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On a coupled problem between the plate equation and the membrane equation on polygons

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RÉSUMÉ. — Nous étudions un problème d'interface sur un domaine polygonal du plan, où les opérateurs laplacien et bilaplacien sont considérés sur chacune des faces respectives. Nous précisons les conditions nécessaires et suffisantes pour que l'opérateur associé soit de Fredholm dans les espaces de Hilbert appropriés. Lorsque c'est le cas, nous donnons une décomposition de la solution variationnelle en parties régulières et singulière.

ABSTRACT. — We study an interface problem on polygonal domains of the plane, where on one face, we consider the biharmonic operator and on the other one, the Laplace operator. We instigate if the associated operator on appropriate Hilbert spaces is a Fredholm operator or not. If it is, we give an expansion of the variational solution into regular and singular parts.

1. Introduction

We introduce a new kind of interface problems on polygonal domains of the plane. The novelty is that the order of the partial differential operators is different on each face. We only study a model problem corresponding to the mechanical example of a coupling between a plate and a membrane. We expect that the methods we developed could be extended to more general problems.

In classical interface problems (see [3, 10] and the references cited there), the variational solution has singularities at the common vertices between the interface and the boundary. Therefore, we can expect the same type

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of results for our problem. Indeed, for interior data in L^2, we can give the decomposition of the variational solution of our problem into a regular part with the optimal regularity and a singular one. The main idea is to use a two steps argument by splitting up two of the interface conditions, and use successively the decomposition results for an inhomogeneous boundary value problem on each face respectively associated with the Laplace operator and the biharmonic one.

For more regular data, we could argue iteratively as before, but this induces too much geometrical conditions (on the angles of the domains). Therefore we prefer a compact perturbation argument as for boundary value problems with non-homogeneous partial differential operators [7, 2]. Indeed, we shall see that the difference of order of the operators on the faces will induce interface conditions with non-homogeneous operators (i.e. it is sum of operators of different order). This argument only holds under some conditions on the Sobolev exponents. One of them is also necessary since we shall show that if this condition fails then the induced operator is not Fredholm.

We finish this paper by solving, in section 6, a differential equation with operator coefficients

$$\frac{\partial u}{\partial t} - Au(t) = e^{\lambda t} t^q f_q \quad \text{in } \mathbb{R}\,, \tag{1.1}$$

where A is a closed operator defined on a Hilbert space X, $\lambda \in \mathbb{C}$, $q \in \mathbb{N} \cup \{0\}$ and $f_q \in X$. As shown in [11] (see also the references cited there), solving (1.1) allows us tu solve explicitly some boundary value problems in a finite cone of $\mathbb{R}^n$ with a right-hand side which is a linear combination of functions of type

$$r^\lambda (\log r)^q \varphi_q(\theta)\,,$$

where (r, θ) are the spherical coordinates, $\lambda \in \mathbb{C}$, $q \in \mathbb{N} \cup \{0\}$ and φ_q is regular enough. This result agrees with those of [6].

2. Formulation of the problem

Let Ω_1, Ω_2 be two bounded simply connected polygonal domains of the plane such that their boundaries have a common side denoted by Γ. We denote by Γ_1 (resp. Γ_2) the boundary of Ω_1 (resp. Ω_2) except Γ, i.e. $\Gamma_j = \partial\Omega_j \setminus \overline{\Gamma}$, for $j = 1, 2$.

For $j = 1, 2$, ν_j will denote unitary outer normal vector on the boundary $\partial\Omega_j$ of Ω_j and τ_j the unitary tangent vector along $\partial\Omega_j$ so that (ν_j, τ_j) is a direct orthonormal basis. Along the common side Γ, we omit the index by setting $(\nu, \tau) = (\nu_2, \tau_2)$. We shall denote S_{jk}, for $k \in \{1, \ldots, N_j\}$, the vertices of Ω_j, numbered according to the trigonometric orientation for Ω_1 and numbered clockwise for Ω_2; ω_{jk} will be the interior angle at S_{jk}. Moreover, for convenience, we assume that $S_{11} = S_{21}$ and $S_{12} = S_{22}$ belong to $\overline{\Gamma}$ and denote them S_1 and S_2 respectively. We also denote by η_{jk}, a cut-off function equal to 1 in a neighbourhood of S_{jk} and equal to 0 in a neighbourhood of the other vertices. As previously, we may suppose that: $\eta_{11} = \eta_{21} =: \eta_1$ and $\eta_{12} = \eta_{22} =: \eta_2$. Finally, γ_j will denote the trace operator on the boundary $\partial\Omega_j$ of Ω_j; $\gamma_{j\Gamma}$ will be the restriction of γ_j to Γ.

For $E > 0$ and $\sigma \in \,]0\,,1[$ (respectively the Young modulus and the Poisson coefficient of the constitutive material of the plate Ω_2), we set $\rho = E/(1 - \sigma^2)$ and we introduce the boundary operator defined only on Γ

$$Mu = \rho\gamma_{2\Gamma}\left(\sigma\Delta u + (1-\sigma)\frac{\partial^2 u}{\partial\nu^2}\right),$$

$$Nu = \rho\gamma_{2\Gamma}\left(\frac{\partial\Delta u}{\partial\nu} + (1-\sigma)\frac{\partial^3 u}{\partial\nu\partial\tau^2}\right).$$

We recall that we use here classical Sobolev spaces, i.e. if Ω is a bounded open set of $\mathbb{R}^2$ and s a non-negativite integer, then

$$H^s(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega)\,,\ \forall\, \alpha \in \mathbb{N}^2 : |\alpha| \le s\right\},$$

its norm being denoted by $\|\cdot\|_{s,\Omega}$. For other definitions, we follow Grisvard's book [4].

We consider the following interface problem: given $f_1 \in H^{s_1-1}(\Omega_1)$, $f_2 \in H^{s_2-2}(\Omega_2)$, $h_1 \in H^{s_2-1/2}(\Gamma)$, $h_2 \in H^{s_2-3/2}(\Gamma) \cup H^{s_1-1/2}(\Gamma)$, for s_1, $s_2 \in \mathbb{N}$ with $s_2 \ge 2$, find $u_1 \in H^{s_1+1}(\Omega_1)$, $u_2 \in H^{s_2+2}(\Omega_2)$, solutions of (2.1)-(2.7) below:

$$\Delta u_1 = f_1 \quad \text{in } \Omega_1\,, \tag{2.1}$$

$$\Delta^2 u_2 = f_2 \quad \text{in } \Omega_2\,, \tag{2.2}$$

$$\gamma_1 u_1 = 0 \quad \text{on } \Gamma_1\,, \tag{2.3}$$

$$\gamma_2 u_2 = \gamma_2\frac{\partial u_2}{\partial\nu_2} = 0 \quad \text{on } \Gamma_2\,, \tag{2.4}$$

$$\gamma_1 u_1 = \gamma_2 u_2 \quad \text{on } \Gamma, \tag{2.5}$$

$$M u_2 = h_1 \quad \text{on } \Gamma, \tag{2.6}$$

$$N u_2 + \gamma_1 \frac{\partial u_1}{\partial \nu} = h_2 \quad \text{on } \Gamma. \tag{2.7}$$

We first give the variational formulation of this problem. We set

$$V = \left\{ \overrightarrow{u} = (u_1, u_2) \in H^1(\Omega_1) \times H^2(\Omega_2) \text{ fulfilling (2.3), (2.4) and (2.5)} \right\}.$$

It is a Hilbert space equipped with the inner product induced by $H^1(\Omega_1) \times H^2(\Omega_2)$ with the norm

$$\|\overrightarrow{u}\|_V = \left(\|u_1\|^2_{1,\Omega_1} + \|u_2\|^2_{2,\Omega_2} \right)^{\frac{1}{2}}.$$

We define the sesquilinear form a on V as follows:

$$a(\overrightarrow{u}, \overrightarrow{v}) = a_1(u_1, v_1) + a_2(u_2, v_2),$$

where we take

$$a_1(u_1, v_1) = \int_{\Omega_1} \nabla u_1 \cdot \nabla \overline{v}_1 \, \mathrm{d}x,$$

$$\begin{aligned} a_2(u_2, v_2) = \rho \int_{\Omega_2} \Big\{ \Delta u_2 \Delta \overline{v}_2 - (1-\sigma) \Big(& \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial^2 \overline{v}_2}{\partial x_2^2} + \\ & + \frac{\partial^2 u_2}{\partial x_2^2} \frac{\partial^2 \overline{v}_2}{\partial x_1^2} - 2 \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \frac{\partial^2 \overline{v}_2}{\partial x_1 \partial x_2} \Big) \Big\} \, \mathrm{d}x \end{aligned}$$

LEMMA 2.1.— *For all $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, h_1, $h_2 \in L^2(\Gamma)$, there exists a unique solution $\overrightarrow{u} \in V$ of*

$$a(\overrightarrow{u}, \overrightarrow{v}) = -\int_{\Omega_1} f_1 \overline{v}_1 \, \mathrm{d}x + \rho \int_{\Omega_2} f_2 \overline{v}_2 \, \mathrm{d}x + \int_\Gamma \left\{ h_1 \gamma_2 \frac{\partial \overline{v}_2}{\partial \nu} - h_2 \gamma_2 \overline{v}_2 \right\} \mathrm{d}\sigma. \tag{2.8}$$

Proof.— In order to apply the Lax-Milgram lemma, we need to show the continuity and the coerciveness of the form a on V.

The continuity is a direct consequence of the continuity of the form a_j on $H^j(\Omega_j)$ and of the Cauchy-Schwarz inequality.

Owing to inequality (2.15) of [12], we deduce that

$$a(\overrightarrow{u}, \overrightarrow{u}) \geq \min\big(1, \rho(1-\sigma)\big) \left\{ \big[u_1\big]^2_{1,\Omega_1} + \big[u_2\big]^2_{2,\Omega_2} \right\}, \quad \forall\, \overrightarrow{u} \in V,$$

where $\big[u_j\big]_{j,\Omega_j}$ denotes the semi-norm of $H^j(\Omega_j)$. But for $\overrightarrow{u} \in V$, the boundary conditions (2.3) and (2.4) respectively fulfilled by u_1 and u_2 imply that the norms and the semi-norms are equivalent (see e.g. Theorem I.1.9 of [8]). Therefore the previous estimate leads to the coerciveness of the form a on V. □

Let us now show that a solution of (2.8) is a weak solution of (2.1)-(2.7). We follow the arguments of section 1.5.3 of [4]: we introduce the spaces

$$E\big(\Delta^j, L^2(\Omega_j)\big) = \left\{ v \in H^j(\Omega_j) : \Delta^j v \in L^2(\Omega_j) \right\}, \quad j = 1, 2,$$

these are Banach spaces for the norms

$$\|v\|_{j,\Omega_j} + \|\Delta^j v\|_{0,\Omega_j}, \quad j = 1, 2.$$

Lemma 1.5.3.9 of [4] proves that $\mathcal{D}(\overline{\Omega}_1)$ is dense in $E\big(\Delta, L^2(\Omega_1)\big)$; analogous arguments lead to the density of $\mathcal{D}(\overline{\Omega}_2)$ into $E\big(\Delta^2, L^2(\Omega_2)\big)$.

LEMMA 2.2.— *The mapping*

$$(u_1, u_2) \rightarrow \left(M u_2, N u_2 + \gamma_{1\Gamma} \frac{\partial u_1}{\partial \nu} \right),$$

which is defined on $\mathcal{D}(\overline{\Omega}_1) \times \mathcal{D}(\overline{\Omega}_2)$, has a unique continuous extension as an operator from $E\big(\Delta, L^2(\Omega_1)\big) \times E\big(\Delta^2, L^2(\Omega_2)\big)$ into $\widetilde{H}^{1/2}(\Gamma)' \times \widetilde{H}^{3/2}(\Gamma)'$ (identifying Γ with a real interval, we recall that $u \in \widetilde{H}^s(\Gamma)$ iff $\widetilde{u}$, the extension of u by 0 outside Γ, remains in $H^s(\mathbb{R})$).

Proof.— Owing to theorem 1.5.2.8 and corollary 1.4.4.10 of [4], given $(w_1, w_2) \in \widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{3/2}(\Gamma)$, there exists $\overrightarrow{v} = (v_1, v_2) \in V$ satisfying

$$\begin{cases} \gamma_1 v_1 = \gamma_2 v_2 = w_2 & \text{on } \Gamma, \\ \gamma_2 \dfrac{\partial v_2}{\partial \nu} = w_1 & \text{on } \Gamma. \end{cases} \tag{2.9}$$

and

$$\|\overrightarrow{v}\|_V \leq C_1 \left\{ \|w_1\|_{\widetilde{H}^{1/2}(\Gamma)} + \|w_2\|_{\widetilde{H}^{3/2}(\Gamma)} \right\}, \tag{2.10}$$

where the constant C_1 is independent of w_1, w_2.

For a fixed $\overrightarrow{u} = (u_1, u_2) \in \mathcal{D}(\overline{\Omega}_1) \times \mathcal{D}(\overline{\Omega}_2)$, let us set

$$\ell(w_1, w_2) = \int_\Gamma \left\{ M u_2 \overline{w}_1 - \left(N u_2 + \gamma_{1\Gamma} \frac{\partial u_1}{\partial \nu} \right) \overline{w}_2 \right\} \mathrm{d}\sigma .$$

By integration by parts, we get (see lemma 2.3 of [12] for the biharmonic operator):

$$\ell(w_1, w_2) = a(\overrightarrow{u}, \overrightarrow{v}) + \int_{\Omega_1} \Delta u_1 \overline{v}_1 \,\mathrm{d}x - \rho \int_{\Omega_2} \Delta^2 u_2 \overline{v}_2 \,\mathrm{d}x . \qquad (2.11)$$

Therefore using the continuity of the form a on V and the estimate (2.10), there exists a constant C_2 independant of w_1, w_2 such that

$$\begin{aligned} \left|\ell(w_1, w_2)\right| \le C_2 &\left\{ \|u_1\|_{E(\Delta, L^2(\Omega_1))} + \|u_2\|_{E(\Delta^2, L^2(\Omega_2))} \right\} \times \\ &\times \left\{ \|w_1\|_{\widetilde{H}^{1/2}(\Gamma)} + \|w_2\|_{\widetilde{H}^{3/2}(\Gamma)} \right\} . \end{aligned}$$

By density, we deduce that ℓ is a continuous linear form on $\widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{3/2}(\Gamma)$. □

Let us notice that the Green formula (2.11) still holds for every $v \in V$ fulfilling (2.9) and every $u_1 \in E\big(\Delta, L^2(\Omega_1)\big)$, $u_2 \in E\big(\Delta^2, L^2(\Omega_2)\big)$, where the left-hand side has to be understood as a duality bracket.

LEMMA 2.3.— *Let $\overrightarrow{u} \in V$ be the unique solution of (2.8). Then $\overrightarrow{u}$ fulfils (2.1) to (2.7).*

Proof.— Applying (2.8) with $(v_1, v_2) \in \mathcal{D}(\Omega_1) \times \mathcal{D}(\Omega_2)$, we see that u_1 (resp. u_2) fulfils (2.1) (resp. (2.2)) in the distributional sense. This also shows that

$$u_1 \in E\big(\Delta, L^2(\Omega_1)\big) , \quad u_2 \in E\big(\Delta^2, L^2(\Omega_2)\big) .$$

Therefore, for arbitrary $(w_1, w_2) \in \widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{3/2}(\Gamma)$, comparing (2.8) with (2.11), when $v \in V$ fulfils (2.9), we deduce that

$$\langle M u_2 \,,\, w_1 \rangle - \left\langle N u_2 + \gamma_1 \frac{\partial u_1}{\partial \nu} \,,\, w_2 \right\rangle = \int_\Gamma \{ h_1 \overline{w}_1 - h_2 \overline{w}_2 \} \,\mathrm{d}\sigma .$$

This obviously implies that $\overrightarrow{u}$ satisfies (2.6) and (2.7). □

3. Regularity for interior data in L^2

In this paragraph, we look for conditions on $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $h_1 \in H^{3/2}(\Gamma)$, $h_2 \in H^{1/2}(\Gamma)$, which ensure that u_1 and u_2 have the optimal regularity, i.e. $u_1 \in H^2(\Omega_1)$, $u_2 \in H^4(\Omega_2)$; indeed we shall prove that u_1 and u_2 admit a decomposition into a regular part with the optimal regularity and a finite sum of singular functions. We shall see that this decomposition result will be determined by analogous decomposition results of two decoupled boundary value problems set in Ω_1 and Ω_2. More precisely, the first one is the Dirichlet problem in Ω_1 with non-homogeneous Dirichlet boundary conditions on Γ, i.e.

$$\begin{cases} \Delta u_1 = f_1 & \text{in } \Omega_1, \\ \gamma_1 u_1 = 0 & \text{on } \Gamma_1, \\ \gamma_1 u_1 = g & \text{on } \Gamma. \end{cases} \tag{3.1}$$

The second one is the following mixed boundary value problem for the biharmonic operator in Ω_2 :

$$\begin{cases} \Delta^2 u_2 = f_2 & \text{in } \Omega_2, \\ \gamma_2 u_2 = \gamma_2 \dfrac{\partial u_2}{\partial \nu_2} = 0 & \text{on } \Gamma_2, \\ M u_2 = h_1 & \text{on } \Gamma, \\ N u_2 = h_2 & \text{on } \Gamma. \end{cases} \tag{3.2}$$

The regularity of the solution of problem (3.1) was given in theorem 5.1.3.5 of [4], while problem (3.2) was studied in theorem 5.2 of [12] (see also [1]). In order to recall these results, let us define the singular exponents and singular functions of problems (3.1) and (3.2).

For problem (3.1), we set

$$\Lambda_{1k} = \left\{ \frac{m\pi}{\omega_{1k}} : m \in \mathbb{N} \right\}, \quad \forall\, k \in \{1, \ldots, N_1\}. \tag{3.3}$$

For $\lambda \in \Lambda_{1k}$, the assosiated singular function is

$$\sigma_{\text{lap}}^{k\lambda}(r,\theta) = \begin{cases} r^\lambda \sin(\lambda\theta) & \text{if } \lambda \notin \mathbb{N}, \\ r^\lambda \{\ln r \sin(\lambda\theta) + \theta \cos(\lambda\theta)\} & \text{if } \lambda \in \mathbb{N}, \end{cases}$$

where (r,θ) are polar coordinates with origin S_{1k} (such that the half-lines $\theta = 0$ and $\theta = \omega_{1k}$ contain the edges containing S_{1k}).

For problem (3.2), in order to avoid too complicated notations, we only recall that the singular exponents are the roots of the following characteristic equation :

$$\sin^2(\lambda-1)\omega_{2k}+\frac{1-\sigma}{3+\sigma}(\lambda-1)\sin^2\omega_{2k}-\frac{4}{(1-\sigma)(3+\sigma)}=0\,,\quad \text{for } k=1,\ 2, \tag{3.4}$$

$$\sin^2(\lambda-1)\omega_{2k}-(\lambda-1)^2\sin^2\omega_{2k}=0\,,\quad \text{for } k\in\{3,\dots,N_2\}\,. \tag{3.5}$$

We only say that there exists a set Λ_{2k} of roots of the equation (3.4) for $k\le 2$ and of (3.5) for $k\ge 3$, repeated according to their multiplicity; to each $\lambda\in\Lambda_{2k}$ corresponds a singular function denoted by $\sigma_{\mathrm{bi}}^{k\lambda}$ (see [12] for more details). For $k=1,\ 2$, the polynomial resolution (cf. § 3.C of [12] and (2.9) of [1]) implies that $\lambda=2$ induces a singular function given by

$$\sigma_{\mathrm{bi}}^{k2}(r,\theta)=r^2\theta+p_k(r,\theta)\,,\quad k=1,\ 2\,,$$

where (r,θ) are polar coordinates with origin S_k and p_k is a polynomial of degree 2 (in the cartesian coordinates). Remark that $\sigma_{\mathrm{bi}}^{k2}\in H^2(\Omega_2)$. Therefore, for convenience, we shall add $\lambda=2$ to Λ_{2k}, for $k=1,\ 2$ and still denoted it by Λ_{2k}.

THEOREM 3.1.— *Let $f_1\in H^{s_1-1}(\Omega_1)$, $g\in H^{s_1+1/2}(\Gamma)$ fulfilling $g(S_1)=g(S_2)=0$, with $s_1\in\mathbb{N}$. Suppose that*

$$s_1\notin\Lambda_{1k}\,,\quad \forall\ k=1,\dots,N_1\,, \tag{3.6}$$

then there exists a unique solution $u_1\in H^1(\Omega_1)$ of problem (3.1) which admits the following expansion:

$$u_1=u_{10}+\sum_{k=1}^{N_1}\sum_{\lambda\in\Lambda_{1k}(s_1)}c_{1k\lambda}\eta_{1k}\sigma_{\mathrm{lap}}^{k\lambda}\,, \tag{3.7}$$

where $u_{10}\in H^{s_1+1}(\Omega_1)$, $c_{1k\lambda}\in\mathbb{C}$ depend continuously on f_1 and g, and $\Lambda_{1k}(s_1)=\Lambda_{1k}\cap\,]\,0\,,\,s_1\,]$.

THEOREM 3.2.— *Let $f_2\in H^{s_2-2}(\Omega_2)$, $h_1\in H^{s_2-1/2}(\Gamma)$, $h_2\in H^{s_2-3/2}(\Gamma)$, with $s_2\in\mathbb{N}$, $s_2\ge 2$. Assume that*

$$\big\{\lambda\in\mathbb{C}:\Re\lambda=s_2+1\big\}\cap\Lambda_{2k}=\emptyset\,,\quad \forall\ k=1,\dots,N_2\,, \tag{3.8}$$

then there exists a unique solution $u_2 \in H^2(\Omega_2)$ *of problem (3.2) such that*

$$u_2 = u_{20} + \sum_{k=1}^{N_2} \sum_{\lambda \in \Lambda_{2k}(s_2)} c_{2k\lambda} \eta_{2k} \sigma_{\text{bi}}^{k\lambda}, \tag{3.9}$$

where $u_{20} \in H^{s_2+2}(\Omega_2)$, $c_{2k\lambda} \in \mathbb{C}$ *depend continuously on* f_2, h_1, h_2. *Here we denote* $\Lambda_{2k}(s_2) = \{\lambda \in \Lambda_{2k} \mid 1 \leq \Re\lambda < s_2 + 1\}$.

Let us now go back to our boundary value problem (2.1)-(2.7). Far from the interface Γ, we see that it corresponds to (3.1) or (3.2); therefore, the regularity of u_1 and u_2 is given by the previous theorems. Analogous arguments as those developed in the sequel show that u_1 and u_2 have the optimal regularity in a neighbourhood of a point of Γ. Therefore, we only have to study the behaviour of u_1 and u_2 in a neighbourhood of the common vertices of Ω_1 and Ω_2.

THEOREM 3.3.— *Let* $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $h_1 \in H^{3/2}(\Gamma)$, $h_2 \in H^{1/2}(\Gamma)$ *and* $\overrightarrow{u} = (u_1, u_2) \in V$ *be the solution of (2.1)-(2.7). For* $k \in \{1, 2\}$, *we have:*

a) if $\omega_{1k} > \pi$, *then* u_1 *admits the following decomposition in a neighbourhood* $\mathcal{V}_k$ *of* S_k:

$$u_1 = u_{10} + c_k \sigma_{\text{lap}}^{k\pi/\omega_{1k}} \quad in \quad \mathcal{V}_k \cap \Omega_1, \tag{3.10}$$

where $u_{10} \in H^2(\Omega_1)$, $c_k \in \mathbb{C}$;

b) if $\omega_{1k} \leq \pi$, *then* $u_1 \in H^2(\mathcal{V}_k \cap \Omega_1)$.

Proof.— We may look $u_1 \in H^1(\Omega_1)$ as the solution of

$$\begin{cases} \Delta u_1 = f_1 & \text{in } \Omega_1, \\ \gamma_1 u_1 = 0 & \text{on } \Gamma_1, \\ \gamma_1 u_1 = \gamma_2 u_2 & \text{on } \Gamma. \end{cases} \tag{3.11}$$

Since $u_2 \in H^2(\Omega_2)$ and fulfils (2.4), theorem 1.6.1.5 of [4] implies that

$$\gamma_{2\Gamma} u_2 \in \widetilde{H}^{3/2}(\Gamma). \tag{3.12}$$

Therefore, applying theorem 3.1 with $s_1 = 1$ to problem (3.11), we get the result except if $\omega_{1k} = \pi$. In this last case, (3.12) and theorem 1.5.1.2 of [4] allow to reduce (3.11) in the neighbourhood $\mathcal{V}_k \cap \Omega_1$ to a homogeneous

Dirichlet problem with interior data in L^2. Classical regularity results on smooth domains leads to the conclusion. □

To study the regularity of u_2, we now use the equations (2.2), (2.4), (2.6) and (2.7), i.e. u_2 is seen as a solution of

$$\begin{cases} \Delta^2 u_2 = f_2 & \text{in } \Omega_2 \cap \mathcal{V}_k, \\ \gamma_2 u_2 = \gamma_2 \dfrac{\partial u_2}{\partial \nu_2} = 0 & \text{on } \Gamma_2 \cap \mathcal{V}_k, \\ M u_2 = 0 & \text{on } \Gamma \cap \mathcal{V}_k, \\ N u_2 = -\gamma_{1\Gamma} \dfrac{\partial u_1}{\partial \nu} & \text{on } \Gamma \cap \mathcal{V}_k, \end{cases} \tag{3.13}$$

in a neighbourhood $\mathcal{V}_k$ of S_k.

If $\omega_{1k} \leq \pi$, then $\gamma_{1\Gamma}\left({\partial u_1}/{\partial \nu}\right) \in H^{1/2}(\Gamma \cap \mathcal{V}_k)$, and we may directly apply theorem 3.2 with $s_2 = 2$ to (3.13). But, if $\omega_{1k} > \pi$, only $\gamma_1\left({\partial u_{10}}/{\partial \nu}\right)$ has the adequate regularity $H^{1/2}$, while the normal derivative of the singular function $\sigma_{\mathrm{lap}}^{k\pi/\omega_{1k}}$ has not. The idea is to compute explicitly the contribution of this singular function. By theorem 6.1 hereafter, there exists a solution $\tau_2^k \in H^2(\Omega_2 \cap \mathcal{V}_k)$ of

$$\begin{cases} \Delta^2 \tau_2^k = 0 & \text{in } \Omega_2 \cap \mathcal{V}_k, \\ \gamma_2 \tau_2^k = \gamma_2 \dfrac{\partial \tau_2^k}{\partial \nu_2} = 0 & \text{on } \Gamma_2 \cap \mathcal{V}_k, \\ M \tau_2^k = 0 & \text{on } \Gamma \cap \mathcal{V}_k, \\ N \tau_2^k = -\gamma_1 \dfrac{\partial \sigma_{\mathrm{lap}}^{k\pi/\omega_{1k}}}{\partial \nu} & \text{on } \Gamma \cap \mathcal{V}_k. \end{cases}$$

Therefore, the function u_{21} defined by

$$u_{21} := u_2 - c_k \tau_2^k ,$$

belongs to $H^2(\Omega_2 \cap \mathcal{V}_k)$ and is a solution of problem (3.2) in $\Omega_2 \cap \mathcal{V}_k$ with data $f_2 \in L^2(\Omega_2 \cap \mathcal{V}_k)$, $h_1 = 0$, $h_2 = -\gamma_{1\Gamma}\left({\partial u_{10}}/{\partial \nu}\right) \in H^{1/2}(\Gamma \cap \mathcal{V}_k)$. Therefore, applying theorem 3.2 with $s_2 = 2$ to u_{21}, we obtain the following theorem.

THEOREM 3.4.— *Let $\overrightarrow{u} = (u_1, u_2) \in V$ be the weak solution of (2.1)-(2.7) with data $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $h_1 \in H^{3/2}(\Gamma)$, $h_2 \in H^{1/2}(\Gamma)$. For $k = 1$ or 2, let us suppose that*

$$\{\lambda \in \mathbb{C} \mid \Re\lambda = 3\} \cap \Lambda_{2k} = \emptyset ,$$

then u_2 admits the following decomposition in a neighbourhood of S_k:

$$u_2 = u_{20} + \sum_{\lambda \in \Lambda_{2k}(4)} c_{2k\lambda} \sigma_{\text{bi}}^{k\lambda} + c_k \tau_2^k , \tag{3.14}$$

where $u_{20} \in H^4(\Omega_2)$, $c_{2k\lambda}$, $c_k \in \mathbb{C}$ and the last term of the right-hand side of (3.14) is zero if $\omega_{1k} \leq \pi$.

4. More regular data

In theorems 3.3 and 3.4, if we increase the regularity of the data, we expect to increase in the same way the regularity of the regular parts. One method is to use the same iterative procedure as in section 3; unfortunately, it imposes too much conditions and is complicated. Therefore, we prefer to use a compact perturbation argument.

We need to introduce the Hilbert spaces

$$\begin{aligned} A^{s_1,s_2} &= \left\{(u_1,u_2) \in H^{s_1+1}(\Omega_1) \times H^{s_2+2}(\Omega_2) \text{ fulfilling (2.3) to (2.5)}\right\}, \\ B^{s_1,s_2} &= H^{s_1-1}(\Omega_1) \times H^{s_2-2}(\Omega_2) \times H^{s_2-1/2}(\Gamma) \times \\ &\qquad \times \left(H^{s_1-1/2}(\Gamma) \cup H^{s_2-3/2}(\Gamma)\right). \end{aligned}$$

The operator $L^{(s_1,s_2)}$ induced by the boundary value problem (2.1)-(2.7) is clearly the following:

$$\begin{aligned} L^{(s_1,s_2)} &: A^{(s_1,s_2)} \to B^{(s_1,s_2)} \\ &(u_1,u_2) \to \left(\Delta u_1,\ \Delta^2 u_2,\ M u_2,\ N u_2 + \gamma_{1\Gamma} \frac{\partial u_1}{\partial \nu}\right) . \end{aligned} \tag{4.1}$$

We look for conditions on s_1, s_2 which ensure that $L^{(s_1,s_2)}$ is a Fredholm operator. To do that we split up $L^{(s_1,s_2)}$ into its principal part $L_0^{(s_1,s_2)}$ and a remainder $L_1^{(s_1,s_2)}$ as follows:

$$\begin{aligned} L_0^{(s_1,s_2)} &: A^{(s_1,s_2)} \to B^{(s_1,s_2)} \\ &(u_1,u_2) \to (\Delta u_1,\ \Delta^2 u_2,\ M u_2,\ N u_2) \end{aligned} \tag{4.2}$$

$$\begin{aligned} L_1^{(s_1,s_2)} &: A^{(s_1,s_2)} \to B^{(s_1,s_2)} \\ &(u_1,u_2) \to \left(0,\ 0,\ 0,\ \gamma_{1\gamma} \frac{\partial u_1}{\partial \nu}\right) . \end{aligned} \tag{4.3}$$

Obviously, we have

$$L^{(s_1,s_2)} = L_0^{(s_1,s_2)} + L_1^{(s_1,s_2)} .$$

THEOREM 4.1.— *If $s_1 \in [\,s_2 - 1\,,\, s_2 + 1\,]$ and the Fredholm conditions (3.6) and (3.8) hold, then $L_0^{(s_1,s_2)}$ is a Fredholm operator and*

$$\operatorname{ind} L_0^{(s_1,s_2)} = -\sum_{j=1}^{2}\sum_{k=1}^{N_j} \operatorname{card} \Lambda_{jk}(s_j) . \tag{4.4}$$

Proof.— Let $(f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)}$. We look for a solution $(u_1, u_2) \in A^{(s_1,s_2)}$ of

$$L_0^{(s_1,s_2)}(u_1, u_2) = (f_1, f_2, h_1, h_2) . \tag{4.5}$$

But this is equivalent to the fact that u_2 is a solution of (3.2) and u_1 is a solution of (3.1) with $g = \gamma_{2\Gamma} u_2$ on Γ. Therefore, applying theorem 3.2 to u_2, we deduce that there exists a unique solution $u_2 \in H^2(\Omega_2)$ of (3.1), which admits the decomposition (3.9). As in theorem 3.4, looking for u_1, we use this decomposition (3.9) of u_2. For all $k \in \{1, 2\}$, $\lambda \in \Lambda_{2k}$, theorem 6.1 below gives the explicit solution $\sigma_{\mathrm{bi}\,1}^{k\lambda} \in H^1(\Omega_1)$ of (4.6) in a neighbourhood $\mathcal{V}_k$ of S_k:

$$\begin{cases} \Delta \sigma_{\mathrm{bi}\,1}^{k\lambda} = 0 & \text{in } \Omega_1, \\ \gamma_1 \sigma_{\mathrm{bi}\,1}^{k\lambda} = 0 & \text{on } \Gamma_1, \\ \gamma_1 \sigma_{\mathrm{bi}\,1}^{k\lambda} = \gamma_2 \sigma_{\mathrm{bi}}^{k\lambda} & \text{on } \Gamma. \end{cases} \tag{4.6}$$

Furthermore, theorem 3.1 proves the existence of a unique solution $v_1 \in H^1(\Omega_1)$ of problem (3.1) with data f_1, $g = \gamma_{2\Gamma} u_{20} \in H^{s_2+3/2}(\Gamma) \hookrightarrow H^{s_1+1/2}(\Gamma)$, since $s_1 \leq s_2 + 1$, which admits the decomposition (3.7). Let us notice that in that decomposition (3.7), the coefficients $c_{1k\lambda}$ depend continuous on f_1 and $\gamma_{2\Gamma} u_{20}$; and therefore continuously on f_1, f_2, h_1, h_2. Setting

$$u_1 = v_1 + \sum_{k=1}^{2} \sum_{\lambda \in \Lambda_{2k}(s_2)} c_{2k\lambda} \eta_k \sigma_{\mathrm{bi}\,1}^{k\lambda} ,$$

we have proven that there exists a unique solution $\overrightarrow{u} = (u_1, u_2) \in V$ of (4.5), which admits the decomposition

$$\overrightarrow{u} = \overrightarrow{u}_0 + \sum_{j=1}^{2}\sum_{k=1}^{N_j} \sum_{\lambda \in \Lambda_{jk}(s_j)} c_{jk\lambda} \eta_{jk} \overrightarrow{\sigma}^{jk\lambda} ,$$

where $\overrightarrow{u}_0 \in A^{(s_1,s_2)}$, $c_{jk\lambda} \in \mathbb{C}$ depend continuously on f_1, f_2, h_1, h_2 and we have set

$$\overrightarrow{\sigma}^{jk\lambda} = (\sigma_{\mathrm{lap}}^{k\lambda}, 0), \quad \forall \lambda \in \Lambda_{1k}, \ k \in \{1, \ldots, N_1\},$$

$$\overrightarrow{\sigma}^{2k\lambda} = (\sigma_{\mathrm{bi}\,1}^{k\lambda}, \sigma_{\mathrm{bi}}^{k\lambda}), \quad \forall \lambda \in \Lambda_{2k}, \ k \in \{1, \ldots, N_2\},$$

with the agreement that $\sigma_{\mathrm{bi}\,1}^{k\lambda} = 0$, if $k \geq 3$.

This establishes that if $(f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)}$ is such that

$$c_{jk\lambda} = 0, \quad \forall \lambda \in \Lambda_{jk}(s_j), \tag{4.8}$$

then it belongs to the range of $L_0^{(s_1,s_2)}$.

Reciprocally, if such a datum belongs to the range, then there exists a $(u_1, u_2) \in A^{(s_1,s_2)}$ solution of (4.5); then u_2 is a solution of (3.2) and u_1 of (3.1) with $g = \gamma_{2\Gamma} u_2$. Due to theorem 3.2 and after theorem 3.1, this implies that it fulfils (4.8). So we have proven that the range of $L_0^{(s_1,s_2)}$ is closed and that (4.4) holds since $L_0^{(s_1,s_2)}$ is clearly injective. □

THEOREM 4.2.— *If $s_1 \in \,]\, s_2 - 1, s_2 + 1]$ and if (3.6) and (3.8) hold, then $L^{(s_1,s_2)}$ is a Fredholm operator and*

$$\operatorname{ind} L^{(s_1,s_2)} = -\sum_{j=1}^{2} \sum_{k=1}^{N_j} \operatorname{card} \Lambda_{jk}(s_j). \tag{4.9}$$

Proof.— the assumption $s_1 > s_2 - 1$ implies that $L_1^{(s_1,s_2)}$ is a compact operator, because $H^{s_1 - 1/2}(\Gamma)$ is compactly imbedded into $H^{s_2 - 3/2}(\Gamma)$. Using a classical perturbation theorem (see theorem IV.5.26 of [5], for instance), we deduce the theorem. □

Since we want to give the asymptotic behaviour of the solution of our problem (2.1)-(2.7), we need the singularities of this problem, i.e. the singularities of $L^{(s_1,s_2)}$. As M. Dauge in [2] for non-homogeneous operators, we compute them by recurrence starting from the singularities of the principal part $L_0^{(s_1,s_2)}$. In view of (4.7), we see that the singularities of $L_0^{(s_1,s_2)}$ are the $\overrightarrow{\sigma}^{jk\lambda}$'s. So we proceed as follows: for $k = 1$ or 2, we set

$$\overrightarrow{\sigma}_0^{jk\lambda} = \overrightarrow{\sigma}^{jk\lambda}, \tag{4.10}$$

and for $p \in \mathbb{N}$, $\overrightarrow{\sigma}_p^{jk\lambda}$ is a solution of (4.11) hereafter in a neighbourhood $\mathcal{V}_k$ of S_k:

$$L_0 \overrightarrow{\sigma}_p^{jk\lambda} = -L_1 \overrightarrow{\sigma}_{p-1}^{jk\lambda}, \quad \text{in } \mathcal{V}_k . \tag{4.11}$$

Splitting up $\overrightarrow{\sigma}_p^{jk\lambda}$ into its components,

$$\overrightarrow{\sigma}_p^{jk\lambda} = \left(\sigma_{p,1}^{jk\lambda}, \sigma_{p,2}^{jk\lambda}\right),$$

problem (4.11) is equivalent to (4.12) and (4.13) hereafter solved in that order using theorem 6.1.

$$\begin{cases} \Delta^2 \sigma_{p,2}^{jk\lambda} = 0 & \text{in } \Omega_2 \cap \mathcal{V}_k, \\ \gamma_2 \sigma_{p,2}^{jk\lambda} = \gamma_2 \dfrac{\partial \sigma_{p,2}^{jk\lambda}}{\partial \nu_2} = 0 & \text{on } \Gamma_2 \cap \mathcal{V}_k, \\ M \sigma_{p,2}^{jk\lambda} = 0 & \text{on } \Gamma \cap \mathcal{V}_k, \\ N \sigma_{p,2}^{jk\lambda} = -\gamma_1 \dfrac{\partial \sigma_{p-1,1}^{jk\lambda}}{\partial \nu} & \text{on } \Gamma \cap \mathcal{V}_k. \end{cases} \tag{4.12}$$

$$\begin{cases} \Delta \sigma_{p,1}^{jk\lambda} = 0 & \text{in } \Omega_1 \cap \mathcal{V}_k, \\ \gamma_1 \sigma_{p,1}^{jk\lambda} = 0 & \text{on } \Gamma_1 \cap \mathcal{V}_k, \\ \gamma_1 \sigma_{p,1}^{jk\lambda} = \gamma_2 \sigma_{p,2}^{jk\lambda} & \text{on } \Gamma \cap \mathcal{V}_k. \end{cases} \tag{4.13}$$

The associated singularity of $L^{(s_1,s_2)}$ is defined by (compare with § 5.C of [2])

$$\overrightarrow{\tau}^{jk\lambda} = \eta_k \sum_{\Re\lambda + 2p \leq s_2 + 1} \overrightarrow{\sigma}_p^{jk\lambda} . \tag{4.14}$$

Let us recall that $\overrightarrow{\tau}^{jk\lambda}$ is called a singularity of $L^{(s_1,s_2)}$ because it belongs to V and not to $A^{(s_1,s_2)}$, while $L^{(s_1,s_2)} \overrightarrow{\tau}^{jk\lambda}$ belongs to $B^{(s_1,s_2)}$. Let us check this last assumption. From (4.11) and (4.14), it is clear that

$$L^{(s_1,s_2)} \overrightarrow{\tau}^{jk\lambda} = L_1 \overrightarrow{\sigma}_{p_{\max}}^{jk\lambda}, \quad \text{in } \mathcal{V}_k ,$$

where $p_{\max}$ is such that

$$\Re\lambda + 2p_{\max} \leq s_2 + 1 < \Re\lambda + 2p_{\max} + 2 . \tag{4.15}$$

Therefore, $L^{(s_1,s_2)} \overrightarrow{\tau}^{jk\lambda} \in B^{(s_1,s_2)}$ iff

$$\eta_{jk} \cdot \gamma_{1\Gamma} \frac{\partial}{\partial \nu} \overrightarrow{\sigma}_{p_{\max},1}^{jk\lambda} \in H^{s_2 - 3/2}(\Gamma) . \tag{4.16}$$

In view of the form of $\overrightarrow{\sigma}^{jk\lambda}$ and theorem 6.1, $\gamma_{1\Gamma}(\partial/\partial_{\nu'})\overrightarrow{\sigma}^{jk\lambda}_{p_{\max},1}$ behaves like $r^{\Re\lambda+2p_{\max}-1}$ in a neighbourhood of S_k. So (4.15) leads to the adequate regularity.

Let us finally notice that the above procedure only concerns the singularities induced by S_1 and S_2 (i.e. the $\overrightarrow{\sigma}^{jk\lambda}$, for $k = 1$ or 2). Indeed, for $k \geq 3$, $L^{(s_1,s_2)} = L_0^{(s_1,s_2)}$ in a neighbourhood of S_{jk}, so the singularities of $L_0^{(s_1,s_2)}$ are those of $L^{(s_1,s_2)}$, i.e.

$$\overrightarrow{\tau}^{jk\lambda} = \eta_{jk}\,\overrightarrow{\sigma}^{jk\lambda}\,, \quad \forall\, k \geq 3\,.$$

We now recall lemma B.1 of [2] concerning the relationship between the index and a singularities space.

LEMMA 4.3 (M. Dauge [2]). — *Let $A_1 \subset A_0$ and $B_1 \subset B_0$ be two pairs of Hilbert spaces such that A_1 is dense in A_0 and B_1 is dense in B_0. Let M_0 be a Fredholm operator from A_0 into B_0, which may be restricted to a semi-Fredholm operator, denoted by M_1, from A_1 into B_1.*

We suppose that there exists a finite dimensional space E having the following properties:

$$E \subset A_0\,, \tag{4.17}$$

$$E \cap A_1 = \{0\}\,, \tag{4.18}$$

$$M_0 E \subset B_1\,. \tag{4.19}$$

Then the following conditions are equivalent:

$$M_1 \text{ is a Fredholm operator and } \dim E = \operatorname{ind} M_0 - \operatorname{ind} M_1\,, \tag{4.20}$$

$$\begin{gathered} \text{for any } u \in A_0 \text{ such that } Mu \in B_1 \\ \text{there exists } v \in A_1 \text{ and } w \in E \text{ such that } u = v + w\,. \end{gathered} \tag{4.21}$$

We are ready to prove the theorem 4.4.

THEOREM 4.4. — *Under the assumptions of theorem 4.2, given $(f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)}$, there exists a unique solution $\overrightarrow{u} \in V$ of problem (2.1)-(2.7), which admits the following decomposition*

$$\overrightarrow{u} = \overrightarrow{u}_0 + \sum_{j=1}^{2}\sum_{k=1}^{N_j}\sum_{\lambda\in\Lambda_{jk}(s_j)} c_{jk\lambda}\,\overrightarrow{\tau}^{jk\lambda}\,, \tag{4.22}$$

where $\overrightarrow{u}_0 \in A^{(s_1,s_2)}$ and $c_{jk\lambda} \in \mathbb{C}$.

Proof.— We apply the previous lemma with

$$\begin{aligned} A_1 &= A^{(s_1,s_2)} & B_1 &= B^{(s_1,s_2)} \\ A_0 &= V & B_0 &= V' \\ M_0 &= \Lambda & M_1 &= L^{(s_1,s_2)}, \end{aligned}$$

where Λ is the natural isomorphism between V and V' defined by

$$\langle \Lambda \vec{u}\,,\, \vec{v}\rangle = a(\vec{u}, \vec{v})\,, \quad \forall\ \vec{u}\,,\ \vec{v} \in V\,.$$

Actually, $B^{(s_1,s_2)}$ is identified with a subspace of B_0 by the following continuous injection: for $\vec{F} = (f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)}$, we set

$$\begin{aligned} \langle \vec{F}\,,\, \vec{v}\rangle := &- \int_{\Omega_1} f_1 v_1 \,\mathrm{d}x + \int_{\Omega_2} f_2 v_2 \,\mathrm{d}x + \\ &+ \int_\Gamma \left\{ h_1 \gamma_1 v_1 - h_2 \gamma_2 \frac{\partial v_2}{\partial \nu} \right\} \mathrm{d}\sigma\,, \quad \forall\ \vec{v} = (v_1, v_2) \in V\,. \end{aligned}$$

The restriction of M_0 to A_1 is clearly M_1 because Green's formula (2.11) implies that

$$\Lambda \vec{u} = L^{(s_1,s_2)} \vec{u}\,, \quad \forall\ \vec{u} \in A^{(s_1,s_2)}\,.$$

The space $A^{(s_1,s_2)}$ is dense in V because we can prove that $\mathcal{D}(\Omega_1 \cup \Omega_2)$ is dense in V. Since Λ is a isomorphism, we deduce that $\Lambda A^{(s_1,s_2)}$ is dense in V'. This implies the density of $B^{(s_1,s_2)}$ since $\Lambda A^{(s_1,s_2)} \subset B^{(s_1,s_2)}$.

Finally, the space E is the vector space spanned by the $\vec{\tau}^{\,jk\lambda}$'s, for $j \in \{1, 2\}$, $k \in \{1, \ldots, N_j\}$, $\lambda \in \Lambda_{jk}(s_j)$. We have previously checked that it fulfils the assumptions (4.17)-(4.19). □

To finish this section, let us show that it is possible to hit the limit case $s_1 = s_2 - 1$.

THEOREM 4.5.— *Let $s_1 = s_2 - 1$, assume that the conditions (3.6) and (3.8) hold and moreover that the Fredholm condition (3.8) holds for $s_2 - 1$ too. Then the conclusion of theorem 4.4 still holds.*

Proof.— We firstly apply theorem 4.4 with the same s_1, but with s_2 replaced by $s_2 - 1$. Therefore, the variational solution $\vec{u}$ of (2.1)-(2.7) admits the decomposition (4.22) with $s_2 - 1$ instead of s_2. So the regular part $\vec{u}_0$ has the optimal regularity in Ω_1 but not in Ω_2. The second component of this regular part is actually solution of a boundary value problem (3.2) with data which are the sum of an optimal regularity part and a contribution of the singularities. As in theorem 3.4, we compute explicitly the solution of this boundary value problem with a singular right-hand side using theorem 6.1. The regular right-hand side induces a decomposition into a new regular part in $H^{s_2+2}(\Omega_2)$ and singularities of the boundary value problem (3.2) for $\Re\lambda \in [\,1\,,\,s_2+1\,[$, due to theorem 3.2.

This allows to show that $L^{(s_1,s_2)}$ is a Fredholm operator of index given by (4.9). At this step, we follow the arguments of theorem 4.4. □

5. The non Fredholm property

The aim of this section is to show that in theorems 4.4 and 4.5, the condition $s_1 \in [\,s_2-1\,,\,s_2+1\,]$ is optimal. In other words, we shall prove that if this condition fails then the operator $L^{(s_1,s_2)}$ is never a Fredholm operator. The proof of this result is again based on a compact perturbation argument.

In the sequel, we shall need the following technical result.

LEMMA 5.1.— *Let X, Y be two Hilbert spaces and A a linear operator from X into Y. Suppose that there exists a finite dimensional subspace E of Y such that*

$$R(A) \supset E^{\perp}\,. \tag{5.1}$$

Then the range of A, $R(A)$, is closed and its codimension is finite.

Proof.— Due to (5.1), $R(A)$ admits the following orthogonal decomposition

$$R(A) = \big(R(A) \cap E\big) \oplus E^{\perp}\,.$$

Since $R(A) \cap E$ is a finite dimensional linear manifold of Y, it is closed. Therefore, the previous decomposition implies that $R(A)$ is closed. □

In the following, we suppose that $s_1 > s_2 + 1$; the case $s_1 < s_2 - 1$ being treated analogously.

We need to introduce a variant of the operator $L^{(s_1,s_2)}$, which take into account the non-homogeneous interface condition (2.5). We set

$$\widehat{A}^{(s_1,s_2)} = \big\{(u_1,u_2) \in H^{s_1+1}(\Omega_1) \times H^{s_2+2}(\Omega_2) \text{ fulfilling (2.3) and (2.4)}\big\}\,,$$

$$\widehat{H}^{s_2+3/2}(\Gamma) = \left\{g \in H^{s_2+3/2}(\Gamma) \mid g(S_1) = g(S_2) = 0\right\},$$
$$\widehat{B}^{(s_1,s_2)} = B^{(s_1,s_2)} \times \widehat{H}^{s_2+3/2}(\Gamma),$$

$$\begin{aligned} \widehat{L}^{(s_1,s_2)} : \widehat{A}^{(s_1,s_2)} &\to \widehat{B}^{(s_1,s_2)} \\ (u_1, u_2) &\to \left(L^{(s_1,s_2)}(u_1,u_2),\, \gamma_{2\Gamma}u_2 - \gamma_{1\Gamma}u_1\right). \end{aligned}$$

Let us notice that theorem 1.6.1.5 of [4] shows that $\widehat{L}^{(s_1,s_2)}$ is well defined.

LEMMA 5.2.— *Suppose that the angles at the ends of Γ are different from π, then $L^{(s_1,s_2)}$ is a Fredholm operator iff $\widehat{L}^{(s_1,s_2)}$ is a Fredholm operator.*

Proof.— Clearly, $L^{(s_1,s_2)}$ and $\widehat{L}^{(s_1,s_2)}$ are injective; therefore the assertion only concerns their ranges.

- Suppose that $\widehat{L}^{(s_1,s_2)}$ is a Fredholm operator. Then there exists a finite dimensional subspace E of $\widehat{B}^{(s_1,s_2)}$ such that

$$R\big(\widehat{L}^{(s_1,s_2)}\big) = E^{\perp}.$$

But this implies that

$$R\big(L^{(s_1,s_2)}\big) \supset (PE)^{\perp},$$

where P is the projection in $\widehat{B}^{(s_1,s_2)}$ on $B^{(s_1,s_2)}$. Since PE has a finite dimension, lemma 5.1 allows us to conclude that $L^{(s_1,s_2)}$ is a Fredholm operator.

- Let us now assume that $L^{(s_1,s_2)}$ is Fredholm. As previously, there exists a finite dimensional subspace E_1 of $B^{(s_1,s_2)}$ such that

$$R\big(L^{(s_1,s_2)}\big) = E_1^{\perp}.$$

Furthermore, using theorem 1.6.1.5 of [4] and the assumptions of the lemma, we can prove that the operator

$$\begin{aligned} T : \widehat{A}^{(s_1,s_2)} &\to \widehat{H}^{s_2+3/2}(\Gamma) \\ (u_1, u_2) &\to \gamma_{2\Gamma}u_2 - \gamma_{1\Gamma}u_1 \end{aligned}$$

is onto. So it admits a continuous right inverse, denoted it by R.

Let us now fix $(\overrightarrow{F}, h) \in \widehat{B}^{(s_1,s_2)}$ satisfying

$$\left(\overrightarrow{F} - L^{(s_1,s_2)}Rh,\ \overrightarrow{v}\right) = 0, \quad \forall\ \overrightarrow{v} \in E_1. \tag{5.2}$$

Then there exists $\overrightarrow{u} \in A^{(s_1,s_2)}$ such that

$$L^{(s_1,s_2)}\overrightarrow{u} = \overrightarrow{F} - L^{(s_1,s_2)}Rh\,.$$

This means $(\overrightarrow{F}, h)$ belongs to the range of $\widehat{L}^{(s_1,s_2)}$ because $\overrightarrow{v}$ defined by

$$\overrightarrow{v} = \overrightarrow{u} + Rh\,,$$

belongs to $\widehat{A}^{(s_1,s_2)}$ and fulfils

$$\widehat{L}^{(s_1,s_2)}\overrightarrow{v} = (\overrightarrow{F}, h)\,.$$

Again, lemma 5.1 implies that $\widehat{L}^{(s_1,s_2)}$ is Fredholm since (5.2) is equivalent to

$$(\overrightarrow{F}, \overrightarrow{v}) - \left(h\,,\ R^* L^{(s_1,s_2)*}\overrightarrow{v}\right) = 0\,, \quad \forall\ \overrightarrow{v} \in E_1\,. \square$$

THEOREM 5.3. — *Suppose that the angles at the ends of* Γ *are different from* π, *if* $s_1 > s_2 + 1$, *then* $L^{(s_1,s_2)}$ *is not a Fredholm operator.*

Proof. — We introduce the operator

$$\begin{aligned} K : \widehat{A}^{(s_1,s_2)} &\to \widehat{B}^{(s_1,s_2)} \\ (u_1, u_2) &\to \left(0,\ 0,\ 0,\ -\gamma_{1\Gamma}\frac{\partial u_1}{\partial \nu}\,,\ \gamma_{1\Gamma} u_1\right)\,. \end{aligned}$$

Setting

$$\widehat{H}^{s_1+1/2}(\Gamma) = \left\{g \in H^{s_1+1/2}(\Gamma) : g(S_k) = 0\,,\ k = 1,\ 2\right\},$$

we remark that

$$R(K) \subset \{0,0,0\} \times H^{s_1+1/2}(\Gamma) \times \widehat{H}^{s_1+1/2}(\Gamma)\,.$$

This implies that K is a compact operator because $\widehat{H}^{s_1+1/2}(\Gamma)$ (resp. $H^{s_1+1/2}(\Gamma)$) is compactly imbedded into $\widehat{H}^{s_2+3/2}(\Gamma)$ (resp. $H^{s_2+3/2}(\Gamma)$).

Let us suppose that $\widehat{L}^{(s_1,s_2)}$ is Fredholm. Then

$$\begin{aligned} \widehat{L}^{(s_1,s_2)} + K : \widehat{A}^{(s_1,s_2)} &\to \widehat{B}^{(s_1,s_2)} \\ (u_1, u_2) &\to \left(\Delta u_1,\ \Delta^2 u_2,\ M u_2,\ N u_2,\ \gamma_{2\Gamma} u_2\right) \end{aligned} \tag{5.3}$$

is also Fredholm. This leads to a contradiction since the kernel of $\widehat{L}^{(s_1,s_2)} + K$ is not finite-dimensional (indeed u_1 does not satisfy any condition on the interface). □

Remark 5.4.— The amplitude 2 in the condition $s_1 \in [\,s_2 - 1\,,\, s_2 + 1\,]$ is exactly the difference between the order of the biharmonic operator and the Laplace operator. This means that if we consider elliptic operators of respective order $2m_1$ and $2m_2$, then the amplitude would be $|2m_1 - 2m_2|$.

6. Logarithmico-polynomial resolution

Theorem 1.3 of [6] gives the existence of a solution to the boundary value problems (4.12) and (4.13). Here, following [11], we give another proof of this result, based on the use the Jordan chains. Indeed, it was shown in [11] how to reduce each of these boundary value problems into an abstract differential equation in a Hilbert space using the change of variable $r = e^t$ and reducing the order. With the particular right-hand side of (4.12) or (4.13), the equivalent differential equation we get is

$$\left(\frac{\partial}{\partial t} - A\right) u(t) = e^{\lambda t} \sum_{q=0}^{Q} t^q f_q\,, \tag{6.1}$$

where A is a closed operator defined in an appropriate Hilbert space X, for some $\lambda \in \mathbb{C}$, $Q \in \mathbb{N} \cup \{0\}$ and $f_q \in X$, for all $q \in \{0, \ldots, Q\}$.

In order to solve (4.12) or (4.13), it is therefore equivalent to solve their corresponding problems (6.1). The case $Q = 0$ was solved in paragraph 4 of [11] in an abstract setting (it was called the polynomial resolution because for $\lambda \in \mathbb{N}$, it corresponds to the resolution of problems (4.12) or (4.13) with polynomial data). We shall extend this technique to the general case $Q \geq 0$.

Let us recall the abstract setting of [11] : X is a Hilbert space, A a closed operator from X into X such that its domain $D(A)$ is also a Hilbert space with its own topology. We assume that there exists a closed subspace Z of X such that $D(A)$ is dense in Z and is compactly imbedded into Z. Finally, the resolvent set of A is assumed to be nonempty.

The first idea to solve (6.1) is to look for a solution u in the same form than the right-hand side, i.e.

$$u(t) = e^{\lambda t} \sum_{q=0}^{Q} t^q \varphi_q\,, \tag{6.2}$$

where $\varphi_q \in D(A)$ are the new unknowns. In that case, problem (6.1) is equivalent to

$$\begin{cases} (\lambda - A)\varphi_q + (q+1)\varphi_{q+1} = f_q\,, & q = 0, \ldots, Q-1, \\ (\lambda - A)\varphi_Q = f_Q\,. \end{cases} \tag{6.3}$$

If λ is not an eigenvalue of A, then for arbitrary $f_q \in X$, (6.3) has unique solutions given by

$$\varphi_q = \sum_{\ell=0}^{Q-q} (-1)^\ell \frac{(q+\ell)!}{\ell!} (\lambda - A)^{-(1+\ell)} f_{q+\ell}\,, \quad \forall\, q = 0, \ldots, Q\,. \tag{6.4}$$

If λ is an eigenvalue of A, the previous technique fails in general. As in [11], we shall use the associated Jordan basis $\left\{ \{\varphi^{\lambda,\mu,k}\}_{k=0}^{\mathcal{K}(\lambda,\mu)-1} \right\}_{\mu=1}^{M(\lambda)}$ and the dual Jordan basis $\left\{ \{\psi^{\lambda,\mu,k}\}_{\mu=0}^{\mathcal{K}(\lambda,\mu)-1} \right\}_{\mu=1}^{M(\lambda)}$. Let us recall that they fulfil (see lemma 2.3 of [11]):

$$(A - \lambda)\varphi^{\lambda,\mu,k} = \varphi^{\lambda,\mu,k-1}\,, \tag{6.5}$$

$$\langle (A-\lambda)u\,,\, \psi^{\lambda,\mu,k} \rangle = \langle u\,,\, \psi^{\lambda,\mu,k+1} \rangle\,, \quad \forall\, u \in D(A)\,, \tag{6.6}$$

$$\langle \varphi^{\lambda,\mu,k}\,,\, \psi^{\lambda,\mu,k} \rangle = \delta_{\mu,\mu'}\delta_{kk'}\,, \tag{6.7}$$

for every $k = 0, \ldots, \mathcal{K}(\lambda,\mu) - 1$, $k' = 0, \ldots, \mathcal{K}(\lambda,\mu') - 1$, μ, $\mu' = 1, \ldots, M(\lambda)$ and the conventions $\varphi^{\lambda,\mu,-1} = 0$ and $\psi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)} = 0$.

For all $q \in \mathbb{N} \cup \{0\}$, let us denote

$$\sigma_q^{\lambda,\mu} = e^{\lambda t} \sum_{\ell=1}^{\mathcal{K}(\lambda,\mu)} \frac{q!}{(\ell+q)!} t^{\ell+q} \varphi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)-\ell}\,. \tag{6.8}$$

Using (6.5), we check that

$$\left(\frac{\partial}{\partial t} - A \right) \sigma_q^{\lambda,\mu} = e^{\lambda t} t^q \varphi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)-1}\,, \quad \forall\, q \in \mathbb{N} \cup \{0\}\,. \tag{6.9}$$

Now, we look for a solution u of (6.1) in the form

$$u(t) = \sum_{q=0}^{Q} \left\{ e^{\lambda t} t^q \varphi_q + \sum_{\mu=1}^{M(\lambda)} c_{\mu q} \sigma_q^{\lambda,\mu} \right\}\,, \tag{6.10}$$

where $\varphi_q \in D(A)$ and $c_{\mu q}$ are unknown. In view of (6.9), problem (6.1) is thus equivalent to

$$(\lambda - A)\varphi_q = f_q - \sum_{\mu=1}^{M(\lambda)} c_{\mu q}\varphi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)-1} - (q+1)\varphi_{q+1}\,, \quad \forall\, q = 0, \ldots, Q\,, \tag{6.11}$$

with the convention $\varphi_{Q+1} = 0$. Since the range of $A - \lambda$ is the orthogonal of $\ker\big((A - \lambda)^*\big) = \mathrm{Sp}\left(\{\psi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)-1}\}_{\mu=1}^{M(\lambda)}\right)$, this problem (6.11) has solutions φ_q, $q = 0, \ldots, Q$ iff

$$\left\langle f_q - \sum_{\mu=1}^{M(\lambda)} c_{\mu q}\varphi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)-1} - (q+1)\varphi_{q+1}\,,\, \psi^{\lambda,\mu',\mathcal{K}(\lambda,\mu')-1}\right\rangle = 0\,, \tag{6.12}$$

for all $\mu' = 1, \ldots, M(\lambda)$, $q = 0, \ldots, Q$.

Using the orthogonal conditions (6.7), (6.12) is equivalent to

$$c_{\mu q} = \left\langle f_q - (q+1)\varphi_{q+1}\,,\, \psi^{\lambda,\mu,\mathcal{K}(\lambda,\mu)-1}\right\rangle\,, \quad \forall\, q = 0, \ldots, Q\,. \tag{6.13}$$

This means that we solve problem (6.11) by recurrence starting with the value $q = Q$. Indeed, for each q, assuming that φ_{q+1} exists, then taking $c_{\mu q}$ given by (6.13), we deduce the existence of at least one solution φ_q of (6.11). Since φ_Q exists (recall that $\varphi_{Q+1} = 0$), we have proven the theorem 6.1.

THEOREM 6.1.— *For all $\lambda \in \mathbb{C}$, $Q \in \mathbb{N} \cup \{0\}$, $\varphi_q \in X$, $q \in \{0, \ldots, Q\}$, there exists a solution $u(t)$ of problem (6.1) in the form*

$$u(t) = \sum_{q=0}^{Q} \left\{ e^{\lambda t} t^q \varphi_q + \sum_{\mu=1}^{M(\lambda)} c_{\mu q}\sigma_q^{\lambda,\mu} \right\}\,, \tag{6.14}$$

where the sum over μ disappears if λ is not an eigenvalue of A; otherwise, the $c_{\mu q}$'s are given by (6.13) and φ_q's are solutions of (6.11).

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