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The structure of the critical set in the general mountain pass principle[(*)]

GUANGCAI FANG[(1)]

RÉSUMÉ. — Nous étudions la structure de l'ensemble critique engendré par le principe général du col de Ghoussoub-Preiss. Ce faisant, nous étendons et simplifions les résultats de structure de Pucci-Serrin établis dans le contexte du principe du col classique de Ambrosetti-Rabinowitz. Le résultat principal est le suivant : si les deux points base sont non critiques, alors l'une des trois assertions suivantes sur l'ensemble critique correspondant est vraie :

1) l'ensemble des maxima locaux stricts contient un sous-ensemble fermé qui sépare les points de base ;
2) il existe un point selle de type passe-montagne ;
3) il y a au moins un couple d'ensembles disjoints non vides de points selle qui sont aussi points limites de minima locaux. Ces ensembles sont connectés par l'ensemble des minima locaux mais pas par celui des points selle.

ABSTRACT. — We study the structure of the critical set generated by the *general mountain pass principle* of Ghoussoub–Preiss. In the process, we extend and simplify the structural results of Pucci–Serrin in the context of the *classical mountain pass theorem* of Ambrosetti–Rabinowitz. The main result states that if the two "base points" are not critical, then one of the following three assertions about the corresponding critical set holds true:

(1) the set of proper local maxima contains a closed subset that separates the two base points;
(2) there is a saddle point of mountain-pass type;
(3) there is at least one pair of nonempty closed disjoint sets of saddle points that are also limiting points of local minima; these sets are connected through the set of local minima but not through the set of saddle points.

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0. Introduction

In this paper, we shall use the information obtained in a recent result of Ghoussoub–Preiss [GP] about the location of min-max critical points, to describe the structure of the critical set in the Mountain pass theorem of Ambrosetti–Rabinowitz [AR]. We will not be using Morse theory and the functionals are only supposed to be C^1. We will use systematically the methodology initiated in [GP] to reprove, simplify and extend various related results established by Hofer [H], Pucci–Serrin [PS1, 2, 3] and Ghoussoub–Preiss [GP].

To state the refined version of the Mountain pass theorem, we need the following:

DEFINITION (a). — *A closed subset H of a Banach space X is said to separate two points u and v in X, if u and v belong to two disjoint connected components of $X \setminus H$.*

We will also need the following compactness condition.

DEFINITION (b). — *A C^1-function $\varphi : X \to \mathbb{R}$ is said to verify the Palais–Smale condition around the set F at the level c (in short $(\mathrm{PS})_{F,c}$) if every sequence $(x_n)_n$ in X verifying*

$$\lim_n \operatorname{dist}(x_n, F) = 0, \quad \lim_n \varphi(x_n) = c \quad \text{and} \quad \lim_n \|\varphi'(x_n)\| = 0$$

has a convergent subsequence.

We will denote by K_c the set of critical points of φ at the level c. We let also Γ_v^u be the set of all continuous paths joining two points u and v in X, that is:

$$\Gamma_v^u = \{g \in C([0,1]; X) \mid g(0) = u \text{ and } g(1) = v\} ,$$

where $C([0,1];X)$ is the space of all X-valued continuous functions on $[0,1]$.

We shall make a systematic use of the following

1. Structure of the critical set in the separating set F

THEOREM 1 (Ghoussoub–Preiss).— *Let $\varphi : X \to \mathbb{R}$ be a C^1-function on a Banach space X. For two points u and v in X, consider the number*

$$c = \inf_{g \in \Gamma_v^u} \max_{0 \le t \le 1} \varphi\big(g(t)\big)$$

and suppose F is a closed subset of X that separates u and v and such that

$$\inf_{x \in F} \varphi(x) \ge c\,.$$

If φ verifies $\big(\mathrm{PS}\big)_{F,c}$, then $F \cap K_c$ is non empty.

Remark (a).— In the case where $F = \{\varphi \ge c\}$, that is when $\max\{\varphi(u), \varphi(v)\} < c$, the above theorem reduces to the well known mountain pass theorem of Ambrosetti–Rabinowitz.

To classify the various types of critical points, we use the following notations:

$$G_c = \{x \in X \mid \varphi(x) < c\} \quad \text{and} \quad L_c = \{x \in X \mid \varphi(x) \ge c\}$$

$$M_c = \{x \in K_c \mid x \text{ is a local minimum of } \varphi\}$$

$$P_c = \{x \in K_c \mid x \text{ is a proper local maximum of } \varphi\}$$

$$S_c = \{x \in K_c \mid x \text{ is a saddle point of } \varphi\}\,.$$

(Recall that x is said to be a saddle point if in each neighborhood of x there exist two points y and z such that $\varphi(y) < \varphi(x) < \varphi(z)$.)

THEOREM 2.— *Under the hypothesis of Theorem 1, assume $F \cap P_c$ contains no compact set that separates u and v, then either $F \cap M_c \neq \emptyset$ or $F \cap S_c \neq \emptyset$.*

Proof.— Suppose that $F \cap M_c = \emptyset = F \cap S_c$. Since F separates u and v we can use a result of Whyburn [Ku, chap. VIII, § 57; III, theorem 1] to find a closed connected subset $\widehat{F} \subset F$ that also separates u and v. Note that $\widehat{F} \cap K_c = \widehat{F} \cap P_c$ and the latter is relatively open in $\widehat{F}$ while $\widehat{F} \cap K_c$ is closed. Since $\widehat{F}$ is connected, then either $\widehat{F} \cap P_c = \emptyset$ or $\widehat{F} \cap P_c = \widehat{F}$. But the first case is impossible since by Theorem 1 we have $\widehat{F} \cap P_c = \widehat{F} \cap K_c \neq \emptyset$. Hence $\widehat{F} \subset P_c$ which is impossible by assumption and the claim is proved. □

COROLLARY 3.— *Under the hypothesis of Theorem 1, assume φ verifies $(\mathrm{PS})_{(F \cup K_c)_\epsilon, c}$ and that $u, v \notin \overline{M_c}$. If P_c does not contain a compact subset that separates u and v, then $S_c \neq \emptyset$.*

In particular, if $\max\{\varphi(u), \varphi(v)\} < c$, φ verifies $(\mathrm{PS})_c$ and X is infinite dimensional, then $S_c \neq \emptyset$.

Proof.— First observe that K_c is the disjoint union of S_c, M_c and P_c. By the $(\mathrm{PS})_{(F \cup K_c)_\epsilon, c}$ condition, we know that K_c is compact. Suppose $S_c = \emptyset$. For each $x \in M_c$, there exists a $B(x, \epsilon_x)$ such that $B(x, \epsilon_x) \subseteq L_c$. Let

$$N = \bigcup_{x \in M_c} B(x, \epsilon_x).$$

Then $M_c \subseteq N \subseteq L_c$. Since u, $v \notin \overline{M_c}$ and $\overline{M_c}$ is compact, we may assume that u, $v \notin \overline{N}$. Now put $F_0 = (F \backslash N) \cup \partial N$. It is clear that $\inf_{x \in F_0} \varphi(x) \geq c$ and that F_0 separates u, v. Moreover, $F_0 \cap (M_c \cup S_c) = \emptyset$. By Theorem 2, $P_c \cap F_0$ and hence P_c must contain a compact subset that separates u, v. A contradiction that completes the proof. □

Remark (b).— Theorem 2 and Corollary 3 improve earlier results by Pucci and Serrin [PS1] and they are due to Ghoussoub–Preiss [GP]. The next Theorem 4 appears in [GP] and improve the results of Hofer [H] and Pucci–Serrin [PS2] that appear in Corollary 5.

DEFINITION (c) (Hofer).— *Say that a point x in K_c is of mountain-pass type if for any neighborhood N of x, the set $\{x \in N \mid \varphi(x) < c\}$ is nonempty and not path connected. We denote by H_c the set of critical points of mountain pass type at the level c.*

THEOREM 4.— *With the hypothesis of Theorem 1, we have:*

(1) Either $F \cap \overline{M_c} \neq \emptyset$ or $F \cap H_c \neq \emptyset$.

(2) If $F \cap P_c$ contains no compact set that separates u and v, then either $F \cap \overline{M_c} \neq \emptyset$ or $F \cap K_c$ contains a saddle point of mountain pass type.

Here is an immediate corollary:

COROLLARY 5.— *With the hypothesis of Theorem 1, assume*

$$\max\{\varphi(u), \varphi(v)\} < c,$$

then:

(1) either $\overline{M_c} \setminus M_c \neq \emptyset$ *or* $H_c \neq \emptyset$*;*

(2) if X *is infinite dimensional, then either* $\overline{M_c} \setminus M_c \neq \emptyset$ *or* K_c *contains a saddle point of mountain pass type.*

Proof.— It is enough to apply Theorem 4 to the set $F = \partial\{\varphi \geq c\}$ and to notice that no local minimum can be on such a set.

To prove Theorem 4, we shall need the following topological lemma. Throughout this paper, we denote by $B(x, \epsilon)$ the open ball in the Banach space X which is centered at x and with radius $\epsilon > 0$.

LEMMA 6.— *Let* F_0 *be a closed subset of* X *that separates two distinct points* u *and* v*. Let* Z_i $(i = 1, 2, \dots, n)$ *be* n *mutually disjoint open subsets of* X *such that* $u, v \notin \bigcup_{i=1}^n \overline{Z_i}$*. Let* G *be an open subset of* $X \setminus F_0$ *and denote by* $Y_i = Z_i \setminus G$*. Then the following holds:*

(i) the set $F_1 = \left[F_0 \setminus (\bigcup_{i=1}^n Z_i)\right] \cup (\bigcup_{i=1}^n \partial Y_i)$ *separates* u *and* v*.*

(ii) if A_i $(i = 1, 2, \dots, n)$ *are* n *nonempty connected components of* G *and for each* i $(1 \leq i \leq n)$ $T_i \subseteq (Z_i \cap \partial A_i)$ *is a relatively open subset of* ∂Y_i *such that* $T_i \cap \partial L = \emptyset$ *for any connected component* L *of* G *with* $L \neq A_i$*, then the set* $F_2 = \left[F_0 \setminus (\bigcup_{i=1}^n Z_i)\right] \cup (\bigcup_{i=1}^n \partial Y_i \setminus T_i)$ *also separates* u *and* v*.*

Proof

(i) Since $G \subseteq X \setminus F_0$, we have

$$F_1 = \left[F_0 \setminus \left(\bigcup_{i=1}^n Y_i\right)\right] \cup \left(\bigcup_{i=1}^n \partial Y_i\right). \tag{1}$$

Clearly F_1 is closed and $u, v \notin F_1$. We need only to show that for any $g \in \Gamma_v^u$,

$$g([0,1]) \cap F_1 \neq \emptyset.$$

If $g([0,1]) \cap (F_0 \setminus \bigcup_{i=1}^n Y_i) \neq \emptyset$, we are done. Otherwise

$$g([0,1]) \cap \left(\bigcup_{i=1}^n Y_i \right) \cap F_0 \neq \emptyset$$

so that if $g([0,1]) \cap (\bigcup_{i=1}^n \partial Y_i) = \emptyset$, then $g([0,1]) \subseteq \bigcup_{i=1}^n Y_i \subseteq \bigcup_{i=1}^n Z_i$ which contradicts that $u, v \notin \bigcup_{i=1}^n \overline{Z_i}$.

(ii) We first prove the following claims: for $i, j = 1, 2, \dots, n$, we have:

(a) $T_i \subseteq Y_i \cap \partial Y_i$, $T_i \cap G = \emptyset$ and $A_i \cap F_2 = \emptyset$;

(b) $T_j \cap \overline{Y_i} = \emptyset$ and $T_i \cap T_j = \emptyset$ if $i \neq j$;

(c) $Z_i \cap (\partial G \setminus T_i) \subseteq \partial Y_i \setminus T_i$.

(a) Since G is open, it is clear from the definition of T_i that $T_i \subseteq Z_i \cap \partial G$ so that $T_i \subseteq Y_i \cap \partial Y_i$ and $T_i \cap G = \emptyset$ for $i = 1, 2, \dots, n$. On the other hand, $A_i \cap \overline{Y_j} \subseteq A_i \cap \overline{(Z_j \setminus G)} \subseteq A_i \cap (\overline{Z_j} \setminus G) = \emptyset$, hence $A_i \cap F_2 = \emptyset$.

(b) If $i, j = 1, 2, \dots, n$ and $i \neq j$, then $T_j \cap \overline{Y_i} \subseteq T_j \cap \overline{Z_i} \subseteq Z_j \cap \overline{Z_i} = \emptyset$ and $T_i \cap T_j \subseteq Z_i \cap Z_j = \emptyset$.

(c) Since G is open, we have that for any $x \in Z_i \cap \partial G \setminus T_i$, $x \notin G$, hence $x \in Z_i \setminus G$ and $x \in Y_i$. Moreover, for any $x \in \partial G \setminus T_i$ and any $\epsilon > 0$ there is $y \in B(x, \epsilon) \cap G$. Clearly $y \notin Y_i$ so that $x \in \partial Y_i$. Since $T_i \cap Z_i \cap (\partial G \setminus T_i) = \emptyset$, we have that $x \in \partial Y_i \setminus T_i$.

Back to the proof of the Lemma, we note first that the set F_2 is closed and is equal to

$$F_2 = \left[F_0 \setminus \left(\bigcup_{i=1}^n Y_i \right) \right] \cup \left(\bigcup_{i=1}^n \partial Y_i \setminus T_i \right). \tag{2}$$

Clearly $u, v \notin F_2$ and we need only to show that for any $g \in \Gamma_v^u$, $g([0,1]) \cap F_2 \neq \emptyset$.

Suppose not, and take $g_0 \in \Gamma_v^u$ such that $g_0([0,1]) \cap F_2 = \emptyset$. We shall work toward a contradiction.

First by (1), we have

$$g_0([0,1]) \cap \left(\bigcup_{i=1}^n T_i \right) \neq \emptyset.$$

Let i_1 be the first $i \in \{1, \ldots, n\}$ such that $g_0([0,1]) \cap T_i \neq \emptyset$. We shall find a $g_{i_1} \in \Gamma_v^u$ such that

$$g_{i_1}([0,1]) \cap F_2 = \emptyset, \quad g_{i_1}([0,1]) \cap T_i = \emptyset \text{ for } 1 \leq i \leq i_1 . \tag{3}$$

To do this, we define the following times:

$$s_1 = \inf\{t \in [0,1] \mid g_0(t) \in Z_{i_1}\}, \quad s_2 = \inf\{t \in [0,1] \mid g_0(t) \in Y_{i_1}\}, \tag{4}$$

$$t_1 = \sup\{t \in [0,1] \mid g_0(t) \in Y_{i_1}\}, \quad t_2 = \sup\{t \in [0,1] \mid g_0(t) \in Z_{i_1}\}. \tag{5}$$

We shall show the following:

(d) $0 < s_1 < s_2 < t_1 < t_2 < 1$;

(e) $g_0(t_1)$ and $g_0(s_2)$ belong to T_{i_1};

(f) $g_0(t) \in A_{i_1}$ for $t \in (s_1, s_2) \cup (t_1, t_2)$.

Indeed, it is clear that $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2$. Since u, $v \notin \bigcup_{i=1}^n \overline{Z_i}$, we have $0 < s_1$ and $t_2 < 1$. On the other hand, $g_0(t_2) \notin Z_{i_1}$ since the latter is open, while $g_0(t_1) \in \partial Y_{i_1} \cap T_{i_1}$ since $g_0([0,1]) \cap F_2 = \emptyset$, hence (a) yields that $g_0(t_1) \in \partial Y_{i_1} \cap T_{i_1} = T_{i_1} \subset Z_{i_1}$. Modulo a similar reasoning for s_1, s_2, (d) and (e) are therefore verified.

To prove (f), we note first that $g_0(t) \in G$ for $t \in (s_1, s_2) \cup (t_1, t_2)$, since otherwise $g_0(t) \in Y_{i_1}$ which contradicts (4) and (5). So, for any $t \in (t_1, t_2)$, $g_0(t) \in U$ for some connected component U of G. If $U \neq A_{i_1}$, we have that $T_{i_1} \cap \partial U = \emptyset$ and since $g_0(t_1) \in T_{i_1}$, we see that $g_0(t_1) \notin \partial U$. Hence there must be $t_3 \in (t_1, t)$ such that $g_0(t_3) \in \partial U \subseteq \partial G \setminus T_{i_1}$. By (c) we see that $g_0(t_3) \in F_2$ which is a contradiction. So $U = A_{i_1}$ and consequently, $g_0(t) \in A_{i_1}$ for all $t \in (t_1, t_2)$, and (f) is proved.

Since now A_{i_1} is path connected, then for $s_1 < s^{i_1} < s_2$, $t_1 < t^{i_1} < t_2$, we can use a path in A_{i_1} to join $g_0(s^{i_1})$ and $g_0(t^{i_1})$. In this way, we get a path $g_{i_1} \in \Gamma_v^u$ such that $g_{i_1}([0,1]) \cap T_{i_1} = \emptyset$ and $g_{i_1}([0,1]) \cap T_i = \emptyset$ for $1 \leq i \leq i_1$, since by (a), $A_{i_1} \cap T_i = \emptyset$ for all $i = 1, 2, \ldots, n$. On the other hand, since $A_{i_1} \cap F_2 = \emptyset$, we get that $g_{i_1}([0,1]) \cap F_2 = \emptyset$ and (3) is established.

Next, let i_2 be the first $i \in \{1, \ldots, n\}$ such that $g_{i_1}([0,1]) \cap T_i \neq \emptyset$. Clearly $i_1 < i_2 \leq n$. In the same way, we can construct $g_{i_2} \in \Gamma_v^u$ such that for $1 \leq i \leq i_2$,

$$g_{i_2}([0,1]) \cap F_2 = \emptyset \quad \text{and} \quad g_{i_2}([0,1]) \cap T_i = \emptyset .$$

By iterating a finite number of times, we will get a $g_n \in \Gamma_v^u$ such that for $1 \leq i \leq n$,

$$g_n\big([0,1]\big) \cap F_2 = \emptyset \quad \text{and} \quad g_n\big([0,1]\big) \cap T_i = \emptyset \,.$$

But this contradicts assertion (i) and the lemma is proved. □

Proof of Theorem 4

(1) Suppose $F \cap K_c$ contains no critical points of mountain-pass type and $F \cap \overline{M_c} = \emptyset$. We claim that:

$$\begin{aligned}&\text{there exist finitely many components of } G_c,\\ &\qquad\qquad\qquad\text{say } C_1, \dots, C_p \text{ and } \varepsilon_1 > 0 \qquad\qquad (*)\\ &\text{such that } G_c \cap \big\{x \mid d(x, F\cap K_c) < \varepsilon_1\big\} \subset C_1 \cup C_2 \cdots \cup C_p .\end{aligned}$$

Indeed, otherwise we could find a sequence x_i in $F \cap K_c$ and a sequence $\big(C_i\big)_i$ of different components of G_c such that $(x_i, C_i) \to 0$. But then any limit point of the sequence x_i would be a critical point for φ of mountain-pass type belonging to F, thus contradicting our initial assumption. Hence $(*)$ is verified.

Let now $M_i = F \cap K_c \cap \overline{C_i}$. Since any point of

$$M_i \cap \left(\bigcup_{j \neq i} \overline{C_j}\right)$$

would be a critical point of Mountain-pass type, we may find for each $i = 1, \dots, p$ an open set N_i such that:

$$M_i \subset N_i \,, \quad \overline{N_i} \cap \overline{N_j} = \emptyset \text{ for } i \neq j$$

Since $F \cap \overline{M_c} = \emptyset$, we may also assume that

$$\left(\bigcup_{i=1}^{p} \overline{N_i}\right) \cap \overline{M_c} = \emptyset \quad \text{and} \quad u,\, v \notin \bigcup_{i=1}^{p} \overline{N_i} \,.$$

Observe that for each i $(1 \leq i \leq n)$, for any $x \in M_i$ there must be $B(x, \epsilon_x)$ such that $B(x, \epsilon_x) \cap U = \emptyset$ for any component U of G_c with $U \neq G_c$. Put

$$T_i = \bigcup_{x \in M_i} B\left(x, \frac{\epsilon_x}{2}\right) \cap \partial C_i \cap N^i \,, \quad Y_i = N_i \backslash G_c \,.$$

Then $T_i \subseteq N^i \cap \partial C_i$ and is relatively open in ∂Y_i. Now

$$\widetilde{F} = \left[F \setminus \left(\bigcup_{i=1}^{p} \overline{N_i} \right) \right] \cup \left(\bigcup_{i=1}^{p} (\partial Y_i \setminus T_i) \right) .$$

By Lemma 6, $\widetilde{F}$ separates u and v, hence $\widetilde{F} \cap K_c \neq \emptyset$ which is clearly a contradiction since $M_i \subset T_i$ and $F \cap \overline{M_c} = \emptyset$.

(2) Since F separates u and v, we can again use the result of Whyburn mentioned above to get a closed connected subset $\widehat{F}$ also separating u and v and such that $\widehat{F} = \partial U = \partial V$ where U and V are two components of $X \setminus \widehat{F}$ containing u and v respectively. Assume $F \cap \overline{M_c} = \emptyset$. The set $K = \widetilde{F} \cap P_c$ is an open subset relative to $\widetilde{F}$. If K is not closed, then any $x \in \overline{K} \setminus K$ is a saddle point since $\widehat{F} \cap M_c = \emptyset$. Moreover, if H is any open neighborhood of x not intersecting M_c and such that $\varphi \leq c$ on H, then both sets $U \cap H$ and $V \cap H$ meet the set $\{\varphi < c\}$. This shows that x is a saddle point of mountain pass type.

Assume now K is closed. Then it is a clopen set in the connected space $\widetilde{F}$. Hence either $K = \widetilde{F}$ or $K = \emptyset$. In the first case $\widetilde{F}$ is then contained in P_c and since it separates u and v, we get a contradiction. In the second case, note that by part (1) $\widetilde{F} \cap K_c$ contains a point of mountain pass type. Such a point is necessarily a saddle point since $\widetilde{F} \cap M_c = \emptyset$ and $\widetilde{F} \cap P_c$. This clearly finishes the proof of Theorem 4. □

For the sequel, we shall need the following concept.

2. Structure of the critical set at the level c

DEFINITION (d). — *For A, B two disjoint subsets of X and any nonempty subset C of X, we say that A, B are connected through C if there is no $F \subseteq C \cup A \cup B$ both relatively closed and open such that $A \subseteq F$ and $F \cap B = \emptyset$.*

When A and B are connected through C, we also say that C connects A and B or that the space $C \cup A \cup B$ is connected between A and B. We refer to Kuratowski [Ku, pp.142-148] for details.

The following theorem is an improvement on the results of Pucci and Serrin [PS3] that are stated in Corollary 8 below.

THEOREM 7.— *With the hypothesis of Theorem 1, we further assume* $u, v \notin K_c$ *and that* φ *verifies* $(\mathbf{PS})_c$. *Then one of the following three assertions concerning the set* K_c *must be true:*

(i) P_c *contains a compact subset that separates* u *and* v*;*

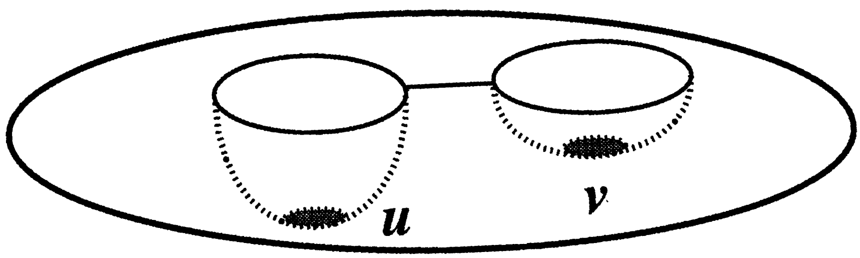

Fig. 1

(ii) K_c *contains a saddle point of mountain-pass type;*

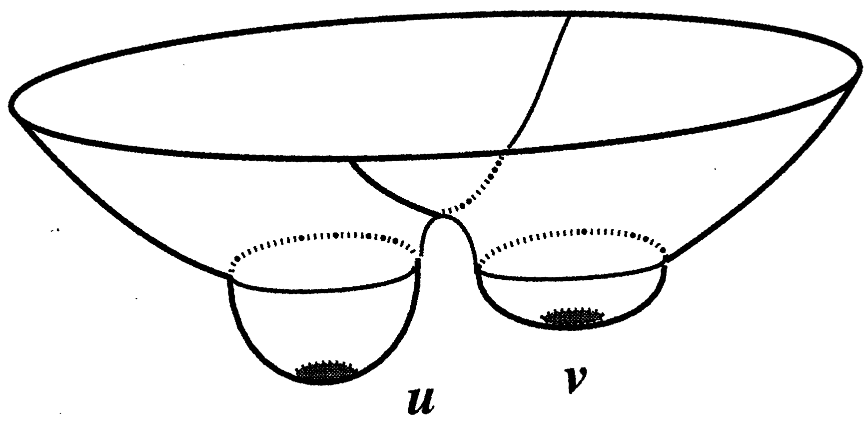

Fig. 2

(iii) there are finitely many components of G_c, *say* C_i $(i = 1, 2, \ldots, n)$ *such that* $S_c = \bigcup_{i=1}^{n} S_c^i$ *and* $S_c^i \cap S_c^j = \emptyset$ *for* $(i \neq j,\ 1 \leq i,\ j \leq n)$ *where* $S_c^i = S_c \cap \overline{C_i}$*; moreover there are at least two of them* $S_c^{i_1}$, $S_c^{i_2}$ $(i_1 \neq i_2,\ 1 \leq i_1,\ i_2 \leq n)$ *such that the sets* $\overline{M_c} \cap S_c^{i_1}$, $\overline{M_c} \cap S_c^{i_2}$ *are nonempty and connected through* M_c.

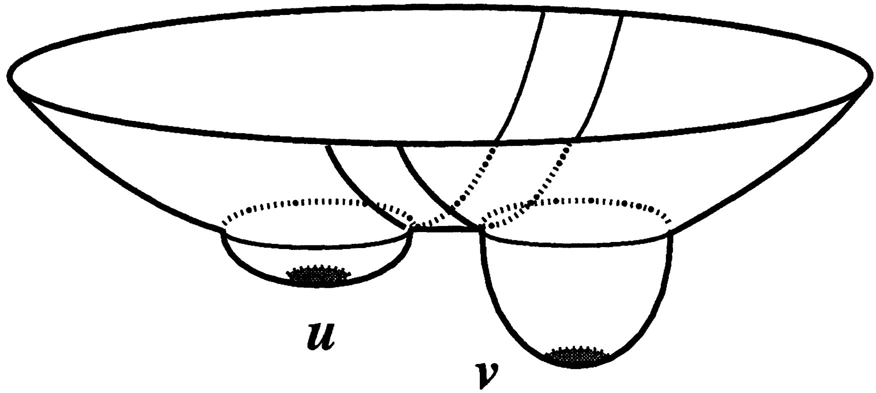

Fig. 3

Corollary 8.— *With the hypothesis of Theorem 1, assume further that* φ *verifies* $(\mathbf{PS})_c$ *and that* $\max\{\varphi(u),\varphi(v)\} < c$. *If* K_c *does not separate* $\mathbf{0}$ *and* e, *then at least one of the following two cases occurs:*

(α) K_c *contains a saddle point of mountain-pass type;*

(β) $\overline{M_c}$ *intersects at least two components of* S_c.

Moreover, if $\overline{M_c}$ *has only a finite number of components, then either*

(α′) K_c *contains a saddle point of mountain-pass type, or*

(β′) at least one component of $\overline{M_c}$ *intersects two or more components of* S_c.

To prove Theorem 7, we shall need the following two topological lemmas. The first is staightforward.

Lemma 9.— *Let* M *be a subset of a Banach space* X. *Suppose* $M = M_1 \cup M_2$ *and* $M_1 \cap M_2 = \emptyset$. *If* M_1 *is both open and closed relative to the subspace* M, *then there exist open sets* D_1, D_2 *of* X *such that*

$$M_1 \subseteq D_1\,, \quad M_2 \subseteq D_2\,, \quad D_1 \cap D_2 = \emptyset\,.$$

Proof.— Since M_1 is both relatively open and closed, so is M_2. Hence there exist open sets E_1, E_2 such that

$$M_1 \subseteq E_1\,, \quad M_2 \subseteq E_2\,, \quad (E_1 \cap E_2) \cap M = \emptyset\,.$$

Set $H_1 = E_1 \backslash (E_1 \cap E_2)$ and $H_2 = E_2 \backslash (E_1 \cap E_2)$. Then $M_1 \subseteq H_1$, $M_2 \subseteq H_2$ and $H_1 \cap H_2 = \emptyset$. Since E_1 is open, for each $x \in \partial E_2 \cap E_1$, $\exists\ \epsilon_x > 0$ such that the ball $B(x, \epsilon_x)$ centered at x with radius ϵ_x is contained in E_1. Let $\epsilon'_x = \text{dist}(x, \partial E_1 \cap E_2)$. Then $\epsilon'_x \geq \epsilon_x > 0$. Set

$$D_1 = H_1 \cup \left(\bigcup_{x \in \partial E_2 \cap E_1} B\left(x, \frac{\epsilon'_x}{4}\right)\right)$$

and

$$D_2 = H_2 \cup \left(\bigcup_{y \in \partial E_1 \cap E_2} B\left(y, \frac{\epsilon'_y}{4}\right)\right) .$$

Clearly, $M_1 \subseteq D_1$, $M_2 \subseteq D_2$ and D_1, D_2 are open. We now claim that $D_1 \cap D_2 = \emptyset$. Indeed, if not, say $z \in D_1 \cap D_2$, then there exist $B(x, \epsilon'_x/4)$, $B(y, \epsilon'_y/4)$ such that $z \in B(x, \epsilon'_x/4) \cap B(y, \epsilon'_y/4)$. Then

$$||x - y|| \leq ||x - z|| + ||z - y|| \leq \frac{\epsilon'_x}{4} + \frac{\epsilon' y}{4} < \max(\epsilon'_x, \epsilon'_y) .$$

On the other hand, $\epsilon'_x \leq ||x - y||$ and $\epsilon'_y \leq ||x - y||$ which imply that $||x - y|| \geq \max(\epsilon'_x, \epsilon'_y)$. A contradiction which completes the proof of the lemma.

LEMMA 10.— *Let S^i $(i = 1, 2, \ldots, n)$ be n mutually disjoint compact subsets of a Banach space X and let M be any nonempty subset of X. If for all i, j $(i \neq j;\ i, j = 1, 2, \ldots, n)$, the sets $S^i \cap \overline{M}$ and $S^j \cap \overline{M}$ are not connected through M, then there are n mutually disjoint open sets N^i $(i = 1, 2, \ldots, n)$ such that*

$$M \subseteq \bigcup_{i=1}^{n} N^i \quad \textit{and} \quad S^i \subseteq N^i \quad \text{for all} \quad i = 1, 2, \ldots, n . \tag{2.1}$$

Proof.— For each i $(i = 1, 2, \ldots, n)$, we denote by M^i the compact set $S^i \cap \overline{M}$. Since by assumption none of the pairs M^i, M^j $(i \neq j;\ i, j = 1, 2, \cdots, n)$ are connected through M, there exist by Lemma 9 open sets O_{ij} and P_{ij} $(O_{ij} = P_{ji},\ i \neq j;\ i, j = 1, 2, \ldots, n)$ such that $M^i \subseteq O_{ij}$, $M^j \subseteq P_{ij}$, $O_{ij} \cap P_{ij} = \emptyset$ $(i \neq j;\ i, j = 1, 2, \ldots, n)$ and

$$M^i \cup M \cup M^j \subseteq O_{ij} \cup P_{ij} \quad (i \neq j\,;\ i, j = 1, 2, \ldots, n) .$$

For each i $(i = 1, 2, \ldots, n)$, let

$$O_i = \bigcap_{\substack{j=1 \\ j\neq i}}^{n} O_{ij}, \quad P_i = \bigcup_{\substack{j=1 \\ j\neq i}}^{n} P_{ij} \tag{2.2}$$

and

$$M_s = \bigcup_{i=1}^{n} M^i, \quad \widetilde{M}^i = \bigcup_{\substack{j=1 \\ j\neq i}}^{n} M^j. \tag{2.3}$$

Then

$$M^i \subseteq O_i, \quad \widetilde{M}^i \subseteq P_i \tag{2.4}$$

and

$$O_i \cap P_i = \emptyset, \quad M_s \cup M \subseteq O_i \cup P_i. \tag{2.5}$$

Put for each i $(i = 1, 2, \ldots, n)$

$$O^i = O_i \cap \left(\bigcap_{j=1, j\neq i}^{n} P_j \right). \tag{2.6}$$

Then by (2.2)-(2.5), we have

$$M^i \subseteq O^i, \quad O^i \cap O^j = \emptyset \quad (i \neq j;\ i, j = 1, 2, \ldots, n). \tag{2.7}$$

It is not generally true that $M_s \cup M \subseteq \bigcup_{i=1}^{n} O^i$. In order to prove the lemma, we let

$$M' = (M_s \cup M) \setminus \left(\bigcup_{i=1}^{n} O^i \right), \quad M'' = (M_s \cup M) \cap \left(\bigcup_{i=1}^{n} O^i \right).$$

Then

$$M_s \cup M = M' \cup M'', \quad M' \cap M'' = \emptyset. \tag{2.8}$$

By (2.5) and (2.6), we see that M'' is both open and closed relative to $M_s \cup M$. Again by Lemma 9, there exist two open sets D' and D'' such that

$$M' \subseteq D', \quad M'' \subseteq D'', \quad D' \cap D'' = \emptyset. \tag{2.9}$$

Now for each i $(i = 1, 2, \ldots, n)$ put $O_D^i = O^i \cap D''$. By (2.7) and (2.9), we have

$$D' \cap \left(\bigcup_{i=1}^{n} O_D^i \right) = \emptyset \,, \tag{2.10}$$
$$M^i \subseteq O_D^i \,, \quad O_D^i \cap O_D^j = \emptyset \quad (i \neq j\,;\; i,\, j = 1,\, 2,\, \ldots,\, n)\,.$$

By the compactness of S^i and M^i, we may introduce

$$a_i = \operatorname{dist}(M^i, X \backslash O_D^i) > 0 \,,$$
$$\delta_1 = \frac{1}{2} \min\{\operatorname{dist}(S^i, S^j) \mid i \neq j\,;\; i,\, j = 1,\, 2,\, \ldots,\, n\} > 0\,. \tag{2.11}$$

Let $\delta_2 = (1/4) \min\{a_i\,,\, \delta_1 \mid i = 1, 2, \ldots, n\}$ and

$$Q_i = \{x \in X \mid \operatorname{dist}(x, M^i) < \delta_2\}\,, \quad S_q^i = S^i \backslash Q_i\,. \tag{2.12}$$

Then $Q_i \subseteq O_D^i$ and $S_q^i \cap \overline{M} = \emptyset$. By (2.11), we see that $\operatorname{dist}(S_q^i, Q_j) \geq \operatorname{dist}(S_q^i, S_q^j) - \delta_2 \geq 3\delta_2$. By the compactness of S_q^i, we may also introduce

$$b_i = \frac{1}{4} \operatorname{dist}(S_q^i, M) > 0\,, \quad \delta_3 = \min\{b_i,\, \delta_2 \mid i = 1,\, 2,\, \ldots,\, n\} > 0\,.$$

Put

$$P = \{x \in X \mid \operatorname{dist}(x, M) < \delta_3\} \tag{2.13}$$

and

$$N_i = Q_i \cup (O_D^i \cap P)\,, \quad R' = D' \cap P\,. \tag{2.14}$$

Then

$$M' \subseteq R'\,, \quad M'' \subseteq \bigcup_{i=1}^{n} N_i\,. \tag{2.15}$$

By (2.10), we have that $R' \cap (\bigcup_{i=1}^{n} N_i) = \emptyset$ and $N_i \cap N_j = \emptyset$ $(i \neq j;$ $i, j = 1, 2, \ldots, n)$. Furthermore

$$\operatorname{dist}(S_q^i, P) \geq \operatorname{dist}(S_q^i, M) - \delta_3 \geq 3\delta_3\,. \tag{2.16}$$

Hence

$$\operatorname{dist}(S_q^i, R') \geq \operatorname{dist}(S_q^i, P) \geq \operatorname{dist}(S_q^i, M) - \delta_3 \geq 3\delta_3\,. \tag{2.17}$$

By (2.14) and (2.16), we also have that

$$\begin{aligned}\operatorname{dist}(S_q^i, N_j) &\geq \min\{\operatorname{dist}(S_q^i, Q_j) \mid \operatorname{dist}(S_q^i, P)\} \\ &\geq \min(3\delta_2\,,\, 3\delta_3) \geq 3\delta_3\,.\end{aligned} \tag{2.18}$$

Now let $N^1 = N_1 \cup \{x \in X \mid \operatorname{dist}(x, S_q^1) < \delta_3\} \cup R'$ and

$$N^i = N_i \cup \{x \in X \mid \operatorname{dist}(x, S_q^i) < \delta_3\} \quad (i \neq 1\,;\, i = 1,\, 2,\, \dots,\, n)\,.$$

By (2.8), (2.12) and (2.15) it follows that

$$S^i \subseteq N^i\,, \quad M \subseteq \bigcup_{i=1}^{n} N^i\,. \tag{2.19}$$

By (2.10), (2.17) and (2.18), we see that

$$N^i \cap N^j = \emptyset \quad (i \neq j\,;\, i,\, j = 1,\, 2,\, \dots,\, n)\,. \tag{2.20}$$

So (2.19) and (2.20) imply that N^i satisfy (2.1). This completes the proof of the lemma. □

3. Proof of Theorem 7

Suppose assertions (ii) and (iii) are not true and let us prove (i). The critical set K_c is the disjoint union of S_c, M_c and P_c. Also by the $(\mathrm{PS})_c$ condition, K_c is compact. It is also clear that S_c is closed and compact. We will assume that $S_c \neq \emptyset$ since otherwise we conclude by Corollary 3. We start with the following:

Claim 1

There exist finitely many components of G_c, say C_i $(i = 1,\, 2,\, \dots,\, n)$ and $\eta_1 > 0$ such that

$$G_c \cap \{x \mid \operatorname{dist}(x, S_c) < \eta_1\} \subseteq \bigcup_{i=1}^{n} C_i\,. \tag{3.1}$$

Indeed, if not, we could find a sequence x_i in S_c and a sequence $(C_i)_i$ of different components of G_c such that $\operatorname{dist}(x_i\,,\, C_i) \to 0$. But then any limit point of the sequence x_i would be a saddle point of mountain-pass type for

φ, thus contradicting our assumption that assertion (ii) is false. Claim 1 is hence proved. We clearly may assume that $C_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$.

Next for each $i = 1, 2, \ldots, n$, let $S_c^i = S_c \cap \overline{C_i}$. They all are compact and mutually disjoint. Also we have that

$$S_c = \bigcup_{i=1}^{n} S_c^i . \tag{3.2}$$

Claim 2

There are n mutually disjoint open sets N^i ($i = 1, 2, \ldots, n$) such that $u, v \notin \bigcup_{i=1}^{n} \overline{N^i}$ and

$$S_c \cup M_c \subseteq \bigcup_{i=1}^{n} N^i \quad \text{and} \quad S_c^i \subseteq N^i \text{ for all } i = 1, 2, \cdots, n . \tag{3.3}$$

Indeed, we have two cases to consider.

Case 1 $M_c = \emptyset$.— This is a trivial case. By the initial assumption that $u, v \notin K_c$, for each i ($i = 1, 2, \ldots, n$) there exists an open neighborhood N^i of S_c^i such that $u, v \notin \overline{N_i}$. Since the S_c^i's are mutually disjoint compact sets, we may take the N^i's in such a way that they are also mutually disjoint.

Case 2 $M_c \neq \emptyset$.— In this case we are in a situation where we have n mutually disjoint compact sets S_c^i ($i = 1, 2, \ldots, n$) and a nonempty set M_c. Moreover all the pairs $S_c^i \cap \overline{M_c}$, $S_c^j \cap \overline{M_c}$ ($i \neq j$; $i, j = 1, 2, l \ldots, n$) are not connected through M_c since assertion (3.3) is assumed false. Applying Lemma 10, we can then find n mutually disjoint open sets N^i such that (3.3) is verified. Since $u, v \notin K_c$, we may clearly assume that $u, v \notin \bigcup_{i=1}^{n} \overline{N^i}$. Claim 2 is proved in both cases.

In order to finish the proof of Theorem 7, we still need the following

Claim 3

There exists a closed set $\hat{F}$ such that $\hat{F}$ separates u, v while

$$\inf_{x \in \widehat{F}} \varphi(x) \geq c \quad \text{and} \quad \widehat{F} \cap (S_c \cup M_c) = \emptyset . \tag{3.4}$$

We let for each i ($i = 1, 2, \ldots, n$):

$$Y_i^c = N^i \backslash G_c . \tag{3.5}$$

Then observe that for each i $(1 \leq i \leq n)$ there must be $B(x, \epsilon_x)$ $(\epsilon_x > 0)$ such that for any connected component U of G_c with $C_i \neq U$, $B(x, \epsilon_x) \cap U = \emptyset$. Otherwise x is a saddle point of mountain-pass type and this contradicts that assertion (ii) is assumed false. Put

$$T_i^c = \bigcup_{x \in S_c^i} B\left(x, \frac{\epsilon_x}{2}\right) \cap \partial C_i \cap N^i \cap \{x \in X \mid \mathrm{dist}(x, S_c^i) < \eta_1\}. \qquad (3.6)$$

Clearly

$$S_c^i \subseteq T_i^c, \quad T_i^c \subseteq N^i \cap \partial C_i \qquad (3.7)$$

and T_i^c is open relative to ∂Y_i. Also $T_i^c \cap \partial U = \emptyset$ for any component U of G_c with $U \neq C_i$. Now let

$$\widehat{F} = \left[(F \cap L_c) \setminus \left(\bigcup_{i=1}^n N^i\right)\right] \cup \left(\bigcup_{i=1}^n \partial Y_i^c \setminus T_i^c\right).$$

Clearly, $\inf_{x \in \widehat{F}} \varphi(x) \geq c$. Since $F \cap L_c$ separates u, v and in view of Claim 1, Claim 2, (3.5) and (3.7), we see that we can apply Lemma 6 with $A_i = C_i$, $G = G_c$, $Z_i = N^i$, $Y_i = Y_i^c$, $T_i = T_i^c$ for all $i = 1, 2, \ldots, n$ to conclude that $\widehat{F}$ separates u, v. On the other hand, since $M_c \cap \left(\overline{G_c} \setminus G_c\right) = \emptyset$, we have by (3.3) and (3.5), that $\partial Y_i^c \cap M_c = \emptyset$. Therefore by (3.2) and (3.6), we have $\bigcup_{i=1}^n (\partial Y_i^c \setminus T_i^c) \cap (S_c \cup M_c) = \emptyset$. Hence $\widehat{F} \cap (M_c \cup S_c) = \emptyset$ and Claim 3 is thus proved.

Finally by Theorem 2, we see that $\widehat{F} \cap P_c$ and hence P_c must contain a compact subset that separates u, v which implies assertion (i). This finishes the proof of the theorem. □

Remark (c). — Theorem 7 is still true if instead of $(\mathrm{PS})_c$, φ verifies only $(\mathrm{PS})_{(F \cup K_c)_\epsilon, c}$. The proof also shows that the condition u, $v \notin K_c$ can be replaced by u, $v \notin S_c \cup \overline{M_c}$.

We have the following surprising corollary concerning the cardinality of the critical set K_c generated by the Mountain pass theorem.

COROLLARY 11. — *Suppose* $\dim(X) \geq 2$. *Then under the hypothesis of Theorem 7, one of the following three assertions must be true:*

(1) K_c *has a saddle point of mountain-pass type;*

(2) the cardinality of P_c *is at least the continuum;*

(3) the cardinality of M_c *is at least the continuum.*