ALEXEI N. SKOROBOGATOV

On a theorem of Enriques - Swinnerton-Dyer


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1. Introduction

Let $k$ be a perfect field. The aim of this note is to give a new proof of the following statement formulated by Enriques [3] in 1897 and proved by Swinnerton-Dyer [11] in 1970.

**Theorem.** Any del Pezzo $k$-surface of degree 5 has a $k$-point.

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This statement is usually used to prove the \( k \)-rationality of such a surface. The proof of [11] is indirect, so it appears that the present proof, which is conceptually very simple, is of some interest\(^{(2)}\).

Recall that a del Pezzo surface of degree 5 is defined as a \( k \)-form of the projective plane \( \mathbb{P}^2 \) with 4 points in general position blown-up (we shall call this a split del Pezzo surface of degree 5; "general position" simply means that no three points are collinear). In fact, we prove that for any del Pezzo \( k \)-surface \( X \) of degree 5 there exists a maximal \( k \)-torus \( T \subset GL(5) \), such that \( X \) is isomorphic to the orbit space of \( T \) on the set of semistable points of the natural action of \( T \) on the Grassmannian \( G(3, 5) \). If \( k \) is infinite we have plenty of semistable \( k \)-points since semistable points form a dense open subset of \( G(m, n) \). For arbitrary \( k \) the existence of a \( k \)-point on \( X \) follows from a simple statement known as the lemma of Lang – Nishimura ([6], [9]).

The action of the group of diagonal matrices of \( GL(n) \) on \( G(m, n) \) is briefly discussed in Section 2. In Section 3 we show that for \( m = 3 \) and \( n = 5 \) the corresponding space of (semi)stable orbits is no other than \( \mathbb{P}^2 \) with four points blown-up, a fact probably well known to experts (cf. [2]). Note that the automorphism group of the split del Pezzo surface of degree 5 is precisely the Weyl group \( W(A_4) \), isomorphic to the group of permutations on five elements. We prove the main theorem in Section 4 (Theorem 4.4) by combining these geometric facts with a formal argument in Galois cohomology.

2. Torus action on Grassmannians: the split case

Let \( V = k \oplus \cdots \oplus k \), \( \dim(V) = n \), be the vector space with a fixed decomposition into the direct sum of one-dimensional subspaces. Let \( SL(n) \) be the group of linear transformations of \( V \) with determinant 1, and \( D \subset SL(n) \) be the subgroup of diagonal matrices. Consider the Grassmannian \( G(m, n) \) of \( m \)-dimensional subspaces of \( V \) with the natural (right) action of \( SL(n) \). The restriction of this action to \( D \) was studied in a series of papers by I. M. Gelfand and his colleagues (see, for example, [4]). Let us recall some of their constructions. In what follows \( A \subset B \) means that \( A \subset B \) and \( A \neq B \).

\(^{(2)}\) After this paper has been sent to a journal, the author became aware that N. I. Shepherd-Barron has recently obtained another simple proof of this theorem ("The rationality of quintic del Pezzo surfaces – A short proof." Bull. London Math. Soc. 24 (1992), pp. 249-250).
Let \( I_n := \{1, 2, \ldots, n\} \). Choose \( e_i \in V \) to be a vector whose \( i \)-th coordinate is non-zero, and all the other coordinates are zeros. For \( I \subseteq I_n \) define \( V_I \subseteq V \) as the subspace generated by \( e_i, i \in I \). Let \( f \) be a function from the subsets of \( I_n \) to non-negative integers. Define a constructible algebraic set \( U_f \subset G(m, n) \) whose points are the subspaces \( S \subset V \) such that \( \dim(S \cap V_I) = f(I) \) for all \( I \subseteq I_n \). We have a decomposition \( G(m, n) = \bigcup_f U_f \). Obviously, \( U_f \) are \( D \)-invariant. The unique dense open set \( U_0 = U_{f_0} \) parametrizes the subspaces \( S \) in general position with respect to all \( V_I \). It is given by

\[
f_0(I) = \max\{0, m + \#I - n\}.
\]

It is often simpler to work not with \( f \) but with another function defined by:

\[
r(I) = m - f(I_n \setminus I).
\]

Let \( S \subset V \) be the subspace corresponding to a point of \( G(m, n) \).

Choose a basis in \( S \), and decompose it with respect to the coordinate system \( V = k \oplus \cdots \oplus k \). Let \( M \) be the resulting matrix. One checks that for a subset \( I \subseteq I_n \) the value \( r(I) \) is the rank of the submatrix of \( M \) of size \((m \times \#I)\) consisting of the columns with numbers in \( I \) (see, e.g. [4, (1.1)]). In particular, the function \( r_0(I) = m - f_0(I_n \setminus I) = \min\{\#I, m\} \) describes the matrices whose every \( m \) columns are linearly independent.

We are interested in “the quotient” of \( G(m, n) \) by \( D \). For this reason we consider stable and semistable points of \( G(m, n) \) with respect to the ample sheaf \( \mathcal{O}(1) \) corresponding to the Plücker embedding. (This makes sense because \( SL(n) \) acts linearly on \( V \), thus \( \mathcal{O}(1) \) has an \( SL(n) \)-linearization, see [8, Chap. 4, § 4].)

**Lemma 2.1.** — The set \( G(m, n)^s \) (resp. \( G(m, n)^ss \)) of stable (resp. semistable) points of \( G(m, n) \) with respect to \( D \) and \( \mathcal{O}(1) \) is the union of \( U_f \) for \( f \) satisfying \( f(I) < (m/n) \#I \) (resp. \( f(I) \leq (m/n) \#I \)) for all \( I \subseteq I_n \).

**Proof.** — This follows from the proof of [8, Prop. 4.3]. □

If \( m \) and \( n \) are coprime then \( (m/n) \#I \) is never an integer for \( \#I < n \), and the lemma implies that \( G(m, n)^s = G(m, n)^ss \).

The condition of stability can be reformulated as follows:

\[
r(I) > (m/n) \#I \quad \text{for all nonempty} \ I \subseteq I_n.
\]  

This implies that \( M \) does not contain a zero column.
By geometric invariant theory [8, (1.10)] there exists a quasiprojective scheme $Y'$, and a morphism $\phi : G(m, n)^s \to Y$ satisfying $\phi(tx) = \phi(x)$, $t \in D$, which is the universal categorical quotient [8, Def. 0.5]. According to the remark following the proof of [8, (1.11)], $Y$ is proper over $k$. Moreover, there is an open set $Y' \subseteq Y$ such that $\phi^{-1}(Y') = G(m, n)^s$, and $\phi : G(m, n)^s \to Y'$ is the universal geometric quotient [8, Def. 0.6]. $Y'$ has the property that every fibre $\phi^{-1}(y)$, $y \in Y'$, is an orbit of $D$. Note that up to an isomorphism, $Y$ and $Y'$ do not depend on the choice of a decomposition $V = k \oplus \cdots \oplus k$, or, equivalently, on the choice of a split maximal torus $D \subseteq SL(n)$.

**Lemma 2.2.** Let $\varepsilon \in GL(n)$ be a diagonal matrix $[\varepsilon_i \delta_{ij}]$, $\varepsilon_i \in k^*$. Define the decomposition $I_n = \bigcup_{r=1}^p J_r$ such that $\varepsilon_i = \varepsilon_j$ if and only if $\{i, j\} \subseteq J_r$ for some $r$. A subspace $S \subseteq V$ is $\varepsilon$-invariant if and only if

$$S = \bigoplus_{r=1}^p (S \cap V_{J_r}).$$

Let $i : SL(n) \to PGL(n)$ be the canonical isogeny such that Ker$(i)$ is the center of $SL(n)$. Let $T := i(D)$.

**Corollary 2.3.** Let $x \in U_f \subseteq G(m, n)$. Then the stabilizer of $x$ in $T$ is trivial if and only if there does not exist a decomposition

$$I_n = \bigcup_{r=1}^p J_r, \quad p \geq 2, \quad \text{such that } \sum_{r=1}^p f(J_r) = m.$$

In particular, this is true for the points of $G(m, n)^s$.

**Proposition 2.4.** The restriction of $\phi$ to $G(m, n)^s \to Y'$ endows $G(m, n)^s$ with the structure of a $Y'$-torsor under $T$. In particular, $Y'$ is smooth.

**Proof.** If $\#I = m$ the condition $r(I) = m$ defines an invariant dense open set $Z_I \subseteq G(m, n)$ (given by the non-vanishing of the corresponding determinant, or in other words, the corresponding Plücker coordinate). These form an open covering of $G(m, n)$. Let us construct a family of invariant open subsets of $Z_I$ such that each of them is a trivial torsor under $T$. 

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In fact, we shall use the constructions of chapter 3 of the book [8]. Assume
$I = \{1, \ldots, m\}$. Define an $R$-partition of $\{1, \ldots, m\}$ as an ordered set of
subsets $S_1, \ldots, S_{n-m}$ which cover $\{1, \ldots, m\}$, and such that ([8, Def. 3.3]):

$$\#(S_i \cap (S_{i-1} \cup \cdots \cup S_1)) = 1 \quad \text{for } i = 2, \ldots, n-m.$$ 

To each $R$-covering we associate an open set $Z_R \subseteq Z_I$ defined as the
intersection of $Z_I$ with all $Z_J$'s such that

$$J = I \cup \{m + j\} \setminus \{i\}, \quad \text{where } i \in S_j.$$ 

One then checks similarly to [loc. cit.] that

$$Z_R \cong T \times A^{(m-1)(n-m-1)}.$$ 

It is not hard to verify that the union of $Z_R$'s for all possible permutations
of $I$, coincides with the subset of $G(m, n)$ consisting of points satisfying the
condition of corollary 2.3 (cf. [8, Prop. 3.3]). These two facts imply the
proposition. □

**Corollary 2.5.** — Let $m$ and $n$ be coprime. Then

$$G(m, n)^s = G(m, n)^ss,$$

and $Y = Y'$ is a smooth projective variety.

**Remark 2.6.** — Let $N$ be the normalizer of $D$ in $SL(n)$, then the Weyl
group $W = W(A_{n-1}) := N/D$ of the root system $A_{n-1}$ is the symmetric
group $\Sigma_n$ permuting the components of the decomposition $V = k \oplus \cdots \oplus k$.
It acts on $D$, and thus on $T$. Clearly $G(m, n)^s$ and $G(m, n)^ss$ are invariant
under $N$, thus $W$ acts by automorphisms on $Y$ and $Y'$.

The following trivial remark will be important in what follows. The group
$\Sigma_n$ of permutations of the components of the decomposition $V = k \oplus \cdots \oplus k$
is naturally a subgroup of $GL(n)$. This makes it possible to identify $W$ with
a subgroup of $GL(n)$. As such, it naturally acts on $G(m, n)$. This action
preserves $G(m, n)^s$ and $G(m, n)^ss$, and the corresponding morphisms to $Y$
and $Y'$ are $W$-equivariant.
3. Del Pezzo surfaces of degree 5: the split case

DEFINITION 3.1. — A split del Pezzo surface of degree 5 is defined as the blowing-up of $\mathbb{P}^2$ in points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and $(1 : 1 : 1)$.

Note that we could as well define a split del Pezzo surfaces of degree 5 as the blowing-up of four points in $\mathbb{P}^2$, no three of them collinear. Indeed, $PGL(3)$ acts transitively on such quadruples. By the universal property of blowing-up [5, II.7.15], there is a unique isomorphism of the corresponding blowings-up extending this action.

PROPOSITION 3.2. — Let $(m, n) = (3, 5)$, then $Y = Y'$ is a split del Pezzo surface of degree 5.

Proof. — The stability condition (1) implies that every two columns are not proportional. Let $I \subset I_5, \# I = 3$. The condition that the columns of $M$ with numbers in $I$ are linearly independent defines a dense open set $Z_I^5 = Z_I \cap G(3, 5)^5$. It is $D$-invariant, so its image $\phi(Z_I^5)$ is also open. Define a dense open set $Z \subset G(3, 5)^5$ as the intersection of the $Z_I^5$’s for all possible three-element subsets of $\{1, 2, 3, 4\}$. Now let $S \subset V$ be the subspace corresponding to a point of $Z$. From the way we defined $Z$ it follows that:

- every three out of the first four columns of $M$ are linearly independent;
- no two columns are proportional.

Changing the basis, and multiplying the columns of $M$ by non-zero numbers (this is the action of $D$), we can arrange that $M$ is of the following form:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & x \\ 0 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 1 & z \end{bmatrix}$$

Here $x, y, z$ are uniquely determined up to multiplication by a common non-zero constant. Conversely, taking any point

$$(x : y : z) \in \mathbb{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)\}$$

one checks immediately that the corresponding matrix $M$ satisfies the stability condition (1), and so the space generated by its rows defines a point in $G(3, 5)^5$. Thus the map which sends $S$ to $(x : y : z) \in \mathbb{P}^2$ is an
isomorphism of $\phi(Z) \subset Y$ onto $\mathbb{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)\}$. By Corollary 2.5, $Y$ is a smooth projective surface, and this isomorphism extends to a birational morphism $\sigma : Y \to \mathbb{P}^2$ (Zariski’s Main Theorem [5, V.5.2]).

Let us denote $L_I = Y \setminus \phi(Z^I)$, $I \subset I_5$, $\#I = 3$. We now prove that:

(a) $L_I \cap L_J = \emptyset$ if and only if $\#(I \cup J) = 4$;

(b) every $L_I$ is isomorphic to $\mathbb{P}^1$.

It follows from (a) and (b) that $Y \setminus \phi(Z)$ is the disjoint union of four smooth proper curves of genus 0. Thus $\sigma^{-1}$ is the blowing-up of the above four points in $\mathbb{P}^2$ (cf. [5, V.5.4]), and the proposition will be proved.

Note that the stability condition (1) has it that $r(K) = 3$ for any 4-element subset $K \subset I_5$. To prove (a) one checks that $\#(I \cup J) = 4$ and $r(I) = r(J) = r(I \cap J) = 2$ automatically imply that $r(I \cup J) = 2$, which is not possible.

In order to prove (b) we can assume by symmetry that $I = \{3, 4, 5\}$. Then $L_{\{3,4,5\}}$ is covered by the following open sets:

\[ A = L_{\{3,4,5\}} \setminus (L_{\{1,2,3\}} \cup L_{\{1,2,4\}}), \]
\[ B = L_{\{3,4,5\}} \setminus (L_{\{1,2,4\}} \cup L_{\{1,2,5\}}), \]
\[ C = L_{\{3,4,5\}} \setminus (L_{\{1,2,3\}} \cup L_{\{1,2,5\}}). \]

Choose a point in $\phi^{-1}(A)$, and a basis in the corresponding vector space $S$. Let $M$ be the matrix obtained by decomposing this basis with respect to the standard basis of $V = k \oplus \cdots \oplus k$. We have $r(\{1, 2, 3\}) = 3$, $r(\{1, 2, 4\}) = 3$. It follows from (a) that $r(\{1, 3, 4\}) = 3$, $r(\{2, 3, 4\}) = 3$. This means that every three out of the first four columns of $M$ are linearly independent. On the other hand, the last three columns are linearly dependent. Now changing the basis, and multiplying the columns of $M$ by non-zero numbers, we can arrange that $M$ is of the following form:

\[ M = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & x
\end{bmatrix} \]

Here $x \in k$ is uniquely defined, and any $x \neq 1$ would do. This proves that $A$ is isomorphic to $\mathbb{P}^1$ minus two points. We leave to the reader the routine verification that $A, B, C$ glue together to produce $\mathbb{P}^1$. This completes the proof of the proposition. □
From now on we fix the notation $Y$ for the split del Pezzo surface of degree 5. Recall that $Y$ contains precisely 10 exceptional curves of the first kind (see, e.g., [7, Chap. 4]).

**Corollary 3.3.** — (of the proof) The genus zero curves $L_I$ are exceptional curves of the first kind on $Y$. There are 10 of these, therefore every exceptional curve of the first kind on $Y$ coincides with $L_I$ for some $I \subset I_5$, $\#I = 3$.

**Proof.** — The curves $L_I$ for $I \subset \{1, 2, 3, 4\}$ can be smoothly blown down as it follows from the proof of Proposition 3.2. By symmetry, the same is true for any $L_I$. □

The following statement seems to be well known to experts (cf. [2, VII]).

**Proposition 3.4.** — The natural map

$$\nu : \text{Aut}(Y) \to \text{Aut}(\text{Pic}(Y))$$

is an isomorphism onto the group of automorphisms of $\text{Pic}(Y)$ leaving invariant the canonical class $K_Y \in \text{Pic}(Y)$ and the scalar product $(\cdot, \cdot)$ given by the intersection index. The group $\nu(\text{Aut}(Y))$ is isomorphic to the Weyl group $W = W(A_4)$, implying $\text{Aut}(Y) \cong W$.

**Proof.** — We know from Remark 2.6 that $W$ acts on $Y$. We prove that $\ker(\nu) = 1$, $\text{Im}(\nu) \cong W$. Indeed, let $\alpha \in \ker(\nu)$, then $\alpha$ fixes the classes $[L_I] \in \text{Pic}(Y)$ of exceptional curves of the first kind. Since $L_I$ is the only curve in its class of linear equivalence, $L_I$ is $\alpha$-invariant. By the proof of Proposition 3.2, the complement in $Y$ to the union of $L_I$, for $I \subset \{1, 2, 3, 4\}$, is isomorphic to $\mathbb{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)\}$. Thus $\alpha$ defines a birational automorphism of $\mathbb{P}^2$, which is in fact biregular by Zariski's Main Theorem. It follows that $\alpha$ comes from an element of $PGL(3)$ fixing the four points as above. Thus $\alpha$ must be the identity map. Next we consider $\nu(\text{Aut}(Y))$. This group fixes the canonical class $K_Y \in \text{Pic}(Y)$. On the other hand, the scalar product $(\cdot, \cdot)$ given by the intersection index, is also $\nu(\text{Aut}(Y))$-invariant. The restriction of $(\cdot, \cdot)$ to the orthogonal complement $K_Y^\perp$ is negative definite, and the elements with norm $-2$ form a root system $A_4$ [7, IV]. By [7, IV.1] the subgroup of $\text{Aut}(\text{Pic}(Y))$ leaving invariant $K_Y$ and $(\cdot, \cdot)$ is the Weyl group $W = W(A_4)$. Thus $\nu(\text{Aut}(Y)) \subseteq W$. By Remark 2.6, $\nu(\text{Aut}(Y))$ contains $\nu(W) \cong W$, implying that $\text{Aut}(Y) \cong W$. □
4. Del Pezzo surfaces of degree 5 and Galois cohomology

Let us recall some standard facts on forms and Galois cohomology [10, 1.5; 2.1; 3.1]. Let $X$ be a variety over $k$. We denote by $\overline{k}$ the algebraic closure of $k$, $\overline{X} := X \times _k \overline{k}$, and $\Gamma := \text{Gal}(\overline{k}/k)$ is the Galois group. The group $\text{Aut}(\overline{X})$ of $\overline{k}$-automorphisms of $\overline{X}$ is equipped with a continuous invariant action of $\Gamma$:

$$a \rightarrow ^s a = (1 \otimes s)a(1 \otimes s^{-1}), \quad s \in \Gamma.$$ 

In what follows this action comes from an action of a finite factor of $\Gamma$, so we shall make this assumption from now on.

If $k \subseteq K \subseteq \overline{k}$, then $\text{Aut}(X \times _k K)$ is the set of fixed elements of $\text{Aut}(\overline{X})$ with respect to the Galois group $\text{Gal}(\overline{k}/k)$. If $K/k$ is a Galois extension, a 1-cocycle $a \in Z^1(K/k, \text{Aut}(X \times _k K))$ is a continuous map

$$a : \text{Gal}(K/k) \rightarrow \text{Aut}(X \times _k K)$$

such that $a_{st} = a_s \cdot ^s a_t$. The cocycles $a$ and $a'$ are cohomologous if there exists $b \in \text{Aut}(X \times _k K)$ such that $a_s' = b^{-1} \cdot a_s \cdot b$. This is an equivalence relation, and the pointed set of orbits is $H^1(K/k, \text{Aut}(X \times _k K))$ (the neutral element comes from the zero cocycle).

A $k$-variety $Z$ is a $K/k$-form of $X$ if $Z \times _k K$ is isomorphic to $X \times _k K$. Let $E(K/k, X)$ be the pointed set of such forms considered up to an isomorphism, with the isomorphism class of $X$ as the neutral element. Let $K/k$ be a finite Galois extension. Then there is a canonical injection of pointed sets

$$\theta : E(K/k, X) \rightarrow H^1(K/k, \text{Aut}(X \times _k K)).$$

Let $Z \in E(K/k, X)$, then a 1-cocycle $a \in \theta(Z)$ can be chosen in the following way. Fix an isomorphism

$$\rho : Z \times _k K \cong X \times _k K,$$

and take $a = (a_s)$ to be the function $\text{Gal}(K/k) \rightarrow \text{Aut}(X \times _k K)$ such that the natural action of $\text{Gal}(K/k)$ on $Z \times _k K$ (via the second factor) translates as its twisted action on $X \times _k K$:

$$\rho(1 \otimes s)\rho^{-1}(x) = a_s(1 \otimes s)x, \quad s \in \text{Gal}(K/k), \ x \in X \times _k K.$$ 

The cohomology class of $a$ does not depend on $\rho.$
If \( X \) is a quasiprojective \( k \)-variety, and \( K/k \) is a finite Galois extension, then \( \theta \) is bijective \([10, \text{III.1.3}]\). In fact, the corresponding form is the quotient scheme \( (X \times_k K)/\text{Gal}(K/k) \) with respect to the twisted action of \( \text{Gal}(K/k) \).

**PROPOSITION 4.1.** — Let \( X \) be a quasiprojective \( k \)-variety. Assume that \( \text{Aut}(X) = \text{Aut}(\overline{X}) \), and that this group is finite. Let \( \text{Inn}(\text{Aut}(X)) \) be the group of inner automorphisms of \( \text{Aut}(X) \), and let

\[
\text{Hom}(\Gamma, \text{Aut}(X))/\text{Inn}(\text{Aut}(X))
\]

be the set of orbits of \( \text{Inn}(\text{Aut}(X)) \) on \( \text{Hom}(\Gamma, \text{Aut}(X)) \) with respect to the natural action. Then there is a canonical bijection of pointed sets

\[
\theta : E(k/k, X) \xrightarrow{\sim} \text{Hom}(\Gamma, \text{Aut}(X))/\text{Inn}(\text{Aut}(X)).
\]

**Proof.** — Since \( \text{Aut}(X) = \text{Aut}(\overline{X}) \), this group has a trivial action of \( \Gamma \). Thus 1-cocycles are no other than homomorphisms, and the equivalence relation of 1-cocycles is just the conjugation. A homomorphism \( \Gamma \to \text{Aut}(\overline{X}) \) has a finite image, thus the corresponding form can be recovered as a quotient scheme, and so \( \theta \) is bijective. \( \square \)

**DEFINITION 4.2.** — A del Pezzo surface of degree 5 is defined as a \( \overline{k}/k \)-form of the split del Pezzo surface of degree 5.

**COROLLARY 4.3.** — There is a natural bijection between the following pointed sets:

(i) the set of isomorphism classes of del Pezzo \( k \)-surfaces of degree 5 with the class of the split surface as the neutral element;

(ii) the pointed set \( H^1(\Gamma, W) \);

(iii) the pointed set of orbits \( \text{Hom}(\Gamma, W)/\text{Inn}(W) \) with the trivial homomorphism as the neutral element.

**Proof.** — By Proposition 3.4 we have \( \text{Aut}(Y) \cong W \), but we also have \( \text{Aut}(\overline{Y}) \cong W \) by the same result, so we are in the situation of Proposition 4.1. \( \square \)
THEOREM 4.4. — *Any del Pezzo $k$-surface of degree 5 has a $k$-point.*

*Proof.* — Let us consider a twisted version of the whole set-up of Section 2. Let us identify $W$ with the group $\Sigma_5$ of permutational matrices in $GL(5)$. Fix a homomorphism $h : \Gamma \to W \cong \Sigma_5$. Define the following action of $\Gamma$ on $V \otimes \bar{k} = \overline{k} \otimes \cdots \otimes \overline{k}$:

$$s(v) = h(s)(1 \otimes s)v, \quad s \in \Gamma, \ v \in V \otimes \bar{k}. \quad (2)$$

This obviously induces an action of $\Gamma$ on $G(3, 5) \times \bar{k}^\times$, and thus on $G(3, 5)^{\Sigma_5} \times \bar{k}^{\Sigma_5}$. By the general theory, we can consider the corresponding $\bar{k}/k$-forms $hG(3, 5)$ and $hG(3, 5)^{\Sigma_5}$.

The map $\phi : G(3, 5)^{\Sigma_5} \to Y$ gives rise to $h\phi : hG(3, 5)^{\Sigma_5} \to hY$ (recall that $W$ normalizes the torus $D$, and hence $\phi$ is $W$-equivariant). Clearly $hY$ is a form of $Y$. Since $\Sigma_5$ normalizes the diagonal torus of $GL(5)$, we get from (2) that the corresponding twisted action of $\Gamma$ on $\overline{Y}$ is given by

$$s(x) = h(s)(1 \otimes s)x, \quad s \in \Gamma, \ x \in \overline{Y}. \quad (2)$$

Thus $hY$ is a del Pezzo surface of degree 5 whose cohomology class is represented by $h \in \text{Hom}(\Gamma, W)$. It follows from Corollary 4.3 that we obtain all del Pezzo surfaces of degree 5 in this way.

Now let us go back to $hG(3, 5)$. This is a homogeneous space of $GL(5)$ twisted by a cocycle $h : \Gamma \to W$. Due to the fact that $W \cong \Sigma_5$ naturally lies in $GL(5)$, the cocycle $h$ lifts to a cocycle with coefficients in $GL(5)$. Any such is a coboundary by Hilbert's Theorem 90. It follows that $hG(3, 5)$ is isomorphic to $G(3, 5)$.

If $k$ is infinite, then $k$-points are Zariski dense on $G(3, 5)$, and so there is a $k$-point on $hG(3, 5)^{\Sigma_5}$, and hence on $hY$. Following [11] we may end the proof in the finite field case by referring to a general theorem of Weil [12] that a smooth projective rational surface defined over a finite field $k$ always has a $k$-point (see also [7, 23.1]). However, a simple general argument is available, which I owe to J.-L. Colliot-Thélène:

**Lemma (Lang [6], Nishimura [9]).** — *If $f : X \to Z$ is a rational map of integral $k$-varieties, where $Z$ is proper and $X$ has a smooth $k$-point, then $Z$ has a $k$-point.*

Applying this with $X = G(3, 5)$ and $Z = hY$ we prove the theorem. □
One can interpret $hG(3,5)^8$ as an “almost universal” torsor on $hY$: it is a torsor under the algebraic $k$-torus dual to the $T$-module $K_Y^T$. (Recall that a universal torsor is a torsor under the dual torus of the whole Picard group $\text{Pic}(Y)$, see the details in [1].) Thus it is not surprising that in our proof $k$-points are first traced on $hG(3,5)^8$: this agrees with the philosophy of the descent theory [1] that the universal torsors over a rational variety in a certain sense “untwist” its arithmetic.

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References