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Extension and Selection theorems in Topological spaces with a generalized convexity structure(*)

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RÉSUMÉ. — Dans un travail précédent on a introduit une structure de convexité abstraite sur un espace topologique. Un espace topologique muni d'une telle structure est appelé un c-espace. Nous montrons ici que le théorème de prolongement de Dugundji, le théorème de sélection de Michael pour les applications multivoques s.c.i, le théorème d'approximation de Cellina pour les applications multivoques s.c.s. et le théorème de point fixe de Kakutani s'adaptent aux c-espaces. Les espaces topologiques supercompacts munis d'une prébase normale et binaire ainsi que les espaces métriques hyperconvexes sont des c-espaces.

ABSTRACT. — In previous papers we introduced a kind of topological convexity called a c-structure. In this paper we prove within this framework Dugundji's extension theorem, Michael's selection theorem for lower semicontinuous multivalued mappings, Cellina's approximate selection theorem for upper semicontinuous multivalued mappings and Kakutani's fixed point theorem. Our proofs follow very closely the original proofs. In the second part we show that the so called supercompact topological spaces with a normal binary subbase and the hyperconvex metric spaces fall within the class we consider.

0. Introduction

In [13], we gave the following definition. A c-structure on a topological space Y is an assignment for each non empty finite subset $A \subseteq Y$ of a non empty contractible subspace $F(A) \subseteq Y$ such that $F(A) \subseteq F(B)$ if $A \subseteq B$. We called (Y, F) a c-space. A non empty subset $E \subseteq Y$ is an

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F-set if $F(A) \subseteq E$ for any non empty finite subset $A \subseteq E$. $(Y, d; F)$ is a metric $l.c$-space if open balls are F-sets and if any neighbourhood $\{y \in Y \mid d(y, E) < r\}$ of any F-set $E \subseteq Y$ is also an F-set.

It is clear by looking at the proofs in [13] that it is enough to assume that the sets $F(A)$ are C^∞ (any continuous function defined on the boundary of a finite dimensional sphere with values in $F(A)$ can be extended to a continuous function on the ball with values in $F(A)$).

For the relevance of such a structure in the theory of minimax inequalities and in fixed point theory one can look at [3], [4], [7] and [8].

Dugundji introduced m-spaces. With a minor change of vocabulary we can define $m.c$-spaces. A c-space is called and $m.c$-space if for any metric space (X, d) and every continuous function $f : X \to Y$ the following is true: for each $x \in X$ and neighbourhood W of $f(x)$ there exists a neighbourhood $\mathcal{U}$ of x and an F-set $E \subseteq Y$ such that $f(\mathcal{U}) \subseteq E \subseteq W$, obviously if each point of Y has a neighbourhood base consisting of F-sets then (Y, F) is an $m.c$-space. Furthermore if Y is a metrizable topological space then by taking $X = Y$ and $f : X \to Y$ the identity function we see that (Y, F) is an $m.c$-space if and only if each point of Y has a neighbourhood base consisting of F-sets [10].

Any metric $l.c$-space is obviously an $m.c$-space. In this paper we generalize in the context of c-spaces Dugundji's extension theorem, Michael's selection theorem and Cellina's approximate selection theorem. As in Cellina's paper we obtain a version of Kakutani's fixed point theorem for compact metric $l.c$-spaces in which $F(E)$ is closed for any finite subset E.

Finally we show that connected metrizable spaces with a normal binary subbase and hyperconvex metric spaces carry a natural c-structure for which they are metric $l.c$-spaces. Hyperconvex spaces have been studied in relation to fixed point theory of nonlinear, non expensive mappings. For their basic properties we refer to the papers of Aronszajn-Panitchpakdi [1], where they were first introduced, and to the papers of Baillon and Sine [2], [15].

Spaces with a normal binary subbase where first introduced by De Groot and Aarts [11] in relation to compactification theory. They have been extensively studied by van Mill and van de Vel, among others. For the basic results concerning these spaces we refer to their papers [16], [17] and [18].

1. Main results

Theorem 1 below is instrumental in establishing these results, we therefore reproduce its proof [13].

If X is a topological space and $\mathcal{R}$ is a covering of X, for each $x \in X$ let

$$\sigma(x;\mathcal{R}) = \{\mathcal{U} \in \mathcal{R} \mid x \in \mathcal{U}\}.$$

THEOREM 1.— *Let X be a paracompact space, $\mathcal{R}$ a locally finite open covering of X, (Y, F) a c-space and $\eta : \mathcal{R} \to Y$ a function. Then there exists a continuous function $g : X \to Y$ such that for each $x \in X$,*

$$g(x) \in F\left(\{\eta(u) \mid u \in \sigma(x;\mathcal{R})\}\right).$$

Proof.— Let $\mathcal{N}(\mathcal{R})$ be the nerve of the covering, $|\mathcal{N}(\mathcal{R})|$ its geometric realization and $\kappa : X \to |\mathcal{N}(\mathcal{R})|$ the continuous function associated with a partition of unity subordinated to $\mathcal{R}$. We will denote by $|\mathcal{N}^{(k)}(\mathcal{R})|$ the geometric realization of the k-skeleton of $\mathcal{N}(\mathcal{R})$ and we identify $\mathcal{R}$ with $|\mathcal{N}^{(0)}(\mathcal{R})|$. The topology of $|\mathcal{N}^{(0)}(\mathcal{R})|$ is discrete, we can therefore identify η with a continuous function $|\mathcal{N}^{(0)}(\mathcal{R})| \to Y$. We will show that η extends to a continuous function $\widehat{\eta} : |\mathcal{N}(\mathcal{R})| \to Y$ such that for any simplex $(\mathcal{U}_0, \ldots, \mathcal{U}_n) \in \mathcal{N}(\mathcal{R})$:

$$\widehat{\eta}\left(\bigcap_{i=0}^{n} \mathrm{St}\,\mathcal{U}_i\right) \subseteq F\left(\{\eta(\mathcal{U}_i) \mid i = 0, \ldots, n\}\right),$$

where $\mathrm{St}\,\mathcal{U}_i \subseteq |\mathcal{N}(\mathcal{R})|$ denotes the star of $\mathcal{U}_i$.

$\mathcal{L} \subseteq |\mathcal{N}(\mathcal{R})|$ is called a subcomplex if it is a simplicial complex whose simplexes are also simplexes of $|\mathcal{N}(\mathcal{R})|$.

Choose a function $\eta_0 : |\mathcal{N}^{(0)}(\mathcal{R})| \to Y$ such that $\eta_0(\mathcal{U}) \in F\{\eta(\mathcal{U})\}$ for each $\mathcal{U} \in \mathcal{R}$. Consider the family of pairs $(\mathcal{L}, \eta_{\mathcal{L}})$ where $\mathcal{L}$ is a subcomplex containing $|\mathcal{N}^{(0)}(\mathcal{R})|$, $\eta_{\mathcal{L}} : \mathcal{L} \to Y$ is a continuous function whose restriction to $|\mathcal{N}^{(0)}(\mathcal{R})|$ is η_0 and

$$\eta_{\mathcal{L}}\left(\bigcap_{i=0}^{n} \mathrm{St}\,\mathcal{U}_i\right) \subseteq F\left(\{\eta(\mathcal{U}_i) \mid i = 0, \ldots, n\}\right),$$

for any simplex $(\mathcal{U}_0, \ldots, \mathcal{U}_n)$ of $\mathcal{L}$. This family is partialy ordered by the relation $(\mathcal{L}; \eta_{\mathcal{L}}) \leq (\mathcal{L}'; \eta_{\mathcal{L}'})$ if $\mathcal{L} \subseteq \mathcal{L}'$ and $\eta_{\mathcal{L}'}\restriction_{\mathcal{L}} = \eta_{\mathcal{L}}$.

Any chain $\{(\mathcal{L}_i; \eta_{\mathcal{L}_i})\}_{i\in I}$ has a maximal element $(\overline{\mathcal{L}}, \eta_{\overline{\mathcal{L}}})$ where $\overline{\mathcal{L}} = \bigcup_{i\in I} \mathcal{L}_i$ and for $p \in \overline{\mathcal{L}}$,

$$\eta_{\overline{\mathcal{L}}}(p) = \eta_{\mathcal{L}_i}(p) \quad \text{if } p \in \mathcal{L}_i\,.$$

$\eta_{\overline{\mathcal{L}}}$ is continuous since on each simplex of $\overline{\mathcal{L}}$ it coincides with one of the $\eta_{\mathcal{L}_i}$. Therefore by the Kuratowski-Zorn lemma this family has a maximal element, let us say $(\widetilde{\mathcal{L}}, \widetilde{\eta})$. We will show that $\widetilde{\mathcal{L}} = |\mathcal{N}(\mathcal{R})|$. If not let k_0 be the first integer such that $|\mathcal{N}^{(k_0)}(\mathcal{R})|$ is not contained in $\widetilde{\mathcal{L}}$. Obviously $k_0 > 0$. There is a simplex $s \subseteq |\mathcal{N}(\mathcal{R})|$ of dimension k_0 which is not contained in $\widetilde{\mathcal{L}}$, its boundary ∂s is the union of the $(k_0 - 1)$ dimensional faces of s and is therefore contained in $\widetilde{\mathcal{L}}$. Let $\mathcal{U}_0, \ldots, \mathcal{U}_{k_0}$ be the vertices of s.

For each j the set $\{\mathcal{U}_i \mid i \neq j\}$ is the set of vertices of a face of s, therefore

$$\eta_{\widetilde{\mathcal{L}}}\left(\bigcap_{i\neq j} \operatorname{St}\mathcal{U}_i\right) \subseteq F\left(\{\eta(\mathcal{U}_i) \mid i \neq j\}\right)\,.$$

Since

$$\bigcup_{j=0}^{k_0} F\left(\{\eta(\mathcal{U}_i) \mid i \neq j\}\right) \subseteq F\left(\{\eta(\mathcal{U}_i) \mid i = 0, \ldots, k_0\}\right)$$

the function

$$\eta_{\widetilde{\mathcal{L}}} : \partial s \to \bigcup_{j=0}^{k_0} F\left(\{\eta(\mathcal{U}_i) \mid i \neq j\}\right)$$

has a continuous extension

$$\widehat{\eta}_0 : s \to F\left(\{\eta(\mathcal{U}_i) \mid i = 0, \ldots, k_0\}\right)\,.$$

$\widehat{\mathcal{L}} = \mathcal{F} \cup s$ is a subcomplex of $|\mathcal{N}(\mathcal{R})|$, now define $\widehat{\eta} : \widehat{\mathcal{L}} \to Y$ by $\widehat{\eta}\restriction_{\widetilde{\mathcal{L}}} = \widetilde{\eta}$ and $\widehat{\eta}\restriction_s = \widehat{\eta}_0$.

Since

$$\widehat{\eta}_0(s) \subseteq F\left(\{\eta(\mathcal{U}_i) \mid i = 0, \ldots, k_0\}\right)$$

the pair $(\widehat{\mathcal{L}}, \widehat{\eta})$ has the required property, which contradicts the maximality of $(\widetilde{\mathcal{L}}, \widetilde{\eta})$. Now consider the function

$$X \;\xrightarrow{\kappa}\; |\mathcal{N}(\mathcal{R})| \;\xrightarrow{\widetilde{\eta}}\; Y\,,$$

we claim that it has the required property. First we have $\kappa^{-1}(\mathrm{St}\,\mathcal{U}) \subseteq \mathcal{U}$ for any $\mathcal{U} \in \mathcal{R}$ and therefore $\{\mathcal{U} \mid \kappa(x) \in \mathrm{St}\,\mathcal{U}\} \subseteq \sigma(x;\mathcal{R})$ for any $x \in X$. By construction of $\tilde{\eta}$ we have

$$\tilde{\eta}\left(\{\mathrm{St}\,\mathcal{U} \mid \kappa(x) \in \mathrm{St}\,\mathcal{U}\}\right) \subseteq F\left(\{\tilde{\eta}(\mathcal{U}) \mid \kappa(x) \in \mathrm{St}\,\mathcal{U}\}\right) .$$

This and the previous inclusion yield

$$\tilde{\eta}(\kappa(x)) \in F\left(\{\tilde{\eta}(\mathcal{U}) \mid \mathcal{U} \in \sigma(x;\mathcal{R})\}\right) .$$

For any $x \in X$, $\kappa(x) \in \bigcap\{\mathrm{St}\,\mathcal{U} \mid u \in \sigma(x;\mathcal{R})\}$ therefore

$$\tilde{\eta}(\kappa(x)) \in F\left(\{\tilde{\eta}(\mathcal{U}) \mid u \in \sigma(x;\mathcal{R})\}\right) . \square$$

We can now state the following generalization of Dugundji's extension theorem.

THEOREM 2.— *Let (X,d) be a metric space, $A \subseteq X$ a closed subspace, (Y,F) an m.c-space and $f : A \to Y$ a continuous function. Then there exists a continuous function $g : X \to Y$ which extends f and such that $g(X) \subseteq E$ for any F-set $E \subseteq Y$ containing $F(A)$.*

Proof.— We will follow Dugundji's proof in [10]. For each $x \in X$ let B_x be an open ball centered at x with radius strictly smaller that $\frac{1}{2}\,d(x,A)$. Let $\mathcal{R}$ be a locally finite open covering of $X \setminus A$ which refines $\{B_x \mid x \in X \setminus A\}$.

With each $\mathcal{U} \in \mathcal{R}$ associate $a_{\mathcal{U}} \in A$ and $x_{\mathcal{U}} \in \mathcal{U}$ such that

$$d(u_{\mathcal{U}}, a_{\mathcal{U}}) < 2d(x_{\mathcal{U}}, A) .$$

Then the following holds : for each $a \in A$ and each neighbourhood $W(a)$ of a in X there is a neighbourhood $V(a)$ of a such that $V(a) \subseteq W(a)$ and for any $\mathcal{U} \in \mathcal{R}$ if $\mathcal{U} \cap V(a) \neq \emptyset$ then $\mathcal{U} \subseteq W(a)$ and $a_{\mathcal{U}} \in A \cap W(a)$.

By the previous theorem there is a continuous function $h : (X \setminus A) \to Y$ such that

$$h(x) \in F\left(\{f(a_{\mathcal{U}}) \mid \mathcal{U} \in \sigma(x;\mathcal{R})\}\right) \quad \text{for any } x \in X \setminus A .$$

We will check as in Dugundji's proof that the function $g : X \to Y$ defined bu $g_{|A} = f$ and $g_{|(X\setminus A)} = h$ is continuous. Since h is continuous on the open set $(X \setminus A)$ we only have to check that g is continuous at each $a \in A$.

Let $W \subseteq Y$ be a neighbourhood of $f(a) = g(a)$ and $C \subseteq Y$ an F-set such that $f\big(W(a) \cap A\big) \subseteq C \subseteq W$ for some neighbourhood $W(a)$ of a.

Let $V(a)$ be a neighbourhood of the point a having the property described earlier with respect to $W(a)$.

If $x \in V(a) \cap A$ then

$$f(x) \in W \quad \text{since } V(a) \subseteq W(a).$$

If $x \in V(a) \setminus A$ then

$$g(x) = h(x) \in F\left(\{f(a_{\mathcal{U}}) \mid \mathcal{U} \in \sigma(x;\mathcal{R})\}\right).$$

If $\mathcal{U} \in \sigma(x;\mathcal{R})$ then

$$x \in \mathcal{U} \cap V(a)$$

and therefore

$$a_{\mathcal{U}} \in W(a) \cap A \quad \text{and} \quad f(a_{\mathcal{U}}) \in C$$

and since C is an F-set

$$h(x) \in F\left(\{f(a_{\mathcal{U}}) \mid \mathcal{U} \in \sigma(x;\mathcal{R})\}\right) \subseteq C$$

which shows that $h\big(V(a) \setminus A\big) \subseteq W$. With the previous inclusion we get $h\big(V(a) \cap A\big) \subseteq W$. The continuity of h is established.

If E is and F-set containing $f(A)$ then $f(a_{\mathcal{U}}) \in E$ for each $\mathcal{U} \in \mathcal{R}$ and

$$F\left(\{f(a_{\mathcal{U}}) \mid \mathcal{U} \in \sigma(x;\mathcal{R})\}\right) \subseteq E,$$

therefore $h(X \setminus A) \subseteq E$. □

Recall that a topological space Y is an absolute extensor for metric spaces, an AE(Metric), if for any metric space X, any closed subspace A of X and any continuous function $f : A \to Y$ there is a continuous function $g : X \to Y$ such that $g_{|A} = f$. By a theorem of Stone a metric space is paracompact.

COROLLARY .— *Any m.c-space or any F-set in an m.c-space (Y, F) is an AE(Metric).*

Recall that a metrizable space Y is an absolute retract, an AR, if for any metric space Z containing Y as a closed subspace there is a continuous retraction $r : Z \to Y$.

A metrizable space Y is an AR if and only if it is an AE(Metric).

COROLLARY .— *A metrizable m.c-space or any F-set in a metrizable m.c-space is an absolute retract.*

In case where Y is a topological vector space and $F(A)$ is the convex hull of the finite set $A \subseteq Y$ the next result was obtained by F. E. Browder [5].

THEOREM 3.— *Let X be a paracompact topological space, (Y, F) a c-space and $T : X \to Y$ a multivalued mapping such that:*

i) for each $x \in X$ and each finite non empty subset $A \subseteq Tx$, $F(A) \subseteq Tx$ and $Tx \neq \emptyset$;

ii) for each $y \in Y$, $T^{-1}y$ is open in X.

Then T has a continuous selection.

Proof.— $\{T^{-1}y; y \in Y\}$ is an open covering of X, Let $\mathcal{R}$ be a locally finite open covering finer than $\{T^{-1}y; y \in Y\}$ and for each $\mathcal{U} \in \mathcal{R}$ choose $\eta(\mathcal{U}) \in Y$ such that $\mathcal{U} \subseteq T^{-1}\eta(\mathcal{U})$. Let $g : X \to Y$ be the continuous function given by theorem 1.

If $x \in \mathcal{U}$ then $\eta(\mathcal{U}) \in Tx$ therefore $\{\eta(\mathcal{U}) \mid \mathcal{U} \in \sigma(x; \mathcal{R})\} \subseteq Tx$ and by i):

$$g(x) \in F\left(\{\eta(\mathcal{U}) \mid \mathcal{U} \in \sigma(x; \mathcal{R})\}\right) \subseteq Tx\,.$$

Notice that if X is compact then we can assume that $\mathcal{R}$ is finite and

$$g(X) \subseteq F\left(\{\eta(\mathcal{U}) \mid \mathcal{U} \in \mathcal{R}\}\right)\,. \square$$

From this theorem we will obtain first an approximate selection theorem for lower semicontinuous multivalued mappings and then a generalization of Michael's selection theorem. These results appear in [13].

THEOREM 4.— *Let X be a paracompact topological space $(Y, d\,; F)$ a metric l.c-space and $T : X \to Y$ a lower semicontinuous multivalued mapping whose values are non empty F-sets. Then for any $\varepsilon > 0$ there is a continuous function $g : X \to Y$ such that for each*

$$x \in X\,, \quad d\big(g(x), Tx\big) < \varepsilon\,.$$

Proof.— Let $B_\varepsilon(y) = \{y' \in Y \mid d(y,y') < \varepsilon\}$. Let

$$Rx = \{y \in Y \mid Tx \cap B_\varepsilon(y) \neq \emptyset\};$$

$y \in Rx$ if and only if $d(y, Tx) < \varepsilon$. Since $(Y, d; F)$ is an $l.c$-space Rx is a non empty F-set $R^{-1}y = \{x \in X \mid Tx \cap B_\varepsilon(y) \neq \emptyset\}$ is open since T is lower semicontinuous.

By theorem 3, R has a continuous selection. □

THEOREM 5.— *Let X be a paracompact topological space, $(Y, d; F)$ a complete metric l.c-space. Then any lower semicontinuous multivalued mapping $T : X \to Y$ whose values are non empty closed F-sets has a continuous selection.*

Proof.— For each $y \in Y$ let

$$V_n y = \left\{y' \in Y \mid d(y,y') < \frac{1}{n}\right\}.$$

By theorem 1, there is a continuous function $f_1 : X \to Y$ such that $Tx \cap V_1 f_1(x) \neq \emptyset$ for each $x \in X$. $T_2 x = Tx \cap V_1 f_1(x)$ defines a lower semicontinuous multivalued mapping whose values are non empty F-sets. If $T_n x$ has been defined we iterate the previous argument and we claim that there is a continuous function $f_n : X \to Y$ such that

$$T_n x \cap V_n f_n(x) \neq \emptyset \quad \text{for each } x \in X$$

and we let

$$T_{n+1} x = T_n x \cap V_n f_n(x).$$

Since

$$d\big(f_n(x), f_{n+1}(x)\big) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

the sequence $\big(f_n(x)\big)_{n \geq 1}$ is uniformly Cauchy, the limit $f(x)$ defines a continuous function. From $Tx \cap V_n f_n(x) \neq \emptyset$ we have

$$d\big(f_n(x), Tx\big) < \frac{1}{2^n} \quad \text{for each } x \in X,$$

and Tx being closed we must have $f(x) \in Tx$. □

COROLLARY .— *Let X be a paracompact topological space, $(Y, d\,; F)$ a complete metric l.c-space such that $F(\{y\}) = \{y\}$ for each $y \in Y$ and $T : X \to Y$ a lower semicontinuous multivalued mapping whose values are non empty closed F-sets. If $A \subseteq X$ is a closed subspace and $f : A \to Y$ is a continuous selection of $T_{|A} : A \to Y$ then there is a continuous selection $g : X \to Y$ of T such that $g_{|A} = f$.*

Proof.— Let

$$\widehat{T}x = \begin{cases} \{f(x)\} & \text{if } x \in A \\ Tx & \text{if } x \in X \setminus A\,. \end{cases}$$

$\widehat{T}x$ is a non empty closed F-set and $\widehat{T} : X \to Y$ is lower semicontinuous. Any continuous selection $g : X \to Y$ of $\widehat{T}$ is an extension of f. □

Cellina gave a simple proof of Kakutani's fixed point theorem from his approximate selection theorem for upper semicontinuous multivalued mapping [6]. A simple adaptation of his proof shows that his result holds in the present context. For a metric space (Y, d) and a subset $E \subseteq Y$ let $B(E, \varepsilon) = \{y \in Y \mid d(E, y) < \varepsilon\}$ and for E, $E' \subseteq Y$ let $d^*(E, E') = \sup\{d(y, E') : y \in E\}$.

Cellina's approximation theorem can now be generalized.

THEOREM 6.— *Let (X, d) be a compact metric space, $(Y, d\,; F)$ a metric l.c-space and $T : X \to Y$ a multivalued mapping such that:*

i) for each $x \in X$, Tx is a non empty F-set;

ii) for each $\varepsilon > 0$ and each $x \in X$ there is $\delta > 0$ such that

$$T\big(B(x, \delta)\big) \subseteq B(Tx, \varepsilon)\,.$$

Then for any $\varepsilon > 0$ there is a continuous function $f : X \to Y$ such that

$$d^*(\text{graph}\, f, T) < \varepsilon\,, \quad f(X) \subseteq F(E)\,,$$

where E is a finite subset of Y, and $f(X)$ is contained in any F-set containing $T(X)$.

Proof.— Let

$$\rho(x, \varepsilon) = \sup\left\{\delta \leq \frac{\varepsilon}{2} \;\middle|\; \exists\, x' \in B(x, \delta) : T\big(B(x, \delta)\big) \subseteq B\left(Tx', \frac{\varepsilon}{2}\right)\right\}$$

Cellina showed that $\inf_{x\in X} \rho(x,\varepsilon) > 0$. Choose α_0, $\alpha_1 \in \mathbb{R}$ such that $0 < \alpha_1 < \alpha_0 < \rho(x,\varepsilon)$ for any $x \in X$ and let $Rx = T\big(B(x,\alpha_1)\big)$. For each $y \in Y$, $R^{-1}y$ is open in X. For each $x \in X$ there is $x' \in X$ and $\delta \in]\,\alpha_1\,,\,\varepsilon/2\,[$ such that $T\big(B(x,\delta)\big) \subseteq B(Tx'\,,\,\varepsilon/2)\big)$.

By compactness of X we have

$$X = \bigcup_{i=0}^{n} R^{-1} y_i$$

for some finite set $\{y_0, \ldots, y_n\} \subseteq Y$ and from theorem 1 we can find a continuous function $f : X \to Y$ such that

$$f(x) \in F\{y_i \mid y_i \in Rx\} = F\left(\{y_i \mid y_i \in T\big(B(x,\delta)\big)\}\right) .$$

Tx' is an F-set therefore $B(Tx'\,,\,\varepsilon/2)$ is also and F-set and

$$\{y_i \mid y_i \in T\big(B(x,\delta)\big)\} \subseteq B\Big(Tx', \frac{\varepsilon}{2}\Big) ;$$

consequently

$$F\big(\{y_i \mid y_i \in Rx\}\big) \subseteq B\Big(Tx', \frac{\varepsilon}{2}\Big) ,$$

and if $E \subseteq Y$ is any F-set containing $T(x)$ it also contains

$$\{y_i \mid y_i \in T\big(B(x,\delta)\big)\}$$

and consequently

$$f(x) \in B\Big(Tx', \frac{\varepsilon}{2}\Big) \cap E .$$

It is also obvious that

$$f(x) \in F\big(\{y_0, \ldots, y_n\}\big) .$$

The inequality $d\big((x, f(x))\,,\,T\big) < \varepsilon$ is established as in Cellina's paper. □

Given a metric $l.c$-space $(Y, d\,;\,F)$, a multivalued mapping $T : Y \to Y$ is a Kakutani mapping if hypothesises i) and ii) of theorem 6 are verified and if Ty is closed for each $y \in Y$. If for each $y \in Y$, Ty is compact, ii) simply means that T is upper semicontinuous.

COROLLARY .— *Let $(Y, d\,;\,F)$ be a compact metric l.c-space such that for any non empty finite subset $A \subseteq Y$ the set $F(A)$ is closed. Then a Kakutani mapping $T : Y \to Y$ has a fixed point.*

Proof.— For any $\varepsilon > 0$ there is a finite set A and a continuous function $f : Y \to Y$ such that $f(Y) \subseteq F(A)$ and the graph of f is within ε of the graph of T.

By what has been established previously $F(A)$ is an absolute retract. $f_{|F(A)}$ has a fixed point x_ε, $(x_\varepsilon, x_\varepsilon)$ is within ε of the graph of T which is closed in $Y \times Y$. Y being compact T must have a fixed point. □

2. Applications

(I) A family of closed subsets of a topological space (Y, τ) will be called a subbase if any closed subspace is an intersection of finite unions of members of the family. A topological space is supercompact if it has a binary subbase $\mathcal{B}$: any non empty subfamily $\mathcal{F} \subseteq \mathcal{B}$ such that any two members of $\mathcal{F}$ meet has a non empty intersection. A family $\mathcal{B}$ is normal if for any pair $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 = \emptyset$ another pair $B'_1, B'_2 \in \mathcal{B}$ can be found such that $B_1 \cap B'_1 = B_2 \cap B'_2 = \emptyset$ and $Y = B'_1 \cup B'_2$. Supercompact spaces, which are obviously compact, have been extensively studied [16], [17] and [18]. A pair $(Y, \mathcal{B})$ where Y is a compact topological space for which $\mathcal{B}$ is a binary normal subbase will be called, following van Mill, normally supercompact. We can assume without loss of generality that $X \in \mathcal{B}$. A normally supercompact space $(Y, \mathcal{B})$ has a rich geometric structure. For any subset $A \subseteq Y$ let $F_{\mathcal{B}}(A) = \bigcap\{B \in \mathcal{B} \mid A \subseteq B\}$, if $A = F_{\mathcal{B}}(A)$ then it is called a $\mathcal{B}$-convex set. Notice that, by definition, $\mathcal{B}$-convex sets are closed. If $F_{\mathcal{B}}(E) \subseteq A$ for any finite subset $E \subseteq A$, A will be called a $\mathcal{B}$-set. If the topological space Y is metrizable then a metric on Y, call it d, compatible with the topology is a $\mathcal{B}$-metric if for any $\mathcal{B}$-convex subspace $C \subseteq Y$ and any $r > 0$, $\{y \in Y \mid d(y, C) \leq r\}$ is a $\mathcal{B}$-convex subspace of Y. For the proofs of the following propositions one can look in [16] and [17].

THEOREM 7.— *Let $(Y, \mathcal{B})$ be a normally supercompact space then:*

i) a subspace $C \subseteq Y$ is $\mathcal{B}$-convex if and only if

$$F_{\mathcal{B}}(\{y_1, y_2\}) \subseteq C$$

for any pair of points y_1, $y_2 \in C$;

ii) for any $\mathcal{B}$-convex subspace $C \subseteq Y$ there is a continuous retraction $p_C : Y \to C$;

iii) if the topology of Y is metrizable then it can be induced by a $\mathcal{B}$-metric d such that for any $y \in Y$ and any $\mathcal{B}$-convex subspace $C \subseteq Y$,

$$d\big(y, p_C(y)\big) = \min\big\{d(y, y') \mid y' \in C\big\}.$$

From ii) if follows that any C^∞ normally supercompact space $(Y, \mathcal{B})$ is a c-space with its natural c-structure $F_{\mathcal{B}}$. We will call a convex subspace $C \subseteq Y$ a $\mathcal{B}$-polytope if $C = F_{\mathcal{B}}(A)$ for some finite set $A \subseteq Y$. If the $\mathcal{B}$-polytopes are C^∞ then $F_{\mathcal{B}}$ defines a c-structure. From i) it follows that the families of $\mathcal{B}$-sets and $\mathcal{B}$-convex sets are identical.

PROPOSITION .— *If $(Y, \mathcal{B})$ is a metrizable normally supercompact space with C^∞ polytopes then for any $\mathcal{B}$-metric it is a complete metric l.c-space.*

Proof.— Completeness is a consequence of compactness. Let d be any $\mathcal{B}$-metric on Y, $E \subseteq Y$ a $\mathcal{B}$-set and $r > 0$. Take points $y_0, \ldots, y_m$ in the set $\big\{y \in Y \mid d(y, E) < r\big\}$. We have to show that

$$F_{\mathcal{B}}\big(\{y_0, \ldots, y_m\}\big) \subseteq \big\{y \in Y \mid d(y, E) < r\big\}.$$

We can find $y'_0, \ldots, y'_m \in E$ such that

$$d(y_i, y'_i) = r_i \leq r' = \max\{r_0, \ldots, r_m\} < r.$$

E is a $\mathcal{B}$-set therefore $F_{\mathcal{B}}\big(\{y'_0, \ldots, y'_m\}\big) \subseteq E$. Since

$$y_i \in \big\{y \in Y \mid d\left(y, F_{\mathcal{B}}\big(\{y'_0, \ldots, y'_m\}\big)\right) \leq r'\big\}$$

and $F_{\mathcal{B}}\big(\{y'_0, \ldots, y'_m\}\big)$ is $\mathcal{B}$-convex and d is a $\mathcal{B}$-metric,

$$\big\{y \in Y \mid d\left(y, F_{\mathcal{B}}\big(\{y'_0, \ldots, y'_m\}\big)\right) \leq r'\big\}$$

is $\mathcal{B}$-convex. Therefore

$$F_{\mathcal{B}}\big(\{y_0, \ldots, y_m\}\big) \subseteq \big\{y \in Y \mid d\left(y, F_{\mathcal{B}}\big(\{y'_0, \ldots, y'_m\}\big)\right) \leq r'\big\}$$

and

$$\big\{y \in Y \mid d\left(y, F_{\mathcal{B}}\big(\{y'_0, \ldots, y'_m\}\big)\right) \leq r'\big\} \subseteq \big\{y \in Y \mid d(y, E) < r\big\}.$$

And open balls are $\mathcal{B}$-sets since $F_{\mathcal{B}}\big(\{y\}\big) = \{y\}$ for each $y \in Y$. □

From the previous proposition we have the following result.

THEOREM 8.— *Let X be a paracompact topological space, $(Y, \mathcal{B})$ a metrizable normally supercompact space whose polytopes are C^∞ and let $T : X \to Y$ be a lower semicontinuous multivalued mapping whose values are non empty $\mathcal{B}$-convex sets. Then:*

i) T has a continuous selection;

ii) any continuous selection $g : F \to Y$ of the restriction of T to a closed subspace F of X can be extended to a continuous selection $f : X \to Y$.

COROLLARY .— *Let $(Y, \mathcal{B})$ be a metrizable normally supercompact space. Then the following statements are equivalent:*

i) Y is an absolute retract,

ii) Y is contractible,

iii) $\mathcal{B}$-polytopes are contractible,

iv) Y is connected,

v) $\mathcal{B}$-polytopes are connected.

Proof

i)→ii) is obvious.
ii)→iii) by ii) of theorem 7
iii)→i) by the previous theorem and obviously i)→iv).

Now let us show iv)→ii). By a theorem of Verbeek [18], Y is locally connected if it is connected and by the Hahn-Mazurkiewicz theorem it is a continuous image of $[0, 1]$. Let $\gamma : [0, 1]$ be a continuous function onto Y. Denote by $H([0, 1])$ and $H(Y)$ the hyperspaces of $[0, 1]$ and Y with the Hausdorff metric and let $H(Y; \mathcal{B})$ be the subspace of $H(Y)$ consisting of $\mathcal{B}$-convex subspaces. It is known that the function $p : H(Y; \mathcal{B}) \times Y \to Y$ which sends the pair (C, y) to $p_C(y)$ is continuous and that the function $F_{\mathcal{B}} : H(Y) \to H(Y; \mathcal{B})$ is also continuous. Now the function $\gamma : [0, 1] \to Y$ induces a continuous function

$$\gamma_* : H([0, 1]) \to H(Y) \quad \text{where} \quad \gamma_*(A) = \{\gamma(a) \mid a \in A\}.$$

Let $\alpha : [0, 1] \to H([0, 1])$ be the continuous function $\alpha(t) = [0, t]$ and consider the function $r : [0, 1] \times Y \to Y$ defined by

$$r(t, y) = p\left(F_{\mathcal{B}}((\gamma_* \circ \alpha)(t)), y\right).$$

It is a continuous function and furthermore

$$r(0,y) = p\left(\{\alpha(0)\}, y\right) = \alpha(0) \quad \text{and} \quad r(1,y) = p(Y,y) = y\,.$$

This shows that Y is contractible. This argument can be found in [18].

iv)$\to$v) is obvious.

Now we show v)$\to$ii). Let C be a $\mathcal{B}$-polytope and $\mathcal{B}\lceil_C = \{B \cap C \mid B \in \mathcal{B}\}$, $\mathcal{B}_C = \{B \in \mathcal{B} \mid C \subseteq B\}$. We check that $\left(C, \mathcal{B}\lceil_C\right)$ is a normally supercompact space. $\mathcal{B}\lceil_C$ is a closed subbase for the induced topology on C: if $F \subseteq C$ is a closed subspace of C, then F is closed in Y and $F = \bigcap_{i\in I} F_i$ where

$$F_i = B_{i\,1} \cup \cdots \cup B_{i\,r_i}\,, \quad B_{i\,j} \in \mathcal{B}$$

$$F = \bigcap_{i\in I}(F_i \cap C)\,, \quad F_i \cap C = (B_{i\,1} \cap C) \cup \cdots \cup (B_{i\,n_i} \cap C)\,.$$

$\mathcal{B}\lceil_C$ is binary: if $\mathcal{A} \subseteq \mathcal{B}\lceil_C$ and $B_1 \cap B_2 \neq \emptyset$ for any pair $B_1, B_2 \in \mathcal{A}$, then the family $\tilde{\mathcal{A}} = \{B \in \mathcal{B} \mid B \cap C \in \mathcal{A}\}$ is also binary, as well as $\tilde{\mathcal{A}} \cup \mathcal{B}_C$, therefore $\bigcap(\tilde{\mathcal{A}} \cup \mathcal{B}_C) \neq \emptyset$ since $\mathcal{B}$ is binary, and $\bigcap(\tilde{\mathcal{A}} \cup \mathcal{B}_C) \subseteq \bigcap \mathcal{A}$.

$\mathcal{B}\lceil_C$ is normal: if $(B_1 \cap C) \cap (B_2 \cap C) = \emptyset$ then $B_1 \cap B_2 \cap (\bigcap \mathcal{B}_C) = \emptyset$, if $B_1 \cap C \neq \emptyset$ and $B_2 \cap C \neq \emptyset$, then $B_1 \cap B_2 = \emptyset$ (the families $\mathcal{B}_C \cup \{B_1\}$ $\mathcal{B}_C \cup \{B_2\}$ have the binary intersection property and therefore $\mathcal{B}_C \cup \{B_1, B_2\}$ would have the binary intersection property if $B_1 \cap B_2 \neq \emptyset$ and this would imply that $B_1 \cap B_2 \cap (\bigcap \mathcal{B}_C) \neq \emptyset$). $\mathcal{B}$ is normal, we can find B_1', $B_2' \in \mathcal{B}$ such that $Y = B_1' \cup B_2'$ and $B_1 \cap B_1' = B_2 \cap B_2' = \emptyset$. Then $C = (C \cap B_1') \cup (C \cap B_2')$ and $(C \cap B_1') \cap (B_1 \cap C) = (C \cap B_2') \cap (B_2 \cap C) = \emptyset$. C is obviously metrizable, we know that iv) implies ii), C is therefore contractible. □

This last result contains a theorem of van Mill : a continuum with a normal binary subbase is an absolute retract.

COROLLARY .— *Let $(Y, \mathcal{B})$ be a connected metrizable normally supercompact space and $T : Y \to Y$ an upper semicontinuous multivalued mapping with non empty closed values such that for any $y \in Y$ and any $y' \notin Ty$ there is $B \in \mathcal{B}$ such that $Ty \subseteq B$ and $y' \notin B$. Then T has a fixed point.*

Proof.— Y is an absolute retract and $\mathcal{B}$-polytopes are retracts of Y, they are therefore closed, and they have therefore the fixed point property. For each $y \in Y$, Ty is a $\mathcal{B}$-set, $T : Y \to Y$ is therefore a Kakutani mapping and must have a fixed point. □

(II) A metric space (Y, d) is hyperconvex if for any collection

$$\{(y_i, r_i)\}_{i \in I} \subseteq Y \times \mathbb{R}_+$$

such that for any i, $j \in I$, $d(y_i, y_j) \leq r_i + r_j$, the collection of closed balls

$$B(y_i, r_i) = \{y \in Y \mid d(y, y_i) \leq r_i\}$$

has non empty intersection. A hyperconvex space is always complete. A hyperconvex space is a non expansive retract of any metric space in which it is embedded. Since any metric space can be embedded in a Banach space it follows that any hyperconvex space is contractible. If $\{B(y_i, r_i)\}_{i \in I}$ is any collection of closed balls in a hyperconvex space then $\bigcap_{i \in I} B(y_i, r_i)$ is itself a hyperconvex space. It follows that for a hyperconvex space (Y, d) the set $\bigcap_{i \in I} B(y_i, r_i)$ is contractible, or empty. For any non empty finite subset $A \subseteq Y$ let

$$F(A) = \bigcap \{B(y, r) \mid A \subseteq B(y, r)\},$$

then $A \mapsto F(A)$ defines a c-structure on (Y, d). R. Sine [15] called $F(A)$ the ball hull of A.

THEOREM 9.— *Any hyperconvex space (Y, d) is a complete metric l.c-space.*

Proof.— For any $y \in Y$, $F(\{y\}) = \{y\}$ therefore singletons are F-sets. We have to show that for any F-set E and, any $r > 0$, $\{y \in Y \mid d(y, E) < r\}$ is an F-set. Let y_0, y_1, ..., $y_n \in Y$ such that $d(y_i, E) < r$ and y_0 belongs to any closed ball containing $\{y_1, \ldots, y_n\}$. We have to show that $d(y_0, E) < r$. Take points $\widehat{y}_1$, ..., $\widehat{y}_n$ in E such that $d(y_i, \widehat{y}_i) < r$ for $i = 1, \ldots, n$. Since $F(\{\widehat{y}_1, \ldots, \widehat{y}_n\}) \subseteq E$ it is enough to show that

$$d\left(y_0, F(\{\widehat{y}_1, \ldots, \widehat{y}_n\})\right) < r.$$

By definition $F(\{\widehat{y}_1, \ldots, \widehat{y}_n\})$ is an intersection of closed balls,

$$F(\{\widehat{y}_1, \ldots, \widehat{y}_n\}) = \bigcap_{i \in I} B(u_i, r_i), \quad (u_i, r_i) \in Y \times \mathbb{R}_+;$$

choose $r' \in \,]\,0, r\,[$ such that

$$d(y_i, \widehat{y}_i) \leq r' \quad \text{for } i = 1, \ldots, n.$$

$\{\widehat{y}_1, \ldots, \widehat{y}_n\} \subseteq B(u_i, r_i)$ for each $i \in I$ therefore

$$\{\widehat{y}_1, \ldots, \widehat{y}_n\} \subseteq B(u_i, r_i + r') \quad \text{for each } i \in I$$

and

$$y_0 \in \bigcap_{i \in I} B(u_i, r_i + r') .$$

By a lemma of R. Sine [15], there is a retraction

$$R : \bigcap_{i \in I} B(u_i, r_i + r') \to \bigcap_{i \in I} B(u_i, r_i)$$

such that $d(y, R(y)) \leq r'$ for any $y \in \bigcap_{i \in I} B(u_i, r_i + r')$. Then

$$R(y_0) \in F(\{\widehat{y}_1, \ldots, \widehat{y}_n\}) \quad \text{and} \quad d(y_0, R(y_0)) \leq r' < r . \square$$

It might be worth noticing that only the following properties are needed to show that (Y, d) is a metric $l.c$-space, assuming that Y is compact:

- Y is C^∞,
- if $\bigcap_{i \in I} B(u_i, r_i)$ is a non empty intersection of closed balls, then for any $r \geq 0$ there is a continuous retraction

$$R : \bigcap_{i \in I} B(u_i, r_i + r) \to \bigcap_{i \in I} B(u_i, r_i)$$

  such that $d(y, R(y)) \leq r$ for any y.

Indeed, since Y is compact, we can choose r large enough such that

$$\bigcap_{i \in I} B(u_i, r_i + r) = Y$$

and $\bigcap_{i \in I} B(u_i, r_i)$ will be a retract of Y and therefore will be C^∞.

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