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## Averaging method for neutral type impulsive differential equations with supremums

SVETLA DIMITROVA MILUSHEVA and DRUMI DIMITROV BAINOV<sup>(1)</sup>

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**RÉSUMÉ.** — La méthode du centrage pour des équations différentielles impulsives du type neutral avec des supremums est justifiée.

**ABSTRACT.** — The averaging method for neutral type impulsive differential equations with supremums is justified.

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### 1. Introduction

The papers of Mil'man, Myshkis [1], [2] mark the beginning of a systematic and profound investigation of impulsive differential equations [3]. The intensive development of the theory of the impulsive differential equations is conditioned by the fact that by means of them processes are successfully simulated which at certain moments of their development undergo a rapid change. Such processes are observed in mechanics, radio engineering, biology, population dynamics, biotechnologies, etc.

Parallel with the development of the theory of impulsive differential equations did the first investigations of ordinary and partial differential equations with maxima [4]-[7] begin in relation to their application to economics and control theory.

In the mathematical simulation in various important branches of control theory, pharmacokinetics, economics, etc. one has to analyse the influence of both the maximum of the function investigated and its impulsive, by jumps changes. Thus, for instance, if the concentration of the medicinal substance in the blood plasma has to be controlled at a venous injection

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(an impulsive change of this concentration) of the medicinal substance, one has to take into account together with it, in view of the optimal therapy, the maximum of this concentration too. An adequate mathematical apparatus for simulation of such processes are the impulsive differential equations with supremums.

In the present pioneer paper the averaging method for neutral type impulsive differential equations with supremums is justified.

## 2. Statement of the problem

Consider the impulsive system of differential equations with supremums of the form

$$\begin{aligned} \dot{x}(t) &= \epsilon X(t, x(t), \bar{x}(t), \tilde{x}(t)), \quad t > 0, t \neq \tau_k, \\ x(t) &= \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), \quad -h \leq t \leq 0, \\ \Delta x &\equiv x(t+0) - x(t-0) = \epsilon I_k(x(t-0)), \quad t = \tau_k, k \in \mathbb{N}, \end{aligned} \quad (1)$$

where  $x = (x_1, \dots, x_n)$ ,  $h$  is a positive constant,

$$\begin{aligned} \bar{x}(t) &= (\bar{x}_1(t), \dots, \bar{x}_n(t)), \quad \tilde{x}(t) = (\tilde{x}_1(t), \dots, \tilde{x}_n(t)), \\ \bar{x}_i(t) &= \sup\{x_i(s) \mid s \in [t-h, t]\}, \quad i = 1, 2, \dots, n, \\ \tilde{x}_i(t) &= \sup\{\dot{x}_i(s) \mid s \in [t-h, t]\}, \quad i = 1, 2, \dots, n, \\ X(t, x, y, z) &: \Omega \rightarrow \mathbb{R}^n, \quad \Omega = \{t \geq 0, x, y \in D \subset \mathbb{R}^n, z \in D_1 \subset \mathbb{R}^n\}, \end{aligned}$$

$\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$  is an initial function,  $I_k(x) : D \rightarrow \mathbb{R}^n$ ,  $k \in \mathbb{N}$ ,  $\tau_k$  are fixed numbers such that  $D = \tau_0 < \tau_1 < \dots \leq \tau_k < \dots$  and  $\lim_{k \rightarrow \infty} \tau_k = \infty$  and  $\epsilon \in (0, \epsilon^*]$  ( $\epsilon^* = \text{const} > 0$ ) is a small parameter.

For any  $x \in D$  let the following limits exist

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x, x, 0) dt = X_0(x), \quad (2)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{0 < \tau_k < T} I_k(x) = I_0(x). \quad (3)$$

Then with the system (1) we associate the averaged system or ordinary differential equations

$$\dot{\xi}(t) = \epsilon [X_0(\xi(t)) + I_0(\xi(t))] \quad (4)$$

with initial condition

$$\xi(0) = \varphi(0). \quad (5)$$

We shall note, for the sake of definiteness, that speaking of a value of a piecewise continuous function at a point of discontinuity we mean the limit from the left of the function (provided it exists) at this point. By the symbol  $\sum_{0 < \tau_k < T}$  we denote summation over all values of  $k$  for which the inequality  $0 < \tau_k < T$  is satisfied, and by the symbol  $\|\cdot\|$  the Euclidean norm in  $\mathbb{R}^n$ .

We shall say that condition (H) is satisfied if the following conditions hold:

H1 The function  $X(t, x, y, z)$  is continuous in the domain

$$\Omega = \{t \geq 0, x, y \in D \subset \mathbb{R}^n, z \in D_1 \subset \mathbb{R}^n\}.$$

The functions  $\varphi(t)$  and  $\dot{\varphi}(t)$  are continuous,  $\varphi_i(t)$  and  $\dot{\varphi}_i(t)$  ( $i = 1, \dots, n$ ) have a finite number of extremums in the interval  $[-h, 0]$ ,  $h = \text{const} > 0$ , and  $\varphi(t) \in D$  and  $\dot{\varphi}(t) \in D_1$  for  $t \in [-h, 0]$ . The functions  $I_k(x)$ ,  $k \in \mathbb{N}$ , are continuous in  $D$ .

H2 There exist positive constants  $M$  and  $\lambda$  such that for any  $t \geq 0$ ,  $x, x', y, y' \in D$ ,  $z, z' \in D_1$ ,  $k \in \mathbb{N}$  the following inequalities hold

$$\begin{aligned} \|X(t, x, y, z)\| + \|I_k(x)\| &\leq M, \\ \|X(t, x, y, z)\| - \|X(t, x', y', z')\| &\leq \lambda(\|x - x'\| + \|y - y'\| + \|z - z'\|), \\ \|I_k(x) - I_k(x')\| &\leq \lambda\|x - x'\|, \end{aligned}$$

and for  $t \in [-h, 0]$  the following inequality is valid

$$\|\varphi(t)\| + \|\dot{\varphi}(t)\| \leq M.$$

H3 For  $x \in D$  there exist the limits (2) and (3).

H4 There exists a positive constant  $\theta$  such that for  $k \in \mathbb{N}$  the inequality  $\tau_k - \tau_{k-1} \geq \theta$  holds, where  $\tau_0 = 0$ .

H5 For any  $\epsilon \in (0, \epsilon^*]$ ,  $\epsilon^* = \text{const} > 0$  system (1) has a unique solution  $x(t)$  which is defined in the interval  $[0, L\epsilon^{-1}]$ ,  $L = \text{const} > 0$ ;  $x_i(t)$  and  $\dot{x}_i(t)$ ,  $i = 1, \dots, n$ , have a finite number of extremums in each interval of finite length;  $x(t)$  and  $\dot{x}(t)$  satisfy respectively the conditions  $x(0+0) = \varphi(0)$  and  $\dot{x}(0+0) = \dot{\varphi}(0)$ .

H6 For  $\epsilon = 1$  the Cauchy problem (4), (5) has a unique solution  $\xi_1(t)$  which is defined in the interval  $[0, L]$  and  $t \in [0, L]$  belongs to the domain  $D$  together with some  $\rho$ -neighbourhood of it ( $\rho = \text{const} > 0$ ).

### 3. Auxiliary assertions

Under some natural constraints imposed on the right-hand sides of the equations in system (1) we shall prove a theorem of proximity of the solutions of the initial system (1) and the averaged system (4) with initial condition (5). In the proof of the main result we shall use the following two lemmas.

LEMMA 1. — For the sequence  $\tau_0, \tau_1, \dots, \tau_k, \dots$  let condition H4 hold. Then for any  $T \geq 0$  and  $t_0 \geq 0$  the following inequality is valid

$$\sum_{t_0 \leq \tau_k < t_0 + T} (\tau_k - t_0) < \frac{3T^2}{2\theta}.$$

*Proof.* — Let  $n \geq 2$  and  $\tau_j < t_0 \leq \tau_{j+1} < \dots < \tau_{j+n} < t_0 + T$ . Then for  $T \geq \theta$  we have

$$\begin{aligned} \sum_{t_0 \leq \tau_k < t_0 + T} (\tau_k - t_0) &= \sum_{k=j+1}^{j+n-1} (\tau_k - t_0) + (\tau_{j+n} - t_0) \leq \\ &\leq \frac{1}{\theta} \sum_{k=j+1}^{j+n-1} (\tau_k - t_0)(\tau_{k+1} - \tau_k) + (\tau_{j+n} - t_0) \leq \\ &\leq \frac{1}{\theta} \int_{\tau_{j+1}}^{\tau_{j+n}} (t - t_0) dt + (\tau_{j+n} - t_0) < \\ &< \frac{T^2}{2\theta} + T \leq \frac{3T^2}{2\theta}. \end{aligned}$$

Let  $\tau_j < t_0 \leq \tau_{j+1} < t_0 + T \leq \tau_{j+2}$ . Then for  $T \geq \theta$  we have

$$\sum_{t_0 \leq \tau_k < t_0 + T} (\tau_k - t_0) = \tau_{j+1} - t_0 < T \leq \frac{3T^2}{2\theta}.$$

This completes the proof of lemma 1.  $\square$

LEMMA 2. — For the sequence  $\tau_1, \dots, \tau_k, \dots$  let condition H4 hold. Let the nonnegative piecewise continuous function  $u(t)$  satisfy for  $t \geq t_0$  the inequality

$$u(t) \leq c + \int_{t_0}^t \gamma \cdot u(\tau) d\tau + \sum_{t_0 < \tau_k < t} \beta \cdot u(\tau_k),$$

in which  $c \geq 0, \beta \geq 0, \gamma > 0$  and  $\tau_k$  are points of discontinuity of the first kind of the function  $u(t)$ . Then for the function  $u(t)$  the estimate

$$u(t) \leq c(1 + \beta)^{i(t_0, t)} \cdot e^{\gamma(t-t_0)},$$

is valid, where  $i(t_0, t)$  is the number of the points  $\tau_k$  belonging to the interval  $[t_0, t)$ .

Lemma 2 is proved by means of the inequality of Gronwall-Bellman and by induction.  $\square$

Lemma 2 is a particular case of the theorems on integral inequalities obtained in [3].

#### 4. Main Results

THEOREM 1. — Let condition (H) hold. Then for any  $\eta > 0$  and  $L > 0$  there exists  $\epsilon_0 \in (0, \epsilon^*]$  ( $\epsilon_0 = \epsilon_0(\eta, L)$ ) such that for  $\epsilon \in (0, \epsilon_0]$  and  $t \in [0, L\epsilon^{-1}]$  the inequality  $\|x(t) - \xi(t)\| < \eta$  holds, where

$$\xi(t) = \xi_1(\epsilon t)$$

is the solution of problem (4), (5).

*Proof.* — From the conditions of theorem 1 it follows that for each compact  $Q \in D$  there exists a continuous function  $\alpha(T)$  which monotonically tends to zero as  $T \rightarrow \infty$  and is such that for  $x \in Q$  the following inequalities hold

$$\left\| \int_0^T [X(\tau, x, x, 0) - X_0(x)] d\tau \right\| \leq T\alpha(T), \tag{6}$$

and

$$\left\| \sum_{0 < \tau_k < T} I_k(x) - I_0(x)T \right\| \leq T\alpha(T). \tag{7}$$

Let  $Q = \{x \in D \mid x = \xi_1(t), t \in [0, L]\}$ . For  $t \in J_\epsilon = [0, L\epsilon^{-1}]$  we have

$$x(t) = x_0 + \epsilon \int_0^t X(\tau, x(\tau), \bar{x}(\tau), \tilde{x}(\tau)) d\tau + \epsilon \sum_{0 < \tau_k < t} I_k(x(\tau_k)), \tag{8}$$

$$\xi(t) = x_0 + \epsilon \int_0^t X_0(\xi(\tau)) d\tau + \epsilon \int_0^t I_0(\xi(\tau)) d\tau, \tag{9}$$

where  $x(t) = \varphi(t)$  and  $\dot{x}(t) = \dot{\varphi}(t)$  for  $t \in [-h, 0]$ .

Subtracting (9) from (8) for  $t \geq 0$  we obtain

$$\begin{aligned} \|x(t) - \xi(t)\| &\leq \epsilon \int_0^t \|X(\tau, x(\tau), \bar{x}(\tau), \tilde{x}(\tau)) - X(\tau, \xi(\tau), \xi(\tau), 0)\| d\tau + \\ &+ \epsilon \sum_{0 < \tau_k < t} \|I_k(x(\tau_k)) - I_k(\xi(\tau_k))\| + \\ &+ \epsilon \left\| \int_0^t [X(\tau, \xi(\tau), \xi(\tau), 0) - X_0(\xi(\tau))] d\tau \right\| + \\ &+ \epsilon \left\| \sum_{0 < \tau_k < t} I_k(\xi(\tau_k)) - \int_0^t I_0(\xi(\tau)) d\tau \right\|. \end{aligned} \tag{10}$$

Denote successively by  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta(t)$  and  $\zeta(t)$  the addends in the right-hand side of (10).

Without loss of generality of the result, in the estimation of the function  $\beta(t)$  we shall assume that

$$h < t, \quad \begin{cases} x_i(\tau_k) = \max(x_i(\tau_k - 0), x_i(\tau_k + 0)) \\ \dot{x}_i(\tau_k) = \max(\dot{x}_i(\tau_k - 0), \dot{x}_i(\tau_k + 0)) \end{cases}$$

for  $i = 1, \dots, n$  and  $k \in \mathbb{N}$ . From the last assumption it follows that the supremums in  $\bar{x}_i(t)$  and  $\tilde{x}_i(t)$ ,  $i = 1, \dots, n$ , are reached.

Denote respectively by  $\bar{s}_i(t, h)$  and  $\tilde{s}_i(t, h)$  the leftmost point of the interval  $[t - h, t]$ ,  $t \geq 0$ , at which respectively  $x_i(s)$  and  $\dot{x}_i(s)$  reach their greatest values in this interval. Then

$$\bar{x}_i(t) = x_i(\bar{s}_i(t, h)) \quad \text{and} \quad \tilde{x}_i(t) = \dot{x}_i(\tilde{s}_i(t, h)), \quad i = 1, \dots, n.$$

Using the conditions of theorem 1, we obtain

$$\begin{aligned} \beta(t) &= \epsilon \int_0^t \|X(\tau, x(\tau), \bar{x}(\tau), \tilde{x}(\tau)) - X(\tau, \xi(\tau), \xi(\tau), 0)\| \, d\tau \leq \\ &\leq 2\epsilon\lambda \int_0^t \|x(\tau) - \xi(\tau)\| \, d\tau + \\ &\quad + \epsilon\lambda \int_0^t [\|\bar{x}(\tau) - x(\tau)\| + \|\tilde{x}(\tau)\|] \, d\tau. \end{aligned} \tag{11}$$

Denote by  $\beta_0(t)$  the second addend in the right-hand side of (11) and obtain the following estimate

$$\begin{aligned} \beta_0(t) &= \epsilon\lambda \int_0^t [\|\bar{x}(\tau) - x(\tau)\| + \|\tilde{x}(\tau)\|] \, d\tau \leq \\ &\leq \epsilon\lambda \int_0^h [\|\bar{x}(\tau)\| + \|x(\tau)\| + \|\tilde{x}(\tau)\|] \, d\tau + \\ &\quad + \epsilon\lambda \int_h^t [\|\bar{x}(\tau) - x(\tau)\| + \|\tilde{x}(\tau)\|] \, d\tau \leq \\ &\leq \epsilon\lambda \int_0^h [\|\varphi(\tau - h)\| + \|\dot{\varphi}(\tau - h)\| + 2\|x(\tau)\| + \|\dot{x}(\tau)\|] \, d\tau + \\ &\quad + \epsilon\lambda \int_h^t \sqrt{\sum_{i=1}^n (x_i(\bar{s}_i(\tau, h)) - x_i(\tau))^2} \, d\tau + \\ &\quad + \epsilon\lambda \int_h^t \sqrt{\sum_{i=1}^n \dot{x}_i^2(\bar{s}_i(\tau, h))} \, d\tau \leq \\ &\leq \epsilon\lambda h \left( 3 + \epsilon + \epsilon \frac{1 + \theta}{\theta} h \right) M + \epsilon^2 \lambda (t - h) M \sqrt{n} + \\ &\quad + \epsilon\lambda \int_h^t \sqrt{\sum_{i=1}^n \left( \epsilon \int_{\bar{s}_i(\tau, h)}^{\tau} X_i(\ell, x(\ell), \bar{x}(\ell), \tilde{x}(\ell)) \, d\ell + \epsilon \sum_{\substack{\bar{s}_i(\tau, h) < \tau_k < \tau \\ \bar{s}_i(\tau, h) < \tau_k < \tau}} I_{ki}(x(\tau_k)) \right)^2} \, d\tau. \end{aligned}$$

Set  $A = \lambda h \left( 3 + \epsilon + \epsilon \frac{1 + \theta}{\theta} h \right) M + \lambda(L - \epsilon h)M\sqrt{n}$ , apply Minkowski's inequality and for  $t \in J_\epsilon$  obtain



$$\begin{aligned}
 \beta_0(t) &\leq \epsilon A + \epsilon^2 \lambda \int_h^t \left\{ \sqrt{\sum_{i=1}^n \left| \int_{\bar{s}_i(\tau, h)}^{\tau} X_i(\ell, x(\ell), \bar{x}(\ell), \bar{x}(\ell)) d\ell \right|^2} \right. \\
 &\quad \left. + \sqrt{\sum_{i=1}^n \left| \sum_{\bar{s}_i(\tau, h) < \tau_k < \tau} I_{ki}(x(\tau_k)) \right|^2} \right\} d\tau \leq \\
 &\leq \epsilon A + \epsilon^2 \lambda \int_h^t \left\{ \sqrt{\sum_{i=1}^n h^2 M^2} + \sqrt{\sum_{i=1}^n \left(\frac{h}{\theta}\right)^2 M^2} \right\} d\tau \leq \\
 &\leq \epsilon A + \epsilon \lambda (L - \epsilon h) \frac{1 + \theta}{\theta} h M \sqrt{n}.
 \end{aligned}$$

For  $\gamma(t)$  for  $t \in J_\epsilon$ , by the conditions of theorem 1, we get the estimate

$$\begin{aligned}
 \gamma(t) &= \epsilon \sum_{0 < \tau_k < t} \|I_k(x(\tau_k)) - I_k(\xi(\tau_k))\| \leq \\
 &\leq \epsilon \lambda \sum_{0 < \tau_k < t} \|x(\tau_k) - \xi(\tau_k)\|.
 \end{aligned}$$

In order to obtain estimates of the functions  $\delta(t)$  and  $\zeta(t)$  for  $t \in J_\epsilon$ , we partition the interval  $J_\epsilon$  into  $m$  equal parts by the points  $t_i = iL/\epsilon m$ ,  $i = 0, 1, \dots, m$ .

Let  $t$  be an arbitrarily chosen and fixed number of the interval  $J_\epsilon$ , and  $t \in (t_s, t_{s+1}]$ , where  $s \in \mathbb{N}$  and  $0 \leq s \leq m - 1$ . Then, using conditions of theorem 1 and inequality (6), we obtain

$$\begin{aligned}
 \delta(t) &= \epsilon \left\| \int_0^t [X(\tau, \xi(\tau), \xi(\tau), 0) - X_0(\xi(\tau))] d\tau \right\| \leq \\
 &\leq \epsilon \left\| \int_0^{t_s} [X(\tau, \xi(\tau), \xi(\tau), 0) - X_0(\xi(\tau))] d\tau \right\| + \\
 &\quad + \epsilon \left\| \int_{t_s}^t [X(\tau, \xi(\tau), \xi(\tau), 0) - X_0(\xi(\tau))] d\tau \right\| \leq \\
 &\leq \epsilon \left\| \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} [X(\tau, \xi(\tau), \xi(\tau), 0) - X_0(\xi(\tau)) + \right. \\
 &\quad \left. - X(\tau, \xi(t_i), \xi(t_i), 0) - X_0(\xi(t_i))] d\tau \right\| +
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon \left\| \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} [X(\tau, \xi(t_i), \xi(t_i), 0) - X_0(\xi(t_i))] d\tau \right\| + \\
 & + \epsilon \int_{t_s}^t [\|X(\tau, \xi(\tau), \xi(\tau), 0)\| + \|X_0(\xi(\tau))\|] d\tau \leq \\
 \leq & \epsilon \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} [\|X(\tau, \xi(\tau), \xi(\tau), 0) - X(\tau, \xi(t_i), \xi(t_i), 0)\| + \\
 & + \|X_0(\xi(\tau)) - X_0(\xi(t_i))\|] d\tau + \\
 & + \epsilon \sum_{i=0}^{s-1} \left\| \int_0^{t_{i+1}} [X(\tau, \xi(t_i), \xi(t_i), 0) - X_0(\xi(t_i))] d\tau \right\| + \\
 & + \epsilon \sum_{i=0}^{s-1} \left\| \int_0^{t_i} [X(\tau, \xi(t_i), \xi(t_i), 0) - X_0(\xi(t_i))] d\tau \right\| + \frac{2ML}{m} \leq \\
 \leq & 3\epsilon\lambda \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} \|\xi(\tau) - \xi(t_i)\| d\tau + \epsilon \sum_{i=0}^{s-1} t_{i+1}\alpha(t_{i+1}) + \\
 & + \epsilon \sum_{i=1}^{s-1} t_i\alpha(t_i) + \frac{2ML}{m} \leq \\
 \leq & 6\epsilon^2\lambda M \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} (\tau - t_i) d\tau + 2\epsilon \sum_{i=1}^s t_i\alpha(t_i) + \frac{2ML}{m} \leq \\
 \leq & \frac{(3s\lambda L + 2m)ML}{m^2} + \frac{2s^2L}{m} \alpha\left(\frac{L}{\epsilon m}\right).
 \end{aligned}$$

We pass to the estimation of  $\zeta(t)$ . Using the conditions of theorem 1 and inequality (7) for  $t \in (t_s, t_{s+1}]$ , we obtain

$$\begin{aligned}
 \zeta(t) & = \epsilon \left\| \sum_{0 < \tau_k < t} I_k(\xi(t_k)) - \int_0^t I_0(\xi(\tau)) d\tau \right\| \leq \\
 & \leq \epsilon \left\| \sum_{0 < \tau_k < t_s} I_k(\xi(t_k)) - \int_0^{t_s} I_0(\xi(\tau)) d\tau \right\| + \\
 & + \epsilon \left\| \sum_{t_s \leq \tau_k < t} I_k(\xi(t_k)) - \int_{t_s}^t I_0(\xi(\tau)) d\tau \right\| \leq \\
 & \leq \epsilon \sum_{0 < \tau_k < t_1} \|I_k(\xi(t_k)) - I_k(\xi(0))\| +
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon \int_0^{t_1} \|I_0(\xi(t_k)) - I_0(\xi(0))\| \, d\tau + \\
 & + \epsilon \sum_{i=1}^{s-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} \|I_k(\xi(\tau_k)) - I_k(\xi(t_i))\| + \right. \\
 & \left. + \int_{t_i}^{t_{i+1}} \|I_0(\xi(\tau)) - I_0(\xi(t_i))\| \, d\tau \right) + \\
 & + \epsilon \left\| \sum_{0 < \tau_k < t_1} I_k(\xi(0)) - \int_0^{t_1} I_0(\xi(0)) \, d\tau \right\| + \\
 & + \epsilon \sum_{i=1}^{s-1} \left\| \sum_{t_i \leq \tau_k < t_{i+1}} I_k(\xi(t_i)) - \int_{t_i}^{t_{i+1}} I_0(\xi(t_i)) \, d\tau \right\| + \\
 & + \epsilon \sum_{t_s \leq \tau_k < t} \|I_k(\xi(t_k))\| + \epsilon \int_{t_s}^t \|I_0(\xi(\tau))\| \, d\tau \leq \\
 \leq & \epsilon \lambda \sum_{0 < \tau_k < t_1} \|\xi(\tau_k) - \xi(0)\| + \epsilon \lambda \int_0^{t_1} \|\xi(\tau) - \xi(0)\| \, d\tau + \\
 & + \epsilon \lambda \sum_{i=1}^{s-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} \|\xi(\tau_k) - \xi(t_i)\| + \int_{t_i}^{t_{i+1}} \|\xi(\tau) - \xi(t_i)\| \, d\tau \right) + \\
 & + t_1 \alpha(t_1) + \epsilon \sum_{i=1}^{s-1} \left\| \sum_{0 < \tau_k < t_{i+1}} I_k(\xi(t_i)) - t_{i+1} I_0(\xi(t_i)) \right\| + \\
 & + \epsilon \sum_{i=1}^{s-1} \left\| \sum_{0 < \tau_k < t_i} I_k(\xi(t_i)) - t_i I_0(\xi(t_i)) \right\| + \\
 & + \epsilon \sum_{i=1}^{s-1} \sum_{\tau_k = t_i} \|I_k(\xi(t_i))\| + \frac{1+\theta}{\theta} \cdot \frac{LM}{m} \leq \\
 \leq & 2\epsilon^2 \lambda M \sum_{i=0}^{s-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} (\tau_k - t_i) + \int_{t_i}^{t_{i+1}} (\tau - t_i) \, d\tau \right) + \\
 & + t_1 \alpha(t_1) + \epsilon \sum_{i=1}^{s-1} t_{i+1} \alpha(t_{i+1}) + \\
 & + \epsilon \sum_{i=1}^{s-1} t_i \alpha(t_i) + \epsilon(s-1)M + \frac{1+\theta}{\theta} \cdot \frac{LM}{m} \leq
 \end{aligned}$$

$$\begin{aligned} &\leq 2\epsilon^2\lambda M \sum_{i=0}^{s-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} (\tau_k - t_i) \right) + \frac{s\lambda L^2 M}{m^2} + \\ &\quad + 2sL\alpha \left( \frac{L}{\epsilon m} \right) + \epsilon(s-1)M + \frac{1+\theta}{\theta} \cdot \frac{LM}{m}. \end{aligned}$$

From the results obtained for  $\delta(t)$  and  $\zeta(t)$  for  $t \in (t_s, t_{s+1}]$  it follows that for any  $t \in J_\epsilon$  the following estimates are valid

$$\begin{aligned} \delta(t) &\leq \frac{(3\lambda L + 2)LM}{m} + 2mL\alpha \left( \frac{L}{3m} \right), \\ \zeta(t) &\leq 2\epsilon^2\lambda M \sum_{i=0}^{m-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} (\tau_k - t_i) \right) + \\ &\quad + \left( \lambda L + \frac{1+\theta}{\theta} \right) \frac{LM}{m} + 2mL\alpha \left( \frac{M}{\epsilon m} \right) + \epsilon m M. \end{aligned}$$

From (10) and the estimates obtained for the functions  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta(t)$  and  $\zeta(t)$  it follows that for  $t \in J_\epsilon$  the following inequality holds

$$\begin{aligned} \|x(t) - \xi(t)\| &\leq b(m, \epsilon) + c(m, \epsilon) + 2\epsilon\lambda \int_0^t \|x(\tau) - \xi(\tau)\| d\tau + \\ &\quad + \epsilon\lambda \sum_{0 < \tau_k < t} \|x(\tau_k) - \xi(\tau_k)\|, \end{aligned} \tag{12}$$

where

$$\begin{aligned} b(m, \epsilon) &= \frac{2(2\lambda L + 1)LM}{m} + \frac{1+\theta}{\theta} \cdot \frac{LM}{m} + \\ &\quad + 2\epsilon^2\lambda M \sum_{i=0}^{m-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} (\tau_k - t_i) \right), \\ c(m, \epsilon) &= \epsilon A + \epsilon\lambda(L - \epsilon h) \frac{1+\theta}{\theta} hM\sqrt{n} + \\ &\quad + 4mL\alpha \left( \frac{L}{\epsilon m} \right) + \epsilon m M. \end{aligned}$$

Choose successively a sufficiently large  $m_0 \in \mathbb{N}$  and a sufficiently small  $\epsilon_1 \in (0, \epsilon^*]$  so that the following inequalities should hold

$$[(3 + 4\theta)\lambda L + 1 + 3\theta] \frac{LM}{\theta m_0} \leq \frac{1}{2} e^{-\lambda L(2 + \frac{1}{\theta})} \min(\eta, \rho), \quad \epsilon_1 m_0 \leq \frac{L}{\theta} \tag{13}$$

For  $m = m_0$  and  $\epsilon \in (0, \epsilon_1]$  apply lemme 1 to the third addend in the right-hand side of  $b(\epsilon, m)$  and obtain

$$2\epsilon^2 \lambda M \sum_{i=0}^{m_0-1} \left( \sum_{t_i \leq \tau_k < t_{i+1}} (\tau_k - t_i) \right) \leq \frac{3\lambda L^2 M}{\theta m_0} \quad (14)$$

From (12), (13) and (14) for  $m = m_0$ ,  $\epsilon \in (0, \epsilon_1]$  and  $t \in J_\epsilon$  there follows the inequality

$$\begin{aligned} \|x(t) - \xi(t)\| &\leq b(m_0, \epsilon) + c(m_0, \epsilon) + 2\epsilon\lambda \int_0^t \|x(\tau) - \xi(\tau)\| d\tau + \\ &+ \epsilon\lambda \sum_{0 < \tau_k < t} \|x(\tau_k) - \xi(\tau_k)\|, \end{aligned} \quad (15)$$

where  $b(m_0, \epsilon) \leq \frac{1}{2} e^{-\lambda L(2 + \frac{1}{\theta})} \min(\eta, \rho)$ .

Apply lemma 2 to (15) and obtain

$$\begin{aligned} \|x(t) - \xi(t)\| &\leq (b(m_0, \epsilon) + c(m_0, \epsilon)) e^{2\epsilon\lambda t} (1 + \epsilon\lambda)^{i(0,t)} \leq \\ &\leq (b(m_0, \epsilon) + c(m_0, \epsilon)) e^{2\epsilon\lambda t + \frac{t}{\theta} \ln(1 + \epsilon\lambda)} \leq \\ &\leq (b(m_0, \epsilon) + c(m_0, \epsilon)) e^{\lambda L(2 + \frac{1}{\theta})}. \end{aligned}$$

Choose a sufficiently small  $\epsilon_0 \in (0, \epsilon_1]$  so that for  $\epsilon \in (0, \epsilon_0]$  to have

$$c(m_0, \epsilon) < \frac{1}{2} e^{-\lambda L(2 + \frac{1}{\theta})} \min(\eta, \rho).$$

Then for  $\epsilon \in (0, \epsilon_0]$  ( $\epsilon_0 = \epsilon_0(\eta, L)$ ) and  $t \in J_\epsilon$  the following inequality should hold

$$\|x(t) - \xi(t)\| < \min(\eta, \rho).$$

Hence, for  $t \in J_\epsilon$ ,  $x(t)$  belongs to the domain  $D$  and the estimate  $\|x(t) - \xi(t)\| < \eta$  is valid.

This completes the proof of theorem 1.  $\square$

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