad (2.9)$$

We shall also need the following well known result:

**Lemma 2.3.** For $\varphi \in W_{1,2}(S) \cap \mathcal{V}_k$ we have:

$$\int_S (\nabla \varphi)^2 \, d\mu \geq \lambda_{k+1} \int_S \varphi^2 \, d\mu. \quad (2.10)$$

**Lemma 2.4.** Let $\Omega_0 = [1, R_0] \times S$ or let $\Omega_0 = [1, \infty) \times S$, let $f$ satisfy in $\Omega_0$:

$$L f = \rho, \quad \rho \in C_{\alpha-2, q}, \quad \sup_{p \in S} |f(1, p)| \leq M,$$

and

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There exists $m_0(a, b, c, \lambda, \alpha, \beta)$ such that the condition:

$$\forall \varphi \in O_{m_0}, \forall r \in [1, \infty), \int_S f(r, p) \varphi(p) \, d\mu = 0$$  \hspace{1cm} (2.11)

implies $f \in C_{\alpha, \varrho}(\Omega_0)$, and:

$$\|f\|_{C_{\alpha, \varrho}(\Omega_0)} \leq C_3 \left( \|\rho\|_{C_{\alpha-2, \varrho}(\Omega_0)} + M \right).$$

If moreover $\rho \in C^{k, \lambda}_{\alpha-2, \varrho}, f|\{1\} \times S = \varphi \in C^{k+2, \lambda}(S)$, and $f|\{R_0\} \times S = 0$ if $\Omega_0 = [1, R_0] \times S$, then:

$$\|f\|_{C^{k+2, \lambda}_{\alpha-2, \varrho}(\Omega_0)} \leq C_4 \left( \|\rho\|_{C^{k, \lambda}_{\alpha-2, \varrho}(\Omega_0)} + \|\varphi\|_{C^{k+2, \lambda}(S)} \right)$$

for some constants $C_3(a, b, c, \lambda_{m_0+1}, \alpha, \beta, q, \lambda), C_4(a, b, c, \Delta_S, k, \alpha, \beta, q, \lambda)$. $C_3$ and $C_4$ are $\beta$-independent if $\Omega = [1, R_0] \times S$.

Proof. — If $\alpha < 0$ or $\beta \leq 0$ let $\tilde{\beta} = 0$, if $\alpha > 0$ and $\Omega_0 = [1, R_0] \times S$ let $\tilde{\beta} = \alpha + 1$, if $\alpha > 0$ and $\Omega_0 = [1, \infty) \times S$ let $\tilde{\beta} = \beta$. The function $\tilde{f} = r^{-\tilde{\beta}}f$ satisfies an equation of the form (2.1):

$$\tilde{L}\tilde{f} = \tilde{\rho}, \quad \tilde{L} = a \frac{d^2}{dr^2} + \frac{b}{r} \frac{d}{dr} + \frac{c + \Delta_S}{r^2}$$

with:

$$\tilde{b} = b + 2\tilde{\beta}a, \quad \tilde{c} = c + a\tilde{\beta}^2 + (b - a)\tilde{\beta}, \quad \tilde{\rho} = r^{-\tilde{\beta}}\rho.$$

Let $F(r) = \int_S \tilde{f}^2(r, p) \, d\mu$. We have:

$$a \frac{d^2 F}{dr^2} = 2 \left\{ a \int_S \left[ \frac{d\tilde{f}}{dr} \right]^2 - \frac{b}{r} \int_S \tilde{f} \frac{d\tilde{f}}{dr} - \frac{c}{r^2} \int_S \tilde{f}^2 - \frac{1}{r^2} \int_S \tilde{f} \Delta_S \tilde{f} + \int_S \tilde{f} \tilde{\rho} \right\}$$

$$\geq -\frac{\tilde{b}}{r} \frac{dF}{dr} - \frac{(2\tilde{c} - 2\lambda_{m_0+1} + 1)}{r^2} F - \int_S \tilde{\rho}^2 r^2,$$

and we have used lemma 2.3 and (2.11) to estimate the $\int \tilde{f} \Delta_S \tilde{f}$ term, together with $2\tilde{f} \tilde{\rho} \geq -\frac{\tilde{f}^2}{r^2} - \tilde{\rho}^2 r^2$, so that:

$$a \frac{d^2 F}{dr^2} + \frac{\tilde{b}}{r} \frac{dF}{dr} + \frac{2\tilde{c} + 1 - 2\lambda_{m_0+1}}{r^2} F \geq -\|\rho\|_{C_{\alpha-2, \varrho}}^2 r^{2\tilde{\alpha} - 2}(1 + \ln r)^2q,$$

$$\tilde{\alpha} = \alpha - \tilde{\beta}.$$
If $\lambda_{m+1} > \{\tilde{a}[\alpha(\tilde{a}+1) + \tilde{b}] + 2\tilde{c} + 1\}/2$, lemma 2.2 and the comparison principle yield $F \to 0$ as $r \to \infty$ if $R_0 = \infty$:

$$\forall r \in [1, \infty), \quad F^{1/2}(r) \leq Cr^{\tilde{a}}(1 + \ln r)^q \left(\|\rho\|_{C_{\alpha-2, q}} + M\right).$$  \quad (2.12)

For $4 \leq R \leq R_0/4$ and $r \in (1/2, 2)$ consider:

$$f_R(r, p) = R^{-\alpha}(1 + \ln r)^{-q}f(Rr, p).$$

$f_R$ satisfies $Lf_R = \rho_R$ with

$$\rho_R(r, p) = R^{-\alpha}(1 + \ln R)^{-q}\rho(Rr, p)$$

and (2.12) implies:

$$\|f_R\|_{L^2((1/2, 2) \times S)} \leq C \left(\|\rho\|_{C_{\alpha-2, q}} + M\right).$$

Interior $L^2$ estimates [Se] and Schauder theory [GT] lead to:

$$\|f_R\|_{C([1/2, 2] \times S)} \leq C \left(\|\rho\|_{C_{\alpha-2, q}} + M\right),$$

$$\|f_R\|_{C^{k+2, \lambda}([1/2, 2] \times S)} \leq C \left(\|\rho\|_{C_{\alpha-2, q}} + M\right).$$ \quad (2.13)

(2.13) and $L_\infty$ or $C^{k, \alpha}$ up to boundary estimates [Se] [GT] give:

$$\|f\|_{C([1, 4] \times S)} \leq C \left(\|\rho\|_{C_{\alpha-2, q}} + M\right),$$

$$\|f\|_{C^{k+2, \lambda}([1, 4] \times S)} \leq C \left(\|\rho\|_{C_{\alpha-2, q}} + \|\varphi\|_{C^{k+2, \lambda}(S)}\right).$$

These estimates combined with the rescaled version of (2.13) yield the claimed result.

We shall state in detail our estimates only when all the $\mu_i^+$ and $\mu_i^-$ are distinct (cf. remark 4 below):

**Theorem 2.1.** Suppose that $\forall i$, $\mu_i^+ \neq \mu_i^-$. Let $f$ be a twice differentiable function satisfying:

$$Lf = \rho, \quad f = O(r^\beta), \quad \rho = O(r^{\alpha-2}(1 + \ln r)^q), \quad \alpha < \beta, \quad \sup_{p \in S}\left|f(1, p)\right| = M,$$
\( \alpha, \beta, q \in \mathbb{R}, \) and if \( \alpha \in \{ \mathbb{R} \mu_i^\pm \} \) we shall suppose \( q \neq 1, \) if \( \beta \in \{ \mathbb{R} \mu_i^\pm \} \) we shall suppose \( q > 0. \) \( f \) can be written in the form:

\[
f = f_H + f_H^- + f_{\log} + f_\rho
\]

where:

\[
Lf_H = Lf_H^- = 0
\]

and:

\[
f_H = \sum_{\mathbb{R} \mu_i^+ < \beta} \left( A_i^+ r^{\mu_i^+} + A_i^- r^{\mu_i^-} \right) \varphi_i, \quad A_i^\pm \in \mathbb{R}
\]

\[
|f_H| \leq C(f) r^{\mu^+}, \quad \mu^+ = \max_{\mathbb{R} \mu_i^+ < \beta} \mathbb{R} \mu_i^+
\]

\[
f_H^- = \sum_{i: \mathbb{R} \mu_i^+ \geq \beta, \alpha \leq \mathbb{R} \mu_i^- < \beta} c_i^- r^{\mu_i^-} \varphi_i
\]

\[
c_i^- = \int_S f(1, p) \varphi_i(p) \, d\mu
\]

\[
+ \frac{1}{a(\mu_i^+ - \mu_i^-)} \int_{[1, \infty) \times S} \left( r^{1-\mu_i^+} - r^{1-\mu_i^-} \right) \varphi_i(p) \rho(r, p) \, dr \, d\mu
\]

\[
|f_H^-| \leq C_H^- \left( M + \| \rho \|_{C_{\alpha-2, q}} \right) r^{\mu^-}, \quad \mu^- = \max_{\mathbb{R} \mu_i^+ \geq \beta, \alpha < \mathbb{R} \mu_i^- < \beta} \mathbb{R} \mu_i^-
\]

\[
C_H^- = C_H^- (a, b, c, \alpha, \beta, \lambda_i)
\]

\[
f_{\log} = \sum_{i: \mathbb{R} \mu_i^- = \alpha} f_i^- (r) \varphi_i + \sum_{i: \mathbb{R} \mu_i^+ = \alpha, \mathbb{R} \mu_i^- \neq \alpha} f_i^+ (r) \varphi_i
\]

\[
|f_{\log}| \leq C_{\log} \| \rho \|_{C_{\alpha-2, q}} r^{\alpha} (1 + \ln r)^{q+1}
\]

\[
|f_\rho| \leq C_0 \left( M + \| \rho \|_{C_{\alpha-2, q}} \right) r^{\alpha} (1 + \ln r)^{q},
\]
where the constants \( C_{\text{log}} \) and \( C_0 \) depend on \( a, b, c, \alpha, q \) and the spectrum of \( S(\ast) \). If \( \rho \in C^{k, \lambda}_{\alpha-2, q} \) then:

\[
\|f_\rho\|_{C^{k+2, \lambda}_{\alpha, q}} \leq C(k, \lambda, \alpha, q, \Delta S, a, b, c) \left( \|\rho\|_{C^{k, \lambda}_{\alpha-2, q}} + \|\varphi\|_{C^{k+2, \lambda}(S)} \right).
\]

\[
\|f_{\log r}\|_{C^{k+2, \lambda}_{\alpha, q+1}} \leq C(k, \lambda, \alpha, q, \Delta S, a, b, c)\|\rho\|_{C^{k, \lambda}_{\alpha-2, q}} \tag{2.24}
\]

where \( \varphi = f \mid_{\{1\} \times S} \).

**Remarks:**

1) The coefficients \( A_i^\pm \) are neither determined by the asymptotic condition \( f = O(r^\beta) \) nor by \( f \mid_{\{1\} \times S} \).

2) It should be stressed that all the sums in the statement of theorem 2.1 are over finite sets of indices.

3) If any sets of indices over which the summations are performed above are empty, then the corresponding term is of course zero. In particular, if \( \{\alpha\} \cap \{\Re \mu_i^\pm\} = \emptyset \) there are no higher powers of \( \ln r \) in \( f \). It must also be noted that though the \( r \) dependence of the higher log terms is fairly arbitrary — dictated by the behaviour of \( \rho \) — the “angular” dependence of these terms is rigidly fixed.

4) If for some \( i_0 \) we have \( \mu_{i_0}^+ = \mu_{i_0}^- \) then \( f \) will differ from (2.14) by a finite number of terms of the form (2.6)—(2.8), with radial behaviour which can “pick-up” up to two powers of \( \ln r \) more than \( \rho \) has if \( \alpha = \mu_{i_0}^\pm \), the details are straightforward and are left to the reader.

5) If \( \alpha \in \{\Re \mu_i^\pm\} \) and \( q = -1 \) the function \( f_{\log} \) will be replaced by \( f_{\log(\log)} \), with (2.22) replaced by:

\[
\left| f_{\log(\log)} \right| \leq C_{\log} \log \log(a, b, c, \alpha, \lambda_i) \|\rho\|_{C^{\alpha-2, q}} r^\alpha \ln(1 + \ln r).
\]

6) If \( \beta \in \{\Re \mu_i^\pm\} \) and \( q < 0 \) the results remain valid except (2.18), where no terms with \( \Re \mu_i^\pm = \beta \) will appear in the summation.

**Proof.** — Let \( m_0 \) be given by lemma 2.4, let \( \varphi_i, i = 1, \ldots, J, \) be an orthonormal basis of \( O_{m_0} \), let:

\[
f_i(r) = \int_S f(r, p) \varphi_i(p) \, d\mu
\]

\[
\rho_i(r) = \int_S \rho(r, p) \varphi_i(p) \, d\mu.
\]

\((\ast)\) In fact only a finite number of \( \lambda_i \)'s matters.
The function:

\[ \tilde{f} = \left( f - \sum_{i=1}^{J} f_i \varphi_i \right) \]

satisfies the equation:

\[ L\tilde{f} = \tilde{\rho}, \quad \tilde{\rho} = \rho - \sum_{i=1}^{J} \rho_i \varphi_i, \]

therefore by lemma 2.4 \[|\tilde{f}| \leq Cr^\alpha(1 + \ln r)^q\|\rho\|_{C^{-2,q}}, \] and to achieve our proof, it is sufficient to estimate the finite number of terms \( \sum f_i \varphi_i \). We have:

\[ L(f_i \varphi_i) = \rho_i \varphi_i \]

therefore the \( f_i \)'s are given by lemma 2.1. The set \( I = \mathbb{N} \cap [0, J] \) can be partitioned as follows:

\[
\begin{align*}
I_0^0 &= \{i \in I : \Re \mu_i^+ \geq \beta, \Re \mu_i^- \geq \beta\} \\
I_0^1 &= \{i \in I : \Re \mu_i^+ \geq \beta, \Re \mu_i^- < \alpha\} \\
I_H^{-1} &= \{i \in I : \Re \mu_i^+ \geq \beta, \alpha < \Re \mu_i^- < \beta\} \\
I_H &= \{i \in I : \Re \mu_i^+ \geq \beta, \Re \mu_i^- \neq \alpha, \Re \mu_i^+ \neq \alpha\} \\
I_0^1_{\log} &= \{i \in I : \Re \mu_i^+ < \beta, \Re \mu_i^- \neq \alpha\} \\
I_0^2_{\log} &= \{i \in I : \Re \mu_i^+ < \beta, \Re \mu_i^- = \alpha\} \\
I_0^3_{\log} &= \{i \in I : \Re \mu_i^+ \geq \beta, \Re \mu_i^- = \alpha\}.
\end{align*}
\]

If \( \alpha \notin \{\Re \mu_i^\pm\} \) the sets \( I_0^a \), \( a = 1, 2, 3 \), are empty and it is sufficient to consider the first three sets above. For \( i \in I_0^0 \) the \( f_i \)'s are of the form (2.5) with \( f_i^- = 0 \), let \( f_0 = \sum_{i \in I_0^0} f_i \varphi_i \), each term satisfies (2.23) therefore their finite sum also will. For \( i \in I_0^1 \) the \( f_i \)'s are of the form (2.4), let \( f_1 = \sum_{i \in I_0^1} f_i \varphi_i \), the constants \( f_i^- \) are uniquely determined by \( \rho \) and \( f \big|_{\{1\} \times S} \) and again each term satisfies (2.23). For \( i \in I_H^{-1} \) the \( f_i \)'s are of the form (2.5), let \( f_2 = \sum_{i \in I_H^{-1}} \text{ [integrals in (2.5)] } \varphi_i \), \( f_H^- = \sum_{i \in I_H^{-1}} \text{ [the } f_i^- \text{ terms in (2.5)] } \varphi_i \), \( f_2 \) satisfies (2.23), and an elementary calculation gives (2.19). For \( i \in I_H \) the \( f_i \)'s are of the form (2.3), let \( f_3 = \sum_{i \in I_H} \text{ [integrals in (2.3)] } \varphi_i \), let \( f_H = \sum_{i \in I_H} \text{ [terms in (2.3)] } \varphi_i \). If \( \alpha \notin \{\Re \mu_i^\pm\} \) we are done by setting \( f_\rho = f_0^0 + f_1^1 + f_2^2 + f_3^3 + \tilde{f} \), and from what has been said the \( C_{a,q} \) estimates
follow. For \( i \in I^1_{\log} \) the \( f_i \)'s are of the form (2.3), and one has to add to \( f_p \) the function:

\[
f_p^4 = \sum_{i \in I^1_{\log}} \left[ \text{integrals in (2.3) with the kernel } s^{1-\mu_i^-} \right] \varphi_i,
\]

\( f_H \) is modified by the appropriate harmonic terms from (2.3), and we set:

\[
f_{\log}^1 = \sum_{i \in I^1_{\log}} \left[ \text{integrals in (2.3) with the kernel } s^{1-\mu_i^+} \right] \varphi_i.
\]

Similarly,

\[
f_{\log}^2 = \sum_{i \in I^2_{\log}} \left[ \text{integrals in (2.3) with the kernel } s^{1-\mu_i^-} \right] \varphi_i

+ \sum_{i \in I^2_{\log}} \left[ \text{integrals in (2.3) with the kernel } s^{1-\mu_i^+} \right] \varphi_i,
\]

\[
f_{\log}^3 = \sum_{i \in I^3_{\log}} \left[ \text{integrals in (2.4) with the kernel } s^{1-\mu_i^-} \right] \varphi_i.
\]

\( f_{\log} = f_{\log}^1 + f_{\log}^2 + f_{\log}^3 \). The \( C_{\alpha,q} \) estimates for \( f_{\log}, f_p, f_H \) and \( f_H^- \) follow directly from the estimate on \( \tilde{f} \) and the representations (2.3)-(2.5), the higher order \( C_{k,\lambda} \) estimates follow from Schauder's theory and a scaling argument, as in the proof of lemma 2.4.

3. An Existence Theorem

**THEOREM 3.1.** — Let \( \rho \in C_{\alpha-2,q}^{k,\lambda}, q > -1 \), let \( I = \{ i : \Re \mu_i^- > \alpha \} \), let \( K = \{ 0 \} \) if \( I = \emptyset \) or \( K = \text{span}\{ \varphi_i, i \in I \} \), and let \( \varphi \in C^{k+2,\lambda}(S) \).

\( a) \) There exists a constant \( C(a,b,c,k,\lambda,\alpha,q,\Delta_S) \) and a function \( f \in C_{\alpha,q+2}^{k+2,\lambda} \) satisfying:

\[
Lf = \rho, \quad f\big|_{\{1\} \times S} - \varphi \in K \tag{3.1}
\]
and satisfying:

\[ \|f\|_{C^{k+2,\lambda}_{\alpha,q+1}} \leq C \left( \|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)} \right). \]  \hspace{1cm} (3.2)

b) If the decay exponents \( \mu^\pm_i \) are all distinct (\( \forall i, \mu_i^+ \neq \mu_i^- \)) or if \( \mu_{i_0}^- = \mu_{i_0}^+ \) but \( \alpha \neq \mu_{i_0}^+ \), then \( f \in C^{k+2,\lambda}_{\alpha,q+1} \) and:

\[ \|f\|_{C^{k+2,\lambda}_{\alpha,q+1}} \leq C \left( \|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)} \right). \]  \hspace{1cm} (3.3)

c) If \( \forall i, \mu_i^+ \neq \mu_i^- \) and \( \alpha \notin \{\Re \mu_i^\pm\} \), then \( f \in C^{k+2,\lambda}_{\alpha,q} \) and:

\[ \|f\|_{C^{k+2,\lambda}_{\alpha,q}} \leq C \left( \|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)} \right). \]  \hspace{1cm} (3.4)

**Remarks:**

1) We are not making any hypotheses about either the sign or the size of \( c \), which allows for many solutions with the same boundary value and prescribed asymptotics to exist. If, however, \( I_1 = \{i : \Re \mu_i^+ \leq \alpha\} = \emptyset \), then \( f \) is uniquely determined by (3.1) and (3.2). If \( q > 0 \) the condition:

\[ \forall i \in I_1, \int_S f(1,p)\varphi_i \, d\mu = 0 \]

can always be imposed and together with (3.1)-(3.2) renders \( f \) unique.

2) (3.2) guarantees the existence of a solution with reasonably well controlled asymptotic behaviour even if we do not know anything about the spectrum of \( \Delta_S \). The existence result above has also been mentioned in [BK] (cf. remark 2 in [BK]).

3) (3.1) essentially says that we are free to specify \( \left. f \right|_{\{1\} \times S} \) up to a finite number of spherical harmonics \( \varphi_i, i \in I \).

4) If \( q = -1 \) points a) and c) still hold, but b) needs not hold if \( \alpha \in \{\Re \mu_i^\pm\} \) — in this case one can derive a \( \left[1 + \ln(1 + \ln r)\right] \) weighted estimate for \( f \).

5) For \( q < -1, q \neq -2 \), a solution will always exist if \( I \) is replaced by \( I' = \{i : \Re \mu_i^- \geq \alpha\} \), (3.2) and point c) holding. If the \( \mu_i^\pm \) are distinct (3.3) will hold as well.
6) If \( q = -2 \) theorem 3.2 still holds with \( I \) replaced by \( I' \) (cf. remark 5) if all the \( \mu_i^\pm \) are distinct or if \( \alpha \neq \mu_i^+ = \mu_i^- \): in the latter case (3.2) has, however, to be replaced by a \([1 + \ln(1 + \ln r)]\) weighted \( C_{\alpha, \lambda}^{k,l} \) estimate.

**Proof.** — Let \( m_0 \) be given by lemma 2.4, let \( \{\varphi_i\}, i = 1, \ldots, J \) be any orthonormal basis of \( O_{m_0} \), let:

\[
\rho_i(r) = \int_S \rho(r, p) \varphi_i(p) \, d\mu,
\]

let \( \bar{\rho} = \rho - \sum \rho_i \varphi_i \), by construction \( \bar{\rho} \bigg|_{\{r\} \times S} \in O_{m_0}^\perp \). Let \( C_{\alpha, \lambda}^{k,l}(R) \) be the Banach space of functions satisfying:

\[
C_{\alpha, \lambda}^{k,l}(R) = \left\{ f \in C_{\alpha, \lambda}^{k,l}([1, R] \times S) : \forall r, f(r, \cdot) \in O_{m_0}^\perp \right\},
\]

with the norm induced from \( C_{\alpha, \lambda}^{k,l} \). If \( c < 0 \) the problem

\[
L(f_R) = \bar{\rho}, \quad f_R \big|_{\{1\} \times S} = P_{O_{m_0}^\perp} \varphi, \quad f_R \big|_{\{R\} \times S} = 0
\]

has a unique solution for all \( R \geq 1 \), where \( P_{O_{m_0}^\perp} \) is the orthogonal projection (in \( L_2(S) \)) on \( O_{m_0}^\perp \). This is also true for \( c > 0 \) which can be established by applying the continuity method [GT] in \( C_{\alpha, \lambda}^{k,l}(R) \) to:

\[
L_t = a \frac{d^2}{dr^2} + b \frac{d}{dr} + tc + \Delta S \frac{d}{r^2}, \quad t \in [0, 1],
\]

the appropriate estimates and injectivity follow from lemma 2.4. Because the estimates of lemma 2.4 are \( R \) independent one can construct, using the family \( f_R \), a solution \( \bar{f} : [1, \infty) \times S \to R \) of:

\[
L\bar{f} = \bar{\rho}
\]

which satisfies the estimates of lemma 2.4. By lemma 2.1 we can also find functions \( f_i \) satisfying \( L(f_i \varphi_i) = \rho_i \varphi_i, i = 1, \ldots, J \), and to achieve the proof we have to show that the \( f_i \)'s can be chosen in a way consistent with (3.1)–(3.2). We shall analyse the terms \( f_i \varphi_i \) in detail only if \( \forall i, \mu_i^+ \neq \mu_i^- \) and if \( q > 0, \alpha \not\in \{\Re \mu_i^\pm\} \), the general result is obtained along similar lines. For \( i \leq J \) such that \( \Re \mu_i^+ < \alpha \) we take \( f_i \) given by (2.3). We have:

\[
f_i^+ + f_i^- = a(\mu_i^+ - \mu_i^-) \int_S \varphi \varphi_i \, d\mu.
\]
If $\mu^+_i \in \mathbb{R}$ set $f^+_i = 0$, $f^-_i$ given by (3.5). If $\Im \mu^+_i \neq 0$ the requirements $f^+_i \in \mathbb{R}, f_i \in \mathbb{R}$ and (3.5) determine $f^+_i$ uniquely. For $i \leq J$ for which $\Re \mu^+_i > \alpha$ and $\Re \mu^-_i < \alpha$ we take $f_i$ to be of the form (2.4) and the coefficient $f^-_i$ is uniquely determined by $\varphi$ and $\rho$. Finally for $i \leq J$ for which $\mu^+_i \in \mathbb{R}$ and $\mu^-_i > \alpha$ we take $f_i$ of the form (2.5) with $f^-_i = 0$ — this does not allow us, however, to fulfill the equality $\int \varphi_i (f|_{\{1\}} - \varphi) = 0$. The case $\Re \mu^+_i > \alpha, \Im \mu^+_i \neq 0$ is analyzed in a similar way. The $C_{\alpha,q}$ a priori estimates on $f = \tilde{f} + \sum f_i \varphi_i$ follow from construction, the higher derivative estimates follow again by a scaling argument.

4. An Asymptotic Estimate For $p$-Harmonic Functions

In this section we shall suppose that $g_{ij}$ is a Riemannian metric on $\mathbb{R}^n \setminus B(1)$, asymptotically euclidean in the following sense:

$$g_{ij} - \delta_{ij} \in C^{-\lambda,\alpha}, \quad \alpha > 0, \quad k \geq 1,$$

(4.1)

$$\exists c \in \mathbb{R}^+, \forall X^i \in \mathbb{R}^n, \forall x \in \mathbb{R}^n \setminus B(1), \quad g_{ij}(x)X^i X^j \geq c \sum (X^i)^2.$$

(4.2)

When $g_{ij} = \delta_{ij}$ the $p$-Laplace equation in $\mathbb{R}^n \setminus B(1)$:

$$\text{div} \left( |\text{grad} f|^{p-2} \text{grad} f \right) = 0, \quad p > 1,$$

(4.3)

with the condition:

the level surfaces of $f$ are, asymptotically, nested spheres

(4.4)

admits as solutions the functions:

$$f_p = \begin{cases} \frac{p-n}{r^{p-1}} & p \neq n \\ \log r & p = n. \end{cases}$$

For $p = \dim M = 3$ it has been shown in [Ch1] (cf. also [Ch2])\(^*\) that under (4.1)-(4.2) there exists a solution $f$ of the problem (4.3)-(4.4) satisfying:

$$f = \log r + \tilde{f}, \quad \tilde{f} \in C_{-\epsilon}^{1,\lambda} \text{ for some } \epsilon > 0,$$

\(^*\) These results are valid with no modification whatever $n$ for $p = n$ and in fact the same methods lead to similar results for other values of $p$, cf. also [KV].
and \( \tilde{f} \in C^{k+1,\lambda}_{-\epsilon}([R_0, \infty) \times S_{n-1}) \) for some \( R_0 \). We shall show that the estimate on the behaviour of \( \tilde{f} \) can be made precise. More generally let us consider:

\[
f = f_p + \tilde{f}, \quad \tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\epsilon}, \quad \epsilon > 0
\] (4.5)

**Theorem 4.1.** — Let \( f \) satisfying (4.5) be \( p \)-harmonic in \( \mathbb{R}^n \setminus B(1) \), let \( g_{ij} \) satisfy (4.1)-(4.2).

1) if \( \alpha < \beta(p, n) = \frac{(p-n)^2 + 4(p-1)(n-1) - n + p}{2(p-1)} \), then \( \tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\alpha}, \) plus eventually a constant if \( \left( \frac{p-n}{p-1} \right) - \epsilon \geq 0 \) and \( \left( \frac{p-n}{p-1} \right) - \alpha < 0 \).

2) if \( \alpha = \beta(p, n) \), then \( \tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\alpha,1}, \) plus eventually a constant if \( \left( \frac{p-n}{p-1} \right) - \epsilon \geq 0 \) and \( \left( \frac{p-n}{p-1} \right) - \alpha < 0 \).

3) if \( \alpha > \beta(p, n) \), then \( \tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\beta}, \) plus eventually a constant if \( \left( \frac{p-n}{p-1} \right) - \epsilon \geq 0 \) and \( \left( \frac{p-n}{p-1} \right) - \beta < 0 \).

**Remarks:**

1) In the physically relevant case \( n = 3 \) one considers \( \alpha \) between 0 and 1. If \( p = 3 \) theorem 4.1 implies:

\[
\tilde{f} = O(r^{-\alpha}) \text{ if } \alpha < 1 \quad \text{ or } \quad \tilde{f} = O\left( \frac{\ln r}{r} \right) \text{ if } \alpha = 1.
\]

2) The estimates of theorem 4.1, case 1), are sharp, which is seen by considering \( g_{ij} \) to be the flat metric written in a non-orthonormal asymptotically flat coordinate system, say \( y^i = x^i + J^i \), \( J^i \in C^{k+1,\lambda}_{1-\alpha} \). The functions \( f_p \) given by (4.4), when expressed in terms of the new coordinates, will exhibit the behaviour described in theorem 4.1. When \( p = n = 3 \) the estimates of case 2) are sharp as well, and it can be shown that the coefficient of the \( \ln r/r \) term is the mass of the metric.
Proof. — (4.5) and the p-Laplace equation:

\[
\left[ g^{ij} + (p - 2) \frac{\nabla^i f \nabla^j f}{|\nabla f|^2} \right] f_{,ij} = 0
\]

imply an equation of the form:

\[
\left[ \delta_{ij} + (p - 2) \frac{x^i x^j}{r^2} \right] \tilde{f}_{,ij} = \rho.
\] (4.6)

A straightforward though somewhat tedious calculation shows that:

\[
\rho \in C^{k-1,\lambda}_{\beta-2}, \quad \beta = -\min(\alpha, 2\varepsilon) + \frac{p-n}{p-1}.
\]

(4.6) written in spherical coordinates gives:

\[
\left[ (p-1) \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + \frac{\Delta_s}{r^2} \right] \tilde{f} = \rho,
\]

the characteristic decay exponents are readily calculated to be:

\[
\mu_i^\pm = \frac{p-n \pm \sqrt{(p-n)^2 + 4(p-1)\lambda_i}}{2(p-1)}, \quad \lambda_i = i(i + n - 2),
\]

(cf. [BGM]) iterating theorem 2.1 if necessary a finite number of times our claim follows.

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