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1. Introduction

We introduce a new kind of interface problems on polygonal domains of the plane. The novelty is that the order of the partial differential operators is different on each face. We only study a model problem corresponding to the mechanical example of a coupling between a plate and a membrane. We expect that the methods we developed could be extended to more general problems.

In classical interface problems (see [3, 10] and the references cited there), the variational solution has singularities at the common vertices between the interface and the boundary. Therefore, we can expect the same type
of results for our problem. Indeed, for interior data in $L^2$, we can give the
decomposition of the variational solution of our problem into a regular part
with the optimal regularity and a singular one. The main idea is to use
a two steps argument by splitting up two of the interface conditions, and
use successively the decomposition results for an inhomogeneous boundary
value problem on each face respectively associated with the Laplace operator
and the biharmonic one.

For more regular data, we could argue iteratively as before, but this
induces too much geometrical conditions (on the angles of the domains).
Therefore we prefer a compact perturbation argument as for boundary
value problems with non-homogeneous partial differential operators \[7, 2\].
Indeed, we shall see that the difference of order of the operators on the faces
will induce interface conditions with non-homogeneous operators (i.e. it is
sum of operators of different order). This argument only holds under some
conditions on the Sobolev exponents. One of them is also necessary since
we shall show that if this condition fails then the induced operator is not
Fredholm.

We finish this paper by solving, in section 6, a differential equation with
operator coefficients

$$\frac{\partial u}{\partial t} - Au(t) = e^{\lambda t} t^q f_q \quad \text{in } \mathbb{R}, \quad (1.1)$$

where $A$ is a closed operator defined on a Hilbert space $X$, $\lambda \in \mathbb{C}$, $q \in \mathbb{N} \cup \{0\}$
and $f_q \in X$. As shown in \[11\] (see also the references cited there), solving
(1.1) allows us to solve explicitly some boundary value problems in a finite
cone of $\mathbb{R}^n$ with a right-hand side which is a linear combination of functions of type

$$r^\lambda (\log r)^q \varphi_q(\theta),$$

where $(r, \theta)$ are the spherical coordinates, $\lambda \in \mathbb{C}$, $q \in \mathbb{N} \cup \{0\}$ and $\varphi_q$ is
regular enough. This result agrees with those of \[6\].

2. Formulation of the problem

Let $\Omega_1$, $\Omega_2$ be two bounded simply connected polygonal domains of the
plane such that their boundaries have a common side denoted by $\Gamma$. We
denote by $\Gamma_1$ (resp. $\Gamma_2$) the boundary of $\Omega_1$ (resp. $\Omega_2$) except $\Gamma$, i.e.
$\Gamma_j = \partial \Omega_j \setminus \overline{\Gamma}$, for $j = 1, 2$. 

\[188\]
For $j = 1, 2$, $\nu_j$ will denote unitary outer normal vector on the boundary $\partial \Omega_j$ of $\Omega_j$ and $\tau_j$ the unitary tangent vector along $\partial \Omega_j$ so that $(\nu_j, \tau_j)$ is a direct orthonormal basis. Along the common side $\Gamma$, we omit the index by setting $(\nu, \tau) = (\nu_2, \tau_2)$. We shall denote $S_{jk}$, for $k \in \{1, \ldots, N_j\}$, the vertices of $\Omega_j$, numbered according to the trigonometric orientation for $\Omega_1$ and numbered clockwise for $\Omega_2$; $\omega_{jk}$ will be the interior angle at $S_{jk}$. Moreover, for convenience, we assume that $S_{11} = S_{21}$ and $S_{12} = S_{22}$ belong to $\Gamma$ and denote them $S_1$ and $S_2$ respectively. We also denote by $\eta_{jk}$, a cut-off function equal to 1 in a neighbourhood of $S_{jk}$ and equal to 0 in a neighbourhood of the other vertices. As previously, we may suppose that: $\eta_{11} = \eta_{21} =: \eta_1$ and $\eta_{12} = \eta_{22} =: \eta_2$. Finally, $\gamma_j$ will denote the trace operator on the boundary $\partial \Omega_j$ of $\Omega_j$; $\gamma_j \Gamma$ will be the restriction of $\gamma_j$ to $\Gamma$.

For $E > 0$ and $\sigma \in ]0, 1[$ (respectively the Young modulus and the Poisson coefficient of the constitutive material of the plate $\Omega_2$), we set $\rho = E/(1 - \sigma^2)$ and we introduce the boundary operator defined only on $\Gamma$

$$
Mu = \rho \gamma_2 \Gamma \left( \sigma \Delta u + (1 - \sigma) \frac{\partial^2 u}{\partial \nu^2} \right),
$$

$$
Nu = \rho \gamma_2 \Gamma \left( \frac{\partial \Delta u}{\partial \nu} + (1 - \sigma) \frac{\partial^3 u}{\partial \nu \partial \tau^2} \right).
$$

We recall that we use here classical Sobolev spaces, i.e. if $\Omega$ is a bounded open set of $\mathbb{R}^2$ and $s$ a non-negative integer, then

$$
H^s(\Omega) = \{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \forall \alpha \in \mathbb{N}^2 : |\alpha| \leq s \},
$$

its norm being denoted by $\| \cdot \|_{s, \Omega}$. For other definitions, we follow Grisvard's book [4].

We consider the following interface problem: given $f_1 \in H^{s_1-1}(\Omega_1)$, $f_2 \in H^{s_2-2}(\Omega_2)$, $h_1 \in H^{s_2-1/2}(\Gamma)$, $h_2 \in H^{s_2-3/2}(\Gamma) \cup H^{s_1-1/2}(\Gamma)$, for $s_1$, $s_2 \in \mathbb{N}$ with $s_2 \geq 2$, find $u_1 \in H^{s_1+1}(\Omega_1)$, $u_2 \in H^{s_2+2}(\Omega_2)$, solutions of (2.1)-(2.7) below:

$$
\Delta u_1 = f_1 \quad \text{in } \Omega_1, \quad (2.1)
$$
$$
\Delta^2 u_2 = f_2 \quad \text{in } \Omega_2, \quad (2.2)
$$
$$
\gamma_1 u_1 = 0 \quad \text{on } \Gamma_1, \quad (2.3)
$$
$$
\gamma_2 u_2 = \gamma_2 \frac{\partial u_2}{\partial \nu_2} = 0 \quad \text{on } \Gamma_2, \quad (2.4)
$$

- 189 -
We first give the variational formulation of this problem. We set

\[ V = \{ \overrightarrow{u} = (u_1, u_2) \in H^1(\Omega_1) \times H^2(\Omega_2) \text{ fulfilling } (2.3), (2.4) \text{ and } (2.5) \}. \]

It is a Hilbert space equipped with the inner product induced by \( H^1(\Omega_1) \times H^2(\Omega_2) \) with the norm

\[ \| \overrightarrow{u} \|_V = \left( \| u_1 \|_{1,\Omega_1}^2 + \| u_2 \|_{2,\Omega_2}^2 \right)^{\frac{1}{2}}. \]

We define the sesquilinear form \( a \) on \( V \) as follows:

\[ a(\overrightarrow{u}', \overrightarrow{v'}) = a_1(u_1, v_1) + a_2(u_2, v_2), \]

where we take

\[ a_1(u_1, v_1) = \int_{\Omega_1} \nabla u_1 \cdot \nabla v_1 \, dx, \]

\[ a_2(u_2, v_2) = \rho \int_{\Omega_2} \left\{ \Delta u_2 \Delta v_2 - (1 - \sigma) \left( \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial^2 \overline{v}_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_2^2} \frac{\partial^2 \overline{v}_2}{\partial x_1^2} - 2 \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \frac{\partial^2 \overline{v}_2}{\partial x_1 \partial x_2} \right) \right\} \, dx \]

**Lemma 2.1.** — For all \( f_1 \in L^2(\Omega_1), \ f_2 \in L^2(\Omega_2), \ h_1, \ h_2 \in L^2(\Gamma), \) there exists a unique solution \( \overrightarrow{u} \in V \) of

\[ a(\overrightarrow{u}', \overrightarrow{v'}) = -\int_{\Omega_1} f_1 \overline{v}_1 \, dx + \rho \int_{\Omega_2} f_2 \overline{v}_2 \, dx + \int_{\Gamma} \left\{ h_1 \gamma_2 \frac{\partial \overline{v}_2}{\partial \nu} - h_2 \gamma_2 \overline{v}_2 \right\} \, d\sigma. \]  

**Proof.** — In order to apply the Lax-Milgram lemma, we need to show the continuity and the coerciveness of the form \( a \) on \( V \).

The continuity is a direct consequence of the continuity of the form \( a_j \) on \( H^j(\Omega_j) \) and of the Cauchy-Schwarz inequality.
Owing to inequality (2.15) of [12], we deduce that

\[ a(\overline{u}, \overline{v}) \geq \min(1, \rho(1 - \sigma)) \left\{ [u_1]_{1, \Omega_1}^2 + [u_2]_{2, \Omega_2}^2 \right\}, \quad \forall \overline{u} \in V, \]

where \([u_j]_{j, \Omega_j}\) denotes the semi-norm of \(H^j(\Omega_j)\). But for \(\overline{v} \in V\), the boundary conditions (2.3) and (2.4) respectively fulfilled by \(u_1\) and \(u_2\) imply that the norms and the semi-norms are equivalent (see e.g. Theorem I.1.9 of [8]). Therefore the previous estimate leads to the coerciveness of the form \(a\) on \(V\). \(\Box\)

Let us now show that a solution of (2.8) is a weak solution of (2.1)-(2.7). We follow the arguments of section 1.5.3 of [4]: we introduce the spaces

\[ E(\Delta^j, L^2(\Omega_j)) = \{ v \in H^j(\Omega_j) : \Delta^j v \in L^2(\Omega_j) \}, \quad j = 1, 2, \]

these are Banach spaces for the norms

\[ \| v \|_{j, \Omega_j} + \| \Delta^j v \|_{0, \Omega_j}, \quad j = 1, 2. \]

Lemma 1.5.3.9 of [4] proves that \(D(\Omega)\) is dense in \(E(\Delta, L^2(\Omega_1))\); analogous arguments lead to the density of \(D(\Omega)\) into \(E(\Delta^2, L^2(\Omega_2))\).

**Lemma 2.2.** — The mapping

\[ (u_1, u_2) \rightarrow \left( Mu_2, Nu_2 + \gamma_{1\Gamma} \frac{\partial u_1}{\partial \nu} \right), \]

which is defined on \(D(\Omega) \times D(\Omega)\), has a unique continuous extension as an operator from \(E(\Delta, L^2(\Omega_1)) \times E(\Delta^2, L^2(\Omega_2))\) into \(\widetilde{H}^{1/2}(\Gamma)' \times \widetilde{H}^{3/2}(\Gamma)\) (identifying \(\Gamma\) with a real interval, we recall that \(u \in \widetilde{H}^s(\Gamma)\) iff \(\overline{u}\), the extension of \(u\) by 0 outside \(\Gamma\), remains in \(H^s(\mathbb{R})\)).

**Proof.** — Owing to theorem 1.5.2.8 and corollary 1.4.4.10 of [4], given \((w_1, w_2) \in \widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{3/2}(\Gamma)\), there exists \(\overline{v} = (v_1, v_2) \in V\) satisfying

\[ \begin{cases} \gamma_1 v_1 = \gamma_2 v_2 = w_2 & \text{on } \Gamma, \\ \gamma_2 \frac{\partial v_2}{\partial \nu} = w_1 & \text{on } \Gamma. \end{cases} \tag{2.9} \]

and

\[ \| \overline{v} \|_V \leq C_1 \left\{ \| w_1 \|_{\widetilde{H}^{1/2}(\Gamma)} + \| w_2 \|_{\widetilde{H}^{3/2}(\Gamma)} \right\}, \tag{2.10} \]

where the constant \(C_1\) is independent of \(w_1, w_2\).
For a fixed $\overline{w} = (u_1, u_2) \in D(\overline{\Omega}_1) \times D(\overline{\Omega}_2)$, let us set

$$\ell(w_1, w_2) = \int_{\Gamma} \left\{ Mu_2 \overline{w}_1 - \left( Nu_2 + \gamma_1 \frac{\partial u_1}{\partial \nu} \right) \overline{w}_2 \right\} \, d\sigma.$$ 

By integration by parts, we get (see lemma 2.3 of [12] for the biharmonic operator):

$$\ell(w_1, w_2) = a(\overline{w}', \overline{w}') + \int_{\Omega_1} \Delta u_1 \overline{v}_1 \, dx - \rho \int_{\Omega_2} \Delta^2 u_2 \overline{v}_2 \, dx. \quad (2.11)$$

Therefore using the continuity of the form $a$ on $V$ and the estimate (2.10), there exists a constant $C_2$ independant of $w_1$, $w_2$ such that

$$|\ell(w_1, w_2)| \leq C_2 \left\{ \| u_1 \|_{E(\Delta, L^2(\Omega_1))} + \| u_2 \|_{E(\Delta^2, L^2(\Omega_2))} \right\} \times \left\{ \| w_1 \|_{\widetilde{H}^{1/2}(\Gamma)} + \| w_2 \|_{\widetilde{H}^{3/2}(\Gamma)} \right\}.$$ 

By density, we deduce that $\ell$ is a continuous linear form on $\widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{3/2}(\Gamma)$. $\Box$

Let us notice that the Green formula (2.11) still holds for every $v \in V$ fulfilling (2.9) and every $u_1 \in E(\Delta, L^2(\Omega_1))$, $u_2 \in E(\Delta^2, L^2(\Omega_2))$, where the left-hand side has to be understood as a duality bracket.

**Lemma 2.3.**— Let $\overline{w} \in V$ be the unique solution of (2.8). Then $\overline{w}$ fulfils (2.1) to (2.7).

**Proof.**— Applying (2.8) with $(v_1, v_2) \in D(\Omega_1) \times D(\Omega_2)$, we see that $u_1$ (resp. $u_2$) fulfils (2.1) (resp. (2.2)) in the distributional sense. This also shows that

$$u_1 \in E(\Delta, L^2(\Omega_1)),$$  
$$u_2 \in E(\Delta^2, L^2(\Omega_2)).$$

Therefore, for arbitrary $(w_1, w_2) \in \widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{3/2}(\Gamma)$, comparing (2.8) with (2.11), when $v \in V$ fulfils (2.9), we deduce that

$$\langle Mu_2, w_1 \rangle - \left( Nu_2 + \gamma_1 \frac{\partial u_1}{\partial \nu}, w_2 \right) = \int_{\Gamma} \left\{ h_1 \overline{w}_1 - h_2 \overline{w}_2 \right\} \, d\sigma.$$ 

This obviously implies that $\overline{w}$ satisfies (2.6) and (2.7). $\Box$
3. Regularity for interior data in $L^2$

In this paragraph, we look for conditions on $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $h_1 \in H^{3/2}(\Gamma)$, $h_2 \in H^{1/2}(\Gamma)$, which ensure that $u_1$ and $u_2$ have the optimal regularity, i.e. $u_1 \in H^2(\Omega_1)$, $u_2 \in H^4(\Omega_2)$; indeed we shall prove that $u_1$ and $u_2$ admit a decomposition into a regular part with the optimal regularity and a finite sum of singular functions. We shall see that this decomposition result will be determined by analogous decomposition results of two decoupled boundary value problems set in $\Omega_1$ and $\Omega_2$. More precisely, the first one is the Dirichlet problem in $\Omega_1$ with non-homogeneous Dirichlet boundary conditions on $\Gamma$, i.e.

$$\begin{aligned}
\Delta u_1 &= f_1 \quad \text{in } \Omega_1, \\
\gamma_1 u_1 &= 0 \quad \text{on } \Gamma_1, \\
\gamma_1 u_1 &= g \quad \text{on } \Gamma.
\end{aligned} \quad (3.1)$$

The second one is the following mixed boundary value problem for the biharmonic operator in $\Omega_2$:

$$\begin{aligned}
\Delta^2 u_2 &= f_2 \quad \text{in } \Omega_2, \\
\gamma_2 u_2 &= \gamma_2 \frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } \Gamma_2, \\
M u_2 &= h_1 \quad \text{on } \Gamma, \\
N u_2 &= h_2 \quad \text{on } \Gamma.
\end{aligned} \quad (3.2)$$

The regularity of the solution of problem (3.1) was given in theorem 5.1.3.5 of [4], while problem (3.2) was studied in theorem 5.2 of [12] (see also [1]). In order to recall these results, let us define the singular exponents and singular functions of problems (3.1) and (3.2).

For problem (3.1), we set

$$\Lambda_{1k} = \left\{ \frac{m \pi}{\omega_{1k}} : m \in \mathbb{N} \right\}, \quad \forall \, k \in \{1, \ldots, N_1\}. \quad (3.3)$$

For $\lambda \in \Lambda_{1k}$, the associated singular function is

$$\sigma_{\text{lap}}^{k\lambda}(r, \theta) = \begin{cases} 
    r^\lambda \sin(\lambda \theta) & \text{if } \lambda \notin \mathbb{N}, \\
    r^\lambda \{\ln r \sin(\lambda \theta) + \theta \cos(\lambda \theta)\} & \text{if } \lambda \in \mathbb{N},
\end{cases}$$

where $(r, \theta)$ are polar coordinates with origin $S_{1k}$ (such that the half-lines $\theta = 0$ and $\theta = \omega_{1k}$ contain the edges containing $S_{1k}$).
For problem (3.2), in order to avoid too complicated notations, we only recall that the singular exponents are the roots of the following characteristic equation:

\[
sin^2(\lambda - 1)\omega_{2k} + \frac{1 - \sigma}{3 + \sigma}(\lambda - 1)\sin^2 \omega_{2k} - \frac{4}{(1 - \sigma)(3 + \sigma)} = 0, \quad \text{for } k = 1, 2, \tag{3.4}
\]

\[
sin^2(\lambda - 1)\omega_{2k} - (\lambda - 1)^2 \sin^2 \omega_{2k} = 0, \quad \text{for } k \in \{3, \ldots, N_2\}. \tag{3.5}
\]

We only say that there exists a set \( \Lambda_{2k} \) of roots of the equation (3.4) for \( k \leq 2 \) and of (3.5) for \( k \geq 3 \), repeated according to their multiplicity; to each \( \lambda \in \Lambda_{2k} \) corresponds a singular function denoted by \( \sigma_{bi}^{k\lambda} \) (see [12] for more details). For \( k = 1, 2 \), the polynomial resolution (cf. § 3.C of [12] and (2.9) of [1]) implies that \( \lambda = 2 \) induces a singular function given by

\[
\sigma_{bi}^{k2}(r, \theta) = r^2\theta + p_k(r, \theta), \quad k = 1, 2,
\]

where \((r, \theta)\) are polar coordinates with origin \( S_\theta \) and \( p_k \) is a polynomial of degree 2 (in the cartesian coordinates). Remark that \( \sigma_{bi}^{k2} \in H^2(\Omega_2) \).

Therefore, for convenience, we shall add \( \lambda = 2 \) to \( \Lambda_{2k} \), for \( k = 1, 2 \) and still denoted it by \( \Lambda_{2k} \).

**Theorem 3.1.** — Let \( f_1 \in H^{s_1-1}(\Omega_1), \ g \in H^{s_1+1/2}(\Gamma) \) fulfilling \( g(S_1) = g(S_2) = 0 \), with \( s_1 \in \mathbb{N} \). Suppose that

\[
s_1 \not\in \Lambda_{1k}, \quad \forall \ k = 1, \ldots, N_1, \tag{3.6}
\]

then there exists a unique solution \( u_1 \in H^1(\Omega_1) \) of problem (3.1) which admits the following expansion:

\[
u_1 = u_{10} + \sum_{k=1}^{N_1} \sum_{\lambda \in \Lambda_{1k}(s_1)} c_{1k\lambda} \eta_{1k} \sigma_{lap}^{k\lambda}, \tag{3.7}
\]

where \( u_{10} \in H^{s_1+1}(\Omega_1) \), \( c_{1k\lambda} \in \mathbb{C} \) depend continuously on \( f_1 \) and \( g \), and \( \Lambda_{1k}(s_1) = \Lambda_{1k} \cap [0, s_1] \).

**Theorem 3.2.** — Let \( f_2 \in H^{s_2-2}(\Omega_2), \ h_1 \in H^{s_2-1/2}(\Gamma), \ h_2 \in H^{s_2-3/2}(\Gamma) \), with \( s_2 \in \mathbb{N}, \ s_2 \geq 2 \). Assume that

\[
\{ \lambda \in \mathbb{C} : \Re \lambda = s_2 + 1 \} \cap \Lambda_{2k} = \emptyset, \quad \forall \ k = 1, \ldots, N_2, \tag{3.8}
\]
On a coupled problem between the plate equation

then there exists a unique solution \( u_2 \in H^2(\Omega_2) \) of problem (3.2) such that

\[
 u_2 = u_{20} + \sum_{k=1}^{N_2} \sum_{\lambda \in \Lambda_{2k}(s_2)} c_{2k\lambda} \eta_{2k} \sigma_{bi}^{k\lambda}, \tag{3.9}
\]

where \( u_{20} \in H^{s_2+2}(\Omega_2), c_{2k\lambda} \in \mathbb{C} \) depend continuously on \( f_2, h_1, h_2 \). Here we denote \( \Lambda_{2k}(s_2) = \{ \lambda \in \Lambda_{2k} \mid 1 \leq \Re \lambda < s_2 + 1 \} \).

Let us now go back to our boundary value problem (2.1)-(2.7). Far from the interface \( \Gamma \), we see that it corresponds to (3.1) or (3.2); therefore, the regularity of \( u_1 \) and \( u_2 \) is given by the previous theorems. Analogous arguments as those developed in the sequel show that \( u_1 \) and \( u_2 \) have the optimal regularity in a neighbourhood of a point of \( \Gamma \). Therefore, we only have to study the behaviour of \( u_1 \) and \( u_2 \) in a neighbourhood of the common vertices of \( \Omega_1 \) and \( \Omega_2 \).

**Theorem 3.3.** — Let \( f_1 \in L^2(\Omega_1), f_2 \in L^2(\Omega_2), h_1 \in H^{3/2}(\Gamma), h_2 \in H^{1/2}(\Gamma) \) and \( \bar{u} = (u_1, u_2) \in V \) be the solution of (2.1)-(2.7). For \( k \in \{1, 2\} \), we have:

a) if \( \omega_1k > \pi \), then \( u_1 \) admits the following decomposition in a neighbourhood \( V_k \) of \( S_k \):

\[
 u_1 = u_{10} + c_k \sigma_{\text{lap}}^{k\pi/\omega_1k} \quad \text{in} \quad V_k \cap \Omega_1, \tag{3.10}
\]

where \( u_{10} \in H^2(\Omega_1), c_k \in \mathbb{C} \);

b) if \( \omega_1k \leq \pi \), then \( u_1 \in H^2(V_k \cap \Omega_1) \).

**Proof.** — We may look \( u_1 \in H^1(\Omega_1) \) as the solution of

\[
 \begin{cases}
 \Delta u_1 = f_1 & \text{in } \Omega_1, \\
 \gamma_1 u_1 = 0 & \text{on } \Gamma_1, \\
 \gamma_1 u_1 = \gamma_2 u_2 & \text{on } \Gamma.
\end{cases} \tag{3.11}
\]

Since \( u_2 \in H^2(\Omega_2) \) and fulfils (2.4), theorem 1.6.1.5 of [4] implies that

\[
 \gamma_2 \Gamma u_2 \in \tilde{H}^{3/2}(\Gamma). \tag{3.12}
\]

Therefore, applying theorem 3.1 with \( s_1 = 1 \) to problem (3.11), we get the result except if \( \omega_1k = \pi \). In this last case, (3.12) and theorem 1.5.1.2 of [4] allow to reduce (3.11) in the neighbourhood \( V_k \cap \Omega_1 \) to a homogeneous
Dirichlet problem with interior data in $L^2$. Classical regularity results on smooth domains leads to the conclusion. □

To study the regularity of $u_2$, we now use the equations (2.2), (2.4), (2.6) and (2.7), i.e. $u_2$ is seen as a solution of

$$\begin{cases}
\Delta^2 u_2 = f_2 & \text{in } \Omega_2 \cap \mathcal{V}_k, \\
\gamma_2 u_2 = \gamma_2 \frac{\partial u_2}{\partial \nu_2} = 0 & \text{on } \Gamma_2 \cap \mathcal{V}_k, \\
M u_2 = 0 & \text{on } \Gamma \cap \mathcal{V}_k, \\
N u_2 = -\gamma_1 \Gamma \frac{\partial u_1}{\partial \nu} & \text{on } \Gamma \cap \mathcal{V}_k,
\end{cases} \tag{3.13}$$

in a neighbourhood $\mathcal{V}_k$ of $S_k$.

If $\omega_{1k} \leq \pi$, then $\gamma_1 \Gamma (\partial u_1 / \partial \nu) \in H^{1/2}(\Gamma \cap \mathcal{V}_k)$, and we may directly apply theorem 3.2 with $s_2 = 2$ to (3.13). But, if $\omega_{1k} > \pi$, only $\gamma_1 (\partial u_{10} / \partial \nu)$ has the adequate regularity $H^{1/2}$, while the normal derivative of the singular function $\sigma_{\text{lap}}^{k\pi / \omega_{1k}}$ has not. The idea is to compute explicitly the contribution of this singular function. By theorem 6.1 hereafter, there exists a solution $\tau_2^k \in H^2(\Omega_2 \cap \mathcal{V}_k)$ of

$$\begin{cases}
\Delta^2 \tau_2^k = 0 & \text{in } \Omega_2 \cap \mathcal{V}_k, \\
\gamma_2 \tau_2^k = \gamma_2 \frac{\partial \tau_2^k}{\partial \nu_2} = 0 & \text{on } \Gamma_2 \cap \mathcal{V}_k, \\
M \tau_2^k = 0 & \text{on } \Gamma \cap \mathcal{V}_k, \\
N \tau_2^k = -\gamma_1 \Gamma \frac{\partial \sigma_{\text{lap}}^{k\pi / \omega_{1k}}}{\partial \nu} & \text{on } \Gamma \cap \mathcal{V}_k.
\end{cases}$$

Therefore, the function $u_{21}$ defined by

$$u_{21} := u_2 - c_k \tau_2^k,$$

belongs to $H^2(\Omega_2 \cap \mathcal{V}_k)$ and is a solution of problem (3.2) in $\Omega_2 \cap \mathcal{V}_k$ with data $f_2 \in L^2(\Omega_2 \cap \mathcal{V}_k)$, $h_1 = 0$, $h_2 = -\gamma_1 \Gamma (\partial u_{10} / \partial \nu) \in H^{1/2}(\Gamma \cap \mathcal{V}_k)$. Therefore, applying theorem 3.2 with $s_2 = 2$ to $u_{21}$, we obtain the following theorem.

**Theorem 3.4.** — Let $\overline{u} = (u_1, u_2) \in V$ be the weak solution of (2.1)-(2.7) with data $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $h_1 \in H^{3/2}(\Gamma)$, $h_2 \in H^{1/2}(\Gamma)$. For $k = 1$ or 2, let us suppose that

$$\{\lambda \in \mathbb{C} \mid \Re \lambda = 3\} \cap \Delta_{2k} = \emptyset,$$

- 196 -
then $u_2$ admits the following decomposition in a neighbourhood of $S_k$:

$$u_2 = u_{20} + \sum_{\lambda \in \Lambda_{2k}(4)} c_{2k\lambda} \sigma_{b\lambda}^{k} + c_k \tau_2^k,$$

(3.14)

where $u_{20} \in H^4(\Omega_2)$, $c_{2k\lambda}, c_k \in \mathbb{C}$ and the last term of the right-hand side of (3.14) is zero if $\omega_{1k} \leq \pi$.

4. More regular data

In theorems 3.3 and 3.4, if we increase the regularity of the data, we expect to increase in the same way the regularity of the regular parts. One method is to use the same iterative procedure as in section 3; unfortunately, it imposes too much conditions and is complicated. Therefore, we prefer to use a compact perturbation argument.

We need to introduce the Hilbert spaces

$$A^{s_1, s_2} = \{ (u_1, u_2) \in H^{s_1 + 1}(\Omega_1) \times H^{s_2 + 2}(\Omega_2) \text{ fulfilling (2.3) to (2.5)} \},$$

$$B^{s_1, s_2} = H^{s_1 - 1}(\Omega_1) \times H^{s_2 - 2}(\Omega_2) \times H^{s_2 - 1/2}(\Gamma) \times \left( H^{s_1 - 1/2}(\Gamma) \cup H^{s_2 - 3/2}(\Gamma) \right).$$

The operator $L^{(s_1, s_2)}$ induced by the boundary value problem (2.1)-(2.7) is clearly the following:

$$L^{(s_1, s_2)} : A^{(s_1, s_2)} \rightarrow B^{(s_1, s_2)}$$

$$(u_1, u_2) \rightarrow \left( \Delta u_1, \Delta^2 u_2, Mu_2, Nu_2 + \gamma_1 \Gamma \frac{\partial u_1}{\partial \nu} \right).$$

(4.1)

We look for conditions on $s_1, s_2$ which ensure that $L^{(s_1, s_2)}$ is a Fredholm operator. To do that we split up $L^{(s_1, s_2)}$ into its principal part $L_0^{(s_1, s_2)}$ and a remainder $L_1^{(s_1, s_2)}$ as follows:

$$L_0^{(s_1, s_2)} : A^{(s_1, s_2)} \rightarrow B^{(s_1, s_2)}$$

$$(u_1, u_2) \rightarrow (\Delta u_1, \Delta^2 u_2, Mu_2, Nu_2)$$

(4.2)

$$L_1^{(s_1, s_2)} : A^{(s_1, s_2)} \rightarrow B^{(s_1, s_2)}$$

$$(u_1, u_2) \rightarrow \left( 0, 0, \gamma_1 \Gamma \frac{\partial u_1}{\partial \nu} \right).$$

(4.3)
Obviously, we have

\[ L^{(s_1,s_2)} = L^{(s_1,s_2)}_0 + L^{(s_1,s_2)}_1. \]

**Theorem 4.1.** If \( s_1 \in [s_2 - 1, s_2 + 1] \) and the Fredholm conditions (3.6) and (3.8) hold, then \( L^{(s_1,s_2)}_0 \) is a Fredholm operator and

\[ \text{ind} L^{(s_1,s_2)}_0 = -\sum_{j=1}^{N_j} \sum_{k=1}^{N_j} \text{card} \Delta_{jk}(s_j). \]  

(4.4)

**Proof.** Let \( (f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)}. \) We look for a solution \((u_1, u_2) \in A^{(s_1,s_2)}\) of

\[ L^{(s_1,s_2)}_0(u_1, u_2) = (f_1, f_2, h_1, h_2). \]

(4.5)

But this is equivalent to the fact that \( u_2 \) is a solution of (3.2) and \( u_1 \) is a solution of (3.1) with \( g = \gamma_2 f' u_2 \) on \( \Gamma \). Therefore, applying theorem 3.2 to \( u_2 \), we deduce that there exists a unique solution \( u_2 \in H^2(\Omega_2) \) of (3.1), which admits the decomposition (3.9). As in theorem 3.4, looking for \( u_1 \), we use this decomposition (3.9) of \( u_2 \). For all \( k \in \{1, 2\}, \lambda \in \Lambda_{2k} \), theorem 6.1 below gives the explicit solution \( \sigma^{k_\lambda}_{b_\lambda} \in H^1(\Omega_1) \) of (4.6) in a neighbourhood \( V_k \) of \( S_k \):

\[
\begin{cases}
\Delta \sigma^{k_\lambda}_{b_\lambda} = 0 & \text{in } \Omega_1, \\
\gamma_1 \sigma^{k_\lambda}_{b_\lambda} = 0 & \text{on } \Gamma_1, \\
\gamma_1 \sigma^{k_\lambda}_{b_\lambda} = \gamma_2 \sigma^{k_\lambda}_{b_\lambda} & \text{on } \Gamma.
\end{cases}
\]

(4.6)

Furthermore, theorem 3.1 proves the existence of a unique solution \( v_1 \in H^1(\Omega_1) \) of problem (3.1) with data \( f_1, g = \gamma_2 f' u_2 \in H^{s_2+3/2}(\Gamma) \hookrightarrow H^{s_1+1/2}(\Gamma) \), since \( s_1 \leq s_2 + 1 \), which admits the decomposition (3.7). Let us notice that in that decomposition (3.7), the coefficients \( c_{1k\lambda} \) depend continuous on \( f_1 \) and \( \gamma_2 u_2 \); and therefore continuously on \( f_1, f_2, h_1, h_2 \). Setting

\[ u_1 = v_1 + \sum_{k=1}^{2} \sum_{\lambda \in \Lambda_{2k}(s_2)} c_{2k\lambda} \eta_k \sigma^{k_\lambda}_{b_\lambda}, \]

we have proven that there exists a unique solution \( \overline{u} = (u_1, u_2) \in V \) of (4.5), which admits the decomposition

\[ \overline{u} = \overline{u}_0 + \sum_{j=1}^{N_j} \sum_{k=1}^{N_j} c_{jk\lambda} \eta_{jk} \overline{\sigma}^{jk\lambda}, \]

- 198 -
On a coupled problem between the plate equation

where \( \overline{u}_0 \in A^{(s_1,s_2)} \), \( c_{jk\lambda} \in \mathbb{C} \) depend continuously on \( f_1, f_2, h_1, h_2 \) and we have set

\[
\overline{\sigma}^{jk\lambda} = (\sigma_{i^k\lambda}^{k\lambda}, 0), \quad \forall \lambda \in \Lambda_{1k}, \ k \in \{1, \ldots, N_1\},
\]

\[
\overline{\sigma}^{2k\lambda} = (\sigma_{bi^k\lambda}^{k\lambda}, \sigma_{bi^k\lambda}^{k\lambda}), \quad \forall \lambda \in \Lambda_{2k}, \ k \in \{1, \ldots, N_2\},
\]

with the agreement that \( \sigma_{bi^k\lambda}^{k\lambda} = 0 \), if \( k \geq 3 \).

This establishes that if \( (f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)} \) is such that

\[
c_{jk\lambda} = 0, \quad \forall \lambda \in \Lambda_{jk}(s_j),
\]

then it belongs to the range of \( L_0^{(s_1,s_2)} \).

Reciprocally, if such a datum belongs to the range, then there exists a \( (u_1, u_2) \in A^{(s_1,s_2)} \) solution of (4.5); then \( u_2 \) is a solution of (3.2) and \( u_1 \) of (3.1) with \( g = \gamma_2 u_2 \). Due to theorem 3.2 and after theorem 3.1, this implies that it fulfils (4.8). So we have proven that the range of \( L_0^{(s_1,s_2)} \) is closed and that (4.4) holds since \( L_0^{(s_1,s_2)} \) is clearly injective. \( \square \)

**Theorem 4.2.** — If \( s_1 \in ]s_2 - 1, s_2 + 1 [ \) and if (3.6) and (3.8) hold, then \( L^{(s_1,s_2)} \) is a Fredholm operator and

\[
\text{ind } L^{(s_1,s_2)} = - \sum_{j=1}^{2} \sum_{k=1}^{N_j} \text{card } \Lambda_{jk}(s_j).
\]

**Proof.** — the assumption \( s_1 > s_2 - 1 \) implies that \( L_1^{(s_1,s_2)} \) is a compact operator, because \( H^{s_1-1/2}(\Gamma) \) is compactly imbedded into \( H^{s_2-3/2}(\Gamma) \). Using a classical perturbation theorem (see theorem IV.5.26 of [5], for instance), we deduce the theorem. \( \square \)

Since we want to give the asymptotic behaviour of the solution of our problem (2.1)-(2.7), we need the singularities of this problem, i.e. the singularities of \( L^{(s_1,s_2)} \). As M. Dauge in [2] for non-homogeneous operators, we compute them by recurrence starting from the singularities of the principal part \( L_0^{(s_1,s_2)} \). In view of (4.7), we see that the singularities of \( L_0^{(s_1,s_2)} \) are the \( \overline{\sigma}^{jk\lambda} \)'s. So we proceed as follows: for \( k = 1 \) or 2, we set

\[
\overline{\sigma}_0^{jk\lambda} = \overline{\sigma}^{jk\lambda}, \quad (4.10)
\]
and for $p \in \mathbb{N}$, $\sigma_p^{j\lambda}$ is a solution of (4.11) hereafter in a neighbourhood $V_k$ of $S_k$:

$$L_0 \sigma_p^{j\lambda} = -L_1 \sigma_{p-1}^{j\lambda}, \quad \text{in} \ V_k.$$  \hspace{1cm} (4.11)

Splitting up $\sigma_p^{j\lambda}$ into its components,

$$\sigma_p^{j\lambda} = (\sigma_{p,1}^{j\lambda}, \sigma_{p,2}^{j\lambda}),$$

problem (4.11) is equivalent to (4.12) and (4.13) hereafter solved in that order using theorem 6.1.

$$\begin{align*}
\begin{cases}
\Delta^2 \sigma_{p,2}^{j\lambda} = 0 & \text{in} \ \Omega_2 \cap V_k, \\
\gamma_2 \sigma_{p,2}^{j\lambda} = \gamma_2 \frac{\partial \sigma_{p,2}^{j\lambda}}{\partial \nu} = 0 & \text{on} \ \Gamma_2 \cap V_k, \\
M \sigma_{p,2}^{j\lambda} = 0 & \text{on} \ \Gamma \cap V_k, \\
N \sigma_{p,2}^{j\lambda} = -\gamma_1 \frac{\partial \sigma_{p-1,1}^{j\lambda}}{\partial \nu} & \text{on} \ \Gamma \cap V_k.
\end{cases} \quad (4.12)
\end{align*}$$

$$\begin{align*}
\begin{cases}
\Delta \sigma_{p,1}^{j\lambda} = 0 & \text{in} \ \Omega_1 \cap V_k, \\
\gamma_1 \sigma_{p,1}^{j\lambda} = 0 & \text{on} \ \Gamma_1 \cap V_k, \\
\gamma_1 \sigma_{p,1}^{j\lambda} = \gamma_2 \sigma_{p,2}^{j\lambda} & \text{on} \ \Gamma \cap V_k.
\end{cases} \quad (4.13)
\end{align*}$$

The associated singularity of $L^{(s_1,s_2)}$ is defined by (compare with § 5.C of [2])

$$\tau^{j\lambda} = \eta_k \sum_{\Re \lambda + 2p \leq s_2 + 1} \sigma_p^{j\lambda}. \hspace{1cm} (4.14)$$

Let us recall that $\tau^{j\lambda}$ is called a singularity of $L^{(s_1,s_2)}$ because it belongs to $V$ and not to $A^{(s_1,s_2)}$, while $L^{(s_1,s_2)} \tau^{j\lambda}$ belongs to $B^{(s_1,s_2)}$. Let us check this last assumption. From (4.11) and (4.14), it is clear that

$$L^{(s_1,s_2)} \tau^{j\lambda} = L_1 \sigma_{p_{\max}}^{j\lambda}, \quad \text{in} \ V_k,$$

where $p_{\max}$ is such that

$$\Re \lambda + 2p_{\max} \leq s_2 + 1 < \Re \lambda + 2p_{\max} + 2. \hspace{1cm} (4.15)$$

Therefore, $L^{(s_1,s_2)} \tau^{j\lambda} \in B^{(s_1,s_2)}$ iff

$$\eta_{jk} \cdot \gamma_1 \Gamma \frac{\partial}{\partial \nu} \sigma_{p_{\max},1}^{j\lambda} \in H^{s_2-3/2}(\Gamma). \hspace{1cm} (4.16)$$
On a coupled problem between the plate equation

In view of the form of $\sigma^{jk\lambda}$ and theorem 6.1, $\gamma_1 \Gamma (\partial / \partial y) \sigma^{jk\lambda}_{\rho_{\text{max},1}}$ behaves like $x^{3\lambda+2\rho_{\text{max},1}^{-1}}$ in a neighbourhood of $S_k$. So (4.15) leads to the adequate regularity.

Let us finally notice that the above procedure only concerns the singularities induced by $S_1$ and $S_2$ (i.e. the $\sigma^{jk\lambda}$, for $k = 1$ or 2). Indeed, for $k \geq 3$, $L^{(s_1,s_2)} = L_0^{(s_1,s_2)}$ in a neighbourhood of $Sjk$, so the singularities of $L_0^{(s_1,s_2)}$ are those of $L^{(s_1,s_2)}$, i.e.

$$\tau^{jk\lambda} = \eta_{jk} \sigma^{jk\lambda}, \quad \forall k \geq 3.$$  

We now recall lemma B.1 of [2] concerning the relationship between the index and a singularities space.

**Lemma 4.3 (M. Dauge [2]).** Let $A_1 \subset A_0$ and $B_1 \subset B_0$ be two pairs of Hilbert spaces such that $A_1$ is dense in $A_0$ and $B_1$ is dense in $B_0$. Let $M_0$ be a Fredholm operator from $A_0$ into $B_0$, which may be restricted to a semi-Fredholm operator, denoted by $M_1$, from $A_1$ into $B_1$.

We suppose that there exists a finite dimensional space $E$ having the following properties:

$$E \subset A_0,$$  

$$E \cap A_1 = \{0\},$$  

$$M_0 E \subset B_1.$$  

Then the following conditions are equivalent:

$$M_1$$ is a Fredholm operator and dim $E = \text{ind } M_0 - \text{ind } M_1$,  

$$\text{for any } u \in A_0 \text{ such that } M u \in B_1$$  

there exists $v \in A_1$ and $w \in E$ such that $u = v + w$.

We are ready to prove the theorem 4.4.

**Theorem 4.4.** Under the assumptions of theorem 4.2, given $(f_1, f_2, h_1, h_2) \in B^{(s_1,s_2)}$, there exists a unique solution $\bar{u} \in V$ of problem (2.1)-(2.7), which admits the following decomposition

$$\bar{u} = \bar{u}_0 + \sum_{j=1}^{N_1} \sum_{k=1}^{N_j} \sum_{\lambda \in \Lambda_{jk}(s_j)} c_{jk\lambda} \tau^{jk\lambda},$$  

where $\bar{u}_0 \in A^{(s_1,s_2)}$ and $c_{jk\lambda} \in \mathbb{C}$. 

- 201 -
Proof. — We apply the previous lemma with

\[
\begin{align*}
A_1 &= A^{(s_1, s_2)} \\
B_1 &= B^{(s_1, s_2)} \\
A_0 &= V \\
B_0 &= V' \\
M_0 &= \Lambda \\
M_1 &= L^{(s_1, s_2)},
\end{align*}
\]

where \( \Lambda \) is the natural isomorphism between \( V \) and \( V' \) defined by

\[
\langle \Lambda \overline{u}, \overline{v} \rangle = a(\overline{u}, \overline{v}), \quad \forall \overline{u}, \overline{v} \in V.
\]

Actually, \( B^{(s_1, s_2)} \) is identified with a subspace of \( B_0 \) by the following continuous injection: for \( \overline{F} = (f_1, f_2, h_1, h_2) \in B^{(s_1, s_2)} \), we set

\[
\langle \overline{F}, \overline{v} \rangle := -\int_{\Omega_1} f_1 v_1 \, dx + \int_{\Omega_2} f_2 v_2 \, dx + \int_{\Gamma} \left\{ h_1 \gamma_1 v_1 - h_2 \gamma_2 \frac{\partial v_2}{\partial \nu} \right\} \, d\sigma,
\]

\( \forall \overline{v} = (v_1, v_2) \in V. \)

The restriction of \( M_0 \) to \( A_1 \) is clearly \( M_1 \) because Green’s formula (2.11) implies that

\[
\Lambda \overline{u} = L^{(s_1, s_2)} \overline{u}, \quad \forall \overline{u} \in A^{(s_1, s_2)}.
\]

The space \( A^{(s_1, s_2)} \) is dense in \( V \) because we can prove that \( D(\Omega_1 \cup \Omega_2) \) is dense in \( V \). Since \( \Lambda \) is a isomorphism, we deduce that \( \Lambda A^{(s_1, s_2)} \) is dense in \( V' \). This implies the density of \( B^{(s_1, s_2)} \) since \( \Lambda A^{(s_1, s_2)} \subset B^{(s_1, s_2)} \).

Finally, the space \( E \) is the vector space spanned by the \( \tau^j k \lambda \)'s, for \( j \in \{1, 2\}, k \in \{1, \ldots, N_j\}, \lambda \in \Lambda_{jk}(s_j) \). We have previously checked that it fulfills the assumptions (4.17)-(4.19). \( \square \)

To finish this section, let us show that it is possible to hit the limit case \( s_1 = s_2 - 1 \).

**Theorem 4.5.** — Let \( s_1 = s_2 - 1 \), assume that the conditions (3.6) and (3.8) hold and moreover that the Fredholm condition (3.8) holds for \( s_2 - 1 \) too. Then the conclusion of theorem 4.4 still holds.

**Proof.** — We firstly apply theorem 4.4 with the same \( s_1 \), but with \( s_2 \) replaced by \( s_2 - 1 \). Therefore, the variational solution \( \overline{u} \) of (2.1)-(2.7) admits the decomposition (4.22) with \( s_2 - 1 \) instead of \( s_2 \). So the regular part \( \overline{u}_0 \) has the optimal regularity in \( \Omega_1 \) but not in \( \Omega_2 \). The second component of this regular part is actually solution of a boundary value problem (3.2)
with data which are the sum of an optimal regularity part and a contribution of the singularities. As in theorem 3.4, we compute explicitly the solution of this boundary value problem with a singular right-hand side using theorem 6.1. The regular right-hand side induces a decomposition into a new regular part in $H^{s_2+2}(\Omega_2)$ and singularities of the boundary value problem (3.2) for $\Re \lambda \in [1, s_2 + 1]$, due to theorem 3.2.

This allows to show that $L^{(s_1, s_2)}$ is a Fredholm operator of index given by (4.9). At this step, we follow the arguments of theorem 4.4. □

5. The non Fredholm property

The aim of this section is to show that in theorems 4.4 and 4.5, the condition $s_1 \in [s_2 - 1, s_2 + 1]$ is optimal. In other words, we shall prove that if this condition fails then the operator $L^{(s_1, s_2)}$ is never a Fredholm operator. The proof of this result is again based on a compact perturbation argument.

In the sequel, we shall need the following technical result.

**Lemma 5.1.** Let $X, Y$ be two Hilbert spaces and $A$ a linear operator from $X$ into $Y$. Suppose that there exists a finite dimensional subspace $E$ of $Y$ such that

$$R(A) \supset E^\perp.$$  

Then the range of $A$, $R(A)$, is closed and its codimension is finite.

**Proof.** Due to (5.1), $R(A)$ admits the following orthogonal decomposition

$$R(A) = (R(A) \cap E) \oplus E^\perp.$$

Since $R(A) \cap E$ is a finite dimensional linear manifold of $Y$, it is closed. Therefore, the previous decomposition implies that $R(A)$ is closed. □

In the following, we suppose that $s_1 > s_2 + 1$; the case $s_1 < s_2 - 1$ being treated analogously.

We need to introduce a variant of the operator $L^{(s_1, s_2)}$, which take into account the non-homogeneous interface condition (2.5). We set

$$\tilde{A}^{(s_1, s_2)} = \{(u_1, u_2) \in H^{s_1+1}(\Omega_1) \times H^{s_2+2}(\Omega_2) \text{ fulfilling (2.3) and (2.4)}\},$$

- 203 -
Let us notice that theorem 1.6.1.5 of [4] shows that $\widehat{L}^{s_2+3/2}(\Gamma)$ is well defined.

**LEMMA 5.2.** — Suppose that the angles at the ends of $\Gamma$ are different from $\pi$, then $L^{(s_1,s_2)}$ is a Fredholm operator iff $\widehat{L}^{(s_1,s_2)}$ is a Fredholm operator.

**Proof.** — Clearly, $L^{(s_1,s_2)}$ and $\widehat{L}^{(s_1,s_2)}$ are injective; therefore the assertion only concerns their ranges.

- Suppose that $\widehat{L}^{(s_1,s_2)}$ is a Fredholm operator. Then there exists a finite dimensional subspace $E$ of $\widehat{B}^{(s_1,s_2)}$ such that
  \[ R(\widehat{L}^{(s_1,s_2)}) = E^\perp. \]

But this implies that
\[ R(L^{(s_1,s_2)}) \supset (PE)^\perp, \]
where $P$ is the projection in $\widehat{B}^{(s_1,s_2)}$ on $B^{(s_1,s_2)}$. Since $PE$ has a finite dimension, lemma 5.1 allows us to conclude that $L^{(s_1,s_2)}$ is a Fredholm operator.

- Let us now assume that $L^{(s_1,s_2)}$ is Fredholm. As previously, there exists a finite dimensional subspace $E_1$ of $B^{(s_1,s_2)}$ such that
  \[ R(L^{(s_1,s_2)}) = E_1^\perp. \]

Furthermore, using theorem 1.6.1.5 of [4] and the assumptions of the lemma, we can prove that the operator
\[ T : \widehat{A}^{(s_1,s_2)} \rightarrow \widehat{H}^{s_2+3/2}(\Gamma) \]
\[ (u_1,u_2) \rightarrow \gamma_2u_2 - \gamma_1u_1 \]
is onto. So it admits a continuous right inverse, denoted it by $R$.

Let us now fix $(\overline{F},h) \in \widehat{B}^{(s_1,s_2)}$ satisfying
\[ (\overline{F} - L^{(s_1,s_2)}Rh, \overline{v}) = 0, \quad \forall \overline{v} \in E_1. \quad \text{(5.2)} \]
Then there exists \( \overline{u} \in A^{(s_1,s_2)} \) such that
\[
L^{(s_1,s_2)} \overline{u} = \overline{F} - L^{(s_1,s_2)} R h.
\]
This means \( (\overline{F}, h) \) belongs to the range of \( L^{(s_1,s_2)} \) because \( \overline{v} \) defined by
\[
\overline{v} = \overline{u} + R h,
\]
belongs to \( A^{(s_1,s_2)} \) and fulfils
\[
L^{(s_1,s_2)} \overline{v} = (\overline{F}, h).
\]
Again, lemma 5.1 implies that \( \widehat{L}^{(s_1,s_2)} \) is Fredholm since (5.2) is equivalent to
\[
(\overline{F}, \overline{v}) - (h, R^* L^{(s_1,s_2)}* \overline{v}) = 0, \quad \forall \overline{v} \in E_1. \square
\]

**Theorem 5.3.** — Suppose that the angles at the ends of \( \Gamma \) are different from \( \pi \), if \( s_1 > s_2 + 1 \), then \( L^{(s_1,s_2)} \) is not a Fredholm operator.

**Proof.** — We introduce the operator
\[
K : \widehat{A}^{(s_1,s_2)} \to \widehat{B}^{(s_1,s_2)}
\]
\[
(u_1, u_2) \to \left( 0, 0, \gamma_1 \Gamma \frac{\partial u_1}{\partial \nu}, \gamma_1 \Gamma u_1 \right).
\]
Setting
\[
\widehat{H}^{s_1+1/2}(\Gamma) = \{ g \in H^{s_1+1/2}(\Gamma) : g(S_k) = 0, \, k = 1, 2 \},
\]
we remark that
\[
R(K) \subset \{ 0, 0, 0 \} \times H^{s_1+1/2}(\Gamma) \times \widehat{H}^{s_1+1/2}(\Gamma).
\]
This implies that \( K \) is a compact operator because \( \widehat{H}^{s_1+1/2}(\Gamma) \) (resp. \( H^{s_1+1/2}(\Gamma) \)) is compactly imbedded into \( \widehat{H}^{s_2+3/2}(\Gamma) \) (resp. \( H^{s_2+3/2}(\Gamma) \)).

Let us suppose that \( \widehat{L}^{(s_1,s_2)} \) is Fredholm. Then
\[
\widehat{L}^{(s_1,s_2)} + K : \widehat{A}^{(s_1,s_2)} \to \widehat{B}^{(s_1,s_2)}
\]
\[
(u_1, u_2) \to \left( \Delta u_1, \Delta^2 u_2, Mu_2, Nu_2, \gamma_2 \Gamma u_2 \right) \quad (5.3)
\]
is also Fredholm. This leads to a contradiction since the kernel of $\mathcal{L}^{(s_1,s_2)} + K$ is not finite-dimensional (indeed $u_1$ does not satisfy any condition on the interface). $\Box$

Remark 5.4. — The amplitude 2 in the condition $s_1 \in [s_2 - 1, s_2 + 1]$ is exactly the difference between the order of the biharmonic operator and the Laplace operator. This means that if we consider elliptic operators of respective order $2m_1$ and $2m_2$, then the amplitude would be $|2m_1 - 2m_2|$.

6. Logarithmico-polynomial resolution

Theorem 1.3 of [6] gives the existence of a solution to the boundary value problems (4.12) and (4.13). Here, following [11], we give another proof of this result, based on the use the Jordan chains. Indeed, it was shown in [11] how to reduce each of these boundary value problems into an abstract differential equation in a Hilbert space using the change of variable $r = e^{t}$ and reducing the order. With the particular right-hand side of (4.12) or (4.13), the equivalent differential equation we get is

$$\left( \frac{\partial}{\partial t} - A \right) u(t) = e^{\lambda t} \sum_{q=0}^{Q} t^q f_q,$$  \hspace{1cm} (6.1)

where $A$ is a closed operator defined in an appropriate Hilbert space $X$, for some $\lambda \in \mathbb{C}$, $Q \in \mathbb{N} \cup \{0\}$ and $f_q \in X$, for all $q \in \{0, \ldots, Q\}$.

In order to solve (4.12) or (4.13), it is therefore equivalent to solve their corresponding problems (6.1). The case $Q = 0$ was solved in paragraph 4 of [11] in an abstract setting (it was called the polynomial resolution because for $\lambda \in \mathbb{N}$, it corresponds to the resolution of problems (4.12) or (4.13) with polynomial data). We shall extend this technique to the general case $Q \geq 0$.

Let us recall the abstract setting of [11]: $X$ is a Hilbert space, $A$ a closed operator from $X$ into $X$ such that its domain $D(A)$ is also a Hilbert space with its own topology. We assume that there exists a closed subspace $Z$ of $X$ such that $D(A)$ is dense in $Z$ and is compactly imbedded into $Z$. Finally, the resolvent set of $A$ is assumed to be nonempty.

The first idea to solve (6.1) is to look for a solution $u$ in the same form than the right-hand side, i.e.

$$u(t) = e^{\lambda t} \sum_{q=0}^{Q} t^q \varphi_q,$$  \hspace{1cm} (6.2)
where $\varphi_q \in D(A)$ are the new unknowns. In that case, problem (6.1) is equivalent to

\[
\begin{cases}
(\lambda - A)\varphi_q + (q + 1)\varphi_{q+1} = f_q, & q = 0, \ldots, Q - 1, \\
(\lambda - A)\varphi_Q = f_Q.
\end{cases}
\tag{6.3}
\]

If $\lambda$ is not an eigenvalue of $A$, then for arbitrary $f_q \in X$, (6.3) has unique solutions given by

\[
\varphi_q = \sum_{\ell=0}^{Q-q} (-1)^{\ell} \frac{(q + \ell)!}{\ell!} (\lambda - A)^{-(1+\ell)} f_{q+\ell}, \quad \forall \ q = 0, \ldots, Q.
\tag{6.4}
\]

If $\lambda$ is an eigenvalue of $A$, the previous technique fails in general. As in [11], we shall use the associated Jordan basis \( \left\{ \varphi_{\lambda,\mu,k}^{(\lambda,\mu)-1} \right\}_{\mu=1}^{M(\lambda)} \) and the dual Jordan basis \( \left\{ \psi_{\lambda,\mu,k}^{(\lambda,\mu)-1} \right\}_{\mu=1}^{M(\lambda)} \). Let us recall that they fulfil (see lemma 2.3 of [11]):

\[
(A - \lambda)\varphi_{\lambda,\mu,k} = \varphi_{\lambda,\mu,k-1},
\tag{6.5}
\]

\[
\langle (A - \lambda)u, \psi_{\lambda,\mu,k} \rangle = \langle u, \psi_{\lambda,\mu,k+1} \rangle, \quad \forall \ u \in D(A),
\tag{6.6}
\]

\[
\langle \varphi_{\lambda,\mu,k}, \psi_{\lambda,\mu,k} \rangle = \delta_{\mu,\mu'}\delta_{kk'},
\tag{6.7}
\]

for every $k = 0, \ldots, K(\lambda, \mu) - 1$, $k' = 0, \ldots, K(\lambda, \mu') - 1$, $\mu, \mu' = 1, \ldots, M(\lambda)$ and the conventions $\varphi_{\lambda,\mu,-1} = 0$ and $\psi_{\lambda,\mu,\lambda(\mu)} = 0$.

For all $q \in \mathbb{N} \cup \{0\}$, let us denote

\[
\sigma_q^{\lambda,\mu} = e^{\lambda t} \sum_{\ell=1}^{K(\lambda, \mu)} \frac{q!}{(\ell + q)!} t^{\ell+q} \varphi_{\lambda,\mu,k}^{(\lambda,\mu)-\ell}.
\tag{6.8}
\]

Using (6.5), we check that

\[
\left( \frac{\partial}{\partial t} - A \right) \sigma_q^{\lambda,\mu} = e^{\lambda t} t^q \varphi_{\lambda,\mu,k}^{(\lambda,\mu)-1}, \quad \forall \ q \in \mathbb{N} \cup \{0\}.
\tag{6.9}
\]

Now, we look for a solution $u$ of (6.1) in the form

\[
\begin{align*}
u(t) &= \sum_{q=0}^{Q} \left\{ e^{\lambda t} t^q \varphi_q + \sum_{\mu=1}^{M(\lambda)} c_{\mu q} \sigma_q^{\lambda,\mu} \right\},
\tag{6.10}
\end{align*}
\]
where \( \varphi_q \in D(A) \) and \( c_{\mu q} \) are unknown. In view of (6.9), problem (6.1) is thus equivalent to

\[
(\lambda - A)\varphi_q = f_q - \sum_{\mu = 1}^{M(\lambda)} c_{\mu q} \varphi_{\lambda, \mu, K(\lambda, \mu) - 1} - (q + 1)\varphi_{q+1}, \quad \forall q = 0, \ldots, Q, (6.11)
\]

with the convention \( \varphi_{Q+1} = 0 \). Since the range of \( A - \lambda \) is the orthogonal of \( \ker((A - \lambda)^*) = \text{Sp}(\{\psi_{\lambda, \mu, K(\lambda, \mu) - 1}^\mu_{\mu = 1}^{M(\lambda)}\}) \), this problem (6.11) has solutions \( \varphi_q, q = 0, \ldots, Q \) iff

\[
\left\langle f_q - \sum_{\mu = 1}^{M(\lambda)} c_{\mu q} \varphi_{\lambda, \mu, K(\lambda, \mu) - 1} - (q + 1)\varphi_{q+1}, \psi_{\lambda, \mu', K(\lambda, \mu') - 1} \right\rangle = 0, \quad (6.12)
\]

for all \( \mu' = 1, \ldots, M(\lambda), q = 0, \ldots, Q \).

Using the orthogonal conditions (6.7), (6.12) is equivalent to

\[
c_{\mu q} = \left\langle f_q - (q + 1)\varphi_{q+1}, \psi_{\lambda, \mu, K(\lambda, \mu) - 1} \right\rangle, \quad \forall q = 0, \ldots, Q. (6.13)
\]

This means that we solve problem (6.11) by recurrence starting with the value \( q = Q \). Indeed, for each \( q \), assuming that \( \varphi_{q+1} \) exists, then taking \( c_{\mu q} \) given by (6.13), we deduce the existence of at least one solution \( \varphi_q \) of (6.11). Since \( \varphi_Q \) exists (recall that \( \varphi_{Q+1} = 0 \)), we have proven the theorem 6.1.

**Theorem 6.1.** — For all \( \lambda \in \mathbb{C}, Q \in \mathbb{N} \cup \{0\}, \varphi_q \in X, q \in \{0, \ldots, Q\} \), there exists a solution \( u(t) \) of problem (6.1) in the form

\[
u(t) = \sum_{q=0}^{Q} \left\{ e^{\lambda t} \varphi_q + \sum_{\mu = 1}^{M(\lambda)} c_{\mu q} \psi_{\lambda, \mu} \right\}, \quad (6.14)
\]

where the sum over \( \mu \) disappears if \( \lambda \) is not an eigenvalue of \( A \); otherwise, the \( c_{\mu q} \)'s are given by (6.13) and \( \varphi_q \)'s are solutions of (6.11).
On a coupled problem between the plate equation

References


