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Estimations of the best constant involving the L^∞ norm in Wente's inequality[*]

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RÉSUMÉ. — Dans ce travail, on s'intéresse à la meilleure constante dans une inégalité due à H. Wente. On montre que cette constante est bornée indépendamment du domaine sur lequel on travaille. En particulier, lorsque le domaine est simplement connexe, la meilleure constante associée à la norme L^∞ est $1/2\pi$.

ABSTRACT. — In this work, we study the best constant in the so called Wente's inequality. Our main result relies on the fact that the constant in Wente's estimate can be bounded from above independently of the domain on which the problem is posed. In particular, if the domain is bounded and simply connected, we show that the best constant involving the L^∞ norm is $1/2\pi$.

Introduction and statement of the results

Let Ω be a smooth and bounded domain in $\mathbb{R}^2$. Given two functions a and b on Ω such that

$$a \in H^1(\Omega) \quad \text{and} \quad b \in H^1(\Omega)\,. \tag{1}$$

We denote by φ the unique solution in $L^2(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = a_{x_1}b_{x_2} - a_{x_2}b_{x_1} & \text{in } \Omega \quad (2)\\ \quad \varphi = 0 & \text{on } \partial\Omega \quad (3)\end{cases}$$

where subscripts denote partial differentiation with respect to coordinates.

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In the context of the equation of the mean curvature, H. Wente in [5], also H. Brezis and J.-M. Coron in [3] have obtained the following striking result.

THEOREM 0 ([3], [5]).— *The solution φ of (2) and (3) is a continuous function on Ω and its gradient belongs to $L^2(\Omega)$. Moreover there exists a constant $C_0(\Omega)$ which depends on Ω such that*

$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} \leq C_0(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)} . \tag{4}$$

F. Bethuel and J.-M. Ghidaglia in [1] showed that under the hypothesis of Theorem 0, there exists a constant C_1 which does not depend of Ω such that (4) holds true:

$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} \leq C_1\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)} . \tag{5}$$

Remarks 1

1 It is clear that $C_0(\Omega)$ in (4) is invariant under translation and dilatation of Ω, it could however depend on its shape. The result in [1] means it is proved that the constant $C_0(\Omega)$ is bounded independently of Ω.

2 The constant $C_0(\Omega)$ is invariant under a conformal transformation of Ω.

We denote by $C_\infty(\Omega)$ the best constant involving the L^∞ norm and by $C_2(\Omega)$ the L^2 norm. Then we have:

$$\|\varphi\|_{L^\infty(\Omega)} \leq C_\infty(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}$$
$$\|\nabla\varphi\|_{L^2(\Omega)} \leq C_2(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)} .$$

Our purpose in this paper is to study $C_\infty(\Omega)$. We first remark that:

$$C_2(\Omega) \leq \sqrt{C_\infty(\Omega)} . \tag{6}$$

Our results are the following theorems.

THEOREM 1.— *Assume that $\Omega = \mathbb{R}^2$, then:*

$$C_\infty(\mathbb{R}^2) = \frac{1}{2\pi} . \tag{7}$$

THEOREM 2.— *Let Ω be a smooth and bounded domain in $\mathbb{R}^2$, then:*

$$C_\infty(\Omega) \geq \frac{1}{2\pi}\,. \tag{8}$$

Moreover if we assume that Ω is a bounded simply connected domain in $\mathbb{R}^2$, then:

$$C_\infty(\Omega) = \frac{1}{2\pi}\,. \tag{9}$$

Proof of Theorem 1.— In [3] H. Brezis and J.-M. Coron showed that $C_\infty(\mathbb{R}^2) \leq 1/2\pi$, the next step is to show that $C_\infty(\mathbb{R}^2) \geq 1/2\pi$. We let

$$a(x_1, x_2) = x_1\, g(r)\,, \quad b(x_1, x_2) = x_2\, g(r) \quad \text{where } r = \sqrt{x_1^2 + x_2^2} \tag{10}$$

and g will be chosen later. We obtain,

$$a_{x_1} b_{x_2} - a_{x_2} b_{x_1} = \frac{1}{2r}\,\frac{\partial\big(r^2\, g^2(r)\big)}{\partial r}\,. \tag{11}$$

By (2), we have

$$-\frac{1}{r}\,\frac{\partial\Big(r\dfrac{\partial\varphi}{\partial r}\Big)}{\partial r} = \frac{1}{2r}\,\frac{\partial\big(r^2\, g^2(r)\big)}{\partial r}\,. \tag{12}$$

Thus we compute

$$\varphi(r) = \frac{1}{2}\int_r^\infty \sigma g^2(\sigma)\,\mathrm{d}\sigma \tag{13}$$

and

$$\|\nabla a\|_{L^2}^2 = \|\nabla b\|_{L^2}^2 = \int_{\mathbb{R}^2} |\nabla a|^2 = \pi\int_0^\infty \sigma^3 g'^2(\sigma)\,\mathrm{d}\sigma\,. \tag{14}$$

Hence

$$C_\infty(\mathbb{R}^2) \geq \frac{1}{2\pi}\,\frac{\displaystyle\int_0^\infty \sigma g^2(\sigma)\,\mathrm{d}\sigma}{\displaystyle\int_0^\infty \sigma^3 g'^2(\sigma)\,\mathrm{d}\sigma}\,. \tag{15}$$

Now we choose $g_\varepsilon = \sigma^{\varepsilon-1}\, e^{-\sigma/2}$, we first observe that

$$a(x_1, x_2) = x_1\, g(r) = r^{\varepsilon-1}\, e^{-r/2} \cos\theta \in H^1(\mathbb{R}^2) \tag{16}$$

and hence

$$\|\nabla a\|_{L^2}^2 = \int_0^\infty r\left[\left(\varepsilon r^{\varepsilon-1}\, e^{-r/2} - \frac{1}{2}\, r^{\varepsilon}\, e^{-r/2}\right) + \left(r^{\varepsilon-1}\, e^{-r/2}\right)^2\right] \mathrm{d}r\,. \tag{17}$$

Then, by (15)

$$C_\infty(\mathbb{R}^2) \geq \frac{1}{2\pi}\, \frac{\displaystyle\int_0^\infty \sigma^{2\varepsilon-1}\, e^{-\sigma}\, \mathrm{d}\sigma}{\displaystyle\int_0^\infty \sigma^3\left((\varepsilon-1)\sigma^{\varepsilon-2} - \frac{1}{2}\,\sigma^{\varepsilon-1}\right)^2 e^{-\sigma}\, \mathrm{d}\sigma} \tag{18}$$

$$\geq \frac{1}{2\pi}\, \frac{\Gamma(2\varepsilon)}{(\varepsilon-1)^2\Gamma(2\varepsilon) - (\varepsilon-1)\Gamma(2\varepsilon+1) + \frac{1}{4}\Gamma(2\varepsilon+2)}\,, \tag{19}$$

where Γ denotes the gamma function. We conclude that

$$C_\infty(\mathbb{R}^2) \geq \frac{1}{2\pi}\, \frac{1}{(\varepsilon-1)^2 - 2\varepsilon(\varepsilon-1) + \frac{\varepsilon}{2}\,(2\varepsilon+1)} \longrightarrow \frac{1}{2\pi} \tag{20}$$

when ε tends to zero, so

$$C_\infty(\mathbb{R}^2) \geq \frac{1}{2\pi}\,.$$

Proof of Theorem 2.— We first assume that Ω is a smooth and bounded domain in $\mathbb{R}^2$, not necessarily simply connected. Here we let also

$$a(x_1, x_2) = x_1\, g(r)\,, \quad b(x_1, x_2) = x_2\, g(r)\,, \quad \text{where } r = \sqrt{x_1^2 + x_2^2}\,,$$

and g will be chosen later.

Next we introduce the following notations

$$L(g) = \frac{\displaystyle\int_0^\infty \sigma\, g^2(\sigma)\, \mathrm{d}\sigma}{\displaystyle\int_0^\infty \sigma^3\, g'^2(\sigma)\, \mathrm{d}\sigma} \quad \text{and} \quad g_n(\sigma) = \sigma^{1/n-1}\, e^{-\sigma/2}\,, \tag{21}$$

we obtain by (15)

$$C_\infty(\mathbb{R}^2) \geq \frac{1}{2\pi} L(g)\,.$$

It was established in the proof of Theorem 1 that:

$$L(g_n) \longrightarrow 1 \tag{22}$$

when n tends to infinity. Let g be fixed, $\psi \in C_0^\infty(\mathbb{R}^2)$ such that:

$$\psi = \begin{cases} 1 & \text{in } |\sigma| \leq 1, \\ 0 & \text{in } |\sigma| \geq 2 \end{cases} \quad \text{and} \quad 0 \leq \psi \leq 1\,. \tag{23}$$

First step

We put $g_\lambda(\sigma) = \psi(\sigma/\lambda)\, g(\sigma)$ and we show the following Lemma.

LEMMA 1.— *With the above notations, we have*

$$L(g_\lambda) \longrightarrow L(g) \tag{24}$$

when λ tends to infinity.

Proof of Lemma 1.— We have as λ tends to infinity,

$$\begin{aligned}\int_0^\infty \sigma\, g^2(\sigma)\,\mathrm{d}\sigma - \int_0^\infty \sigma\, g_\lambda^2(\sigma)\,\mathrm{d}\sigma &= \int_0^\infty \sigma\left(1 - \psi^2\left(\frac{\sigma}{\lambda}\right)\right) g^2(\sigma)\,\mathrm{d}\sigma \\ &\leq \int_\lambda^\infty \sigma g^2(\sigma)\,\mathrm{d}\sigma \longrightarrow 0\,.\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^\infty \sigma^3\, g_\lambda'^2(\sigma)\,\mathrm{d}\sigma - \int_0^\infty \sigma^3 g'^2(\sigma)\,\mathrm{d}\sigma = \\ &= \int_0^\infty \sigma^3\left(\frac{1}{\lambda}\psi'\left(\frac{\sigma}{\lambda}\right) g(\sigma) + \psi\left(\frac{\sigma}{\lambda}\right) g'(\sigma)\right)^2 \mathrm{d}\sigma - \int_0^\infty \sigma^3 g'^2(\sigma)\,\mathrm{d}\sigma \\ &= \int_0^\infty \frac{\sigma^3}{\lambda^2}\psi'^2\left(\frac{\sigma}{\lambda}\right) g^2(\sigma)\,\mathrm{d}\sigma + 2\int_0^\infty \frac{\sigma^3}{\lambda}\psi'\left(\frac{\sigma}{\lambda}\right)\psi\left(\frac{\sigma}{\lambda}\right) g(\sigma) g'(\sigma)\,\mathrm{d}\sigma \\ &\quad - \int_0^\infty \sigma^3\left(1 - \psi^2\left(\frac{\sigma}{\lambda}\right)\right) g'^2(\sigma)\,\mathrm{d}\sigma \\ &= \text{(I)} + \text{(II)} + \text{(III)}\,,\end{aligned}$$

where

$$\big|(\mathrm{III})\big| \leq \int_\lambda^\infty \sigma^3\Big(1-\psi^2\Big(\frac{\sigma}{\lambda}\Big)\Big)\, g'^2(\sigma)\,\mathrm{d}\sigma \leq \int_\lambda^\infty \sigma^3\, g'^2(\sigma)\,\mathrm{d}\sigma \longrightarrow 0\,,$$

$$(\mathrm{I}) \leq \int_\lambda^{2\lambda} \frac{\sigma^3}{\lambda^2}\,\psi'^2\Big(\frac{\sigma}{\lambda}\Big) g^2(\sigma)\,\mathrm{d}\sigma \leq C\int_\lambda^{2\lambda} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma \longrightarrow 0\,,$$

$$\begin{aligned}\big|(\mathrm{II})\big| &\leq 2\int_\lambda^{2\lambda} \frac{\sigma^3}{\lambda}\,\Big|\psi'\Big(\frac{\sigma}{\lambda}\Big)\Big|\,\psi\Big(\frac{\sigma}{\lambda}\Big)\big|g'(\sigma)\big|\,\big|g(\sigma)\big|\,\mathrm{d}\sigma\\ &\leq C\int_\lambda^{2\lambda} \sigma^2\big|g'(\sigma)\big|\,\big|g(\sigma)\big|\,\mathrm{d}\sigma\,.\end{aligned}$$

By the inequality of Cauchy–Schwarz, we obtain

$$(\mathrm{II}) \leq C\left(\int_\lambda^{2\lambda} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma\right)^{1/2}\left(\int_\lambda^{2\lambda} \sigma^3\, g'^2(\sigma)\,\mathrm{d}\sigma\right)^{1/2},$$

then

$$L(g_\lambda) \longrightarrow L(g) \quad \text{as } \lambda \text{ tends to infinity}\,.$$

Returning to g_n, we prove that for n fixed $L(g_{n\lambda}) \to L(g_n)$ as λ tends to infinity. By (24),

$$\exists\,\lambda_n \quad \text{such that} \quad \left|L\Big(\psi\Big(\frac{\sigma}{\lambda_n}\Big) g_n(\sigma)\Big) - L(g_n)\right| \leq \frac{1}{n} \tag{25}$$

then by (22)

$$L\Big(\psi\Big(\frac{\sigma}{\lambda_n}\Big) g_n(\sigma)\Big) \longrightarrow 1 \quad \text{as } n\to\infty\,. \tag{26}$$

Denote $\widehat{g}_n(\sigma) = \psi(\sigma/\lambda_n) g_n(\sigma)$, we have $\operatorname{supp}\, \widehat{g}_n \subset [\,0\,,\,2\lambda_n\,]$. Let $h_n(\sigma) = \widehat{g}_n(4\lambda_n\sigma)$, we have $\operatorname{supp} h_n \subset [\,0\,,\,1/2\,]$. We remark that L is invariant by dilatation. Let $g_\alpha(\sigma) = g(\alpha\sigma)$, an easy computation shows that $L(g_\alpha) = L(g)$, then

$$L(h_n) \longrightarrow 1 \quad \text{as } n\to\infty\,. \tag{27}$$

Denote by

$$a_n(x_1,x_2) = x_1 h_n(r)\,, \quad b_n(x_1,x_2) = x_2 h_n(r)\,, \quad \text{where } r=\sqrt{x_1^2+x_2^2}\,. \tag{28}$$

We introduce the solution ψ_n of the following problem

$$\begin{cases} -\Delta\psi_n = a_{nx_1} b_{nx_2} - a_{nx_2} b_{nx_1} & \text{in } \Omega \qquad (29) \\ \psi_n = 0 & \text{on } \partial\Omega. \qquad (30) \end{cases}$$

We have

$$\psi_n(r) = \frac{1}{2}\int_r^1 \sigma h_n^2(\sigma)\,\mathrm{d}\sigma \quad \text{so} \quad |\psi_n|_\infty = \frac{1}{2}\int_0^1 \sigma h_n^2(\sigma)\,\mathrm{d}\sigma\,.$$

Since

$$\|\nabla a_n\|_{L^2(B_1)}\|\nabla b_n\|_{L^2(B_1)} = \pi\int_0^1 \sigma^3 h_n'^2(\sigma)\,\mathrm{d}\sigma\,,$$

then, by (27)

$$C_\infty(B_1) \geq \frac{1}{2\pi}\,L(h_n) \longrightarrow \frac{1}{2\pi} \quad \text{and} \quad C_\infty(B_1) \geq \frac{1}{2\pi}\,.$$

Second step

Let Ω be a smooth, nonempty and bounded domain in $\mathbb{R}^2$, then there exists $r_0 > 0$, $z_0 \in \Omega$ such that $B(z_0, r_0) \subset \Omega$.

If we note $\widehat{a}_n(z) = a_n\big((z - z_0)/r_0\big)$, $\widehat{b}_n(z) = b_n\big((z - z_0)/r_0\big)$, we have

$$\widehat{a}_n\,,\ \widehat{b}_n \in H_0^1\big(B(z_0, r_0)\big)\,,$$

since

$$\widehat{a}_n = \widehat{b}_n = 0 \text{ on } \Omega \setminus B(z_0, r_0) \text{ then } \widehat{a}_n\,,\ \widehat{b}_n \in H_0^1(\Omega)\,.$$

Let $\widehat{\psi}_n$ be a solution of the following problem

$$\begin{cases} -\Delta\widehat{\psi}_n = \widehat{a}_{nx_1}\widehat{b}_{nx_2} - \widehat{a}_{nx_2}\widehat{b}_{nx_1} & \text{in } \Omega \qquad (31) \\ \widehat{\psi}_n = 0 & \text{on } \partial\Omega \qquad (32) \end{cases}$$

we shall prove the following result.

LEMMA 2.— *Under the above hypothesis, we have :*

$$\widehat{\psi}_n(z) = \psi_n\left(\frac{z - z_0}{r_0}\right). \tag{33}$$

Proof of Lemma 2.— Let $\Omega_1 = B\big(z_0, (3/4)r_0\big), \Omega_2 = \mathbb{R}^2 \backslash B\big(z_0, (1/2)r_0\big)$, $\chi_1 \in C_0^\infty(\mathbb{R}^2)$, $\operatorname{supp}\chi_1 \subset \Omega_1$, $\chi_1 = 1$ on $B\big(z_0, r_0/2\big)$ and $\chi_2 = 1 - \chi_1$, so $\operatorname{supp}\chi_2 \subset \Omega_2$.

For $\eta \in C_0^\infty(\Omega)$,

$$-\int_\Omega \Delta\psi_n\Big(\frac{z-z_0}{r_0}\Big)\,\eta(z)\,\mathrm{d}z = -\int_\Omega \psi_n\Big(\frac{z-z_0}{r_0}\Big)\,\Delta\eta(z)\,\mathrm{d}z$$

since $\operatorname{supp}\psi_n \subset B\big(z_0, r_0/2\big)$ and $\chi_2 = 0$ in $B\big(z_0, r_0/2\big)$, we have

$$\begin{aligned}
-\int_\Omega \Delta\psi_n\Big(\frac{z-z_0}{r_0}\Big)\,\eta(z)\,\mathrm{d}z &= -\int_\Omega \psi_n\Big(\frac{z-z_0}{r_0}\Big)\,\Delta\big(\chi_1\eta(z)+\chi_2\eta(z)\big)\,\mathrm{d}z\\
&= -\int_{B(z_0,r_0)} \psi_n\Big(\frac{z-z_0}{r_0}\Big)\,\Delta\big(\chi_1\eta(z)\big)\,\mathrm{d}z\\
&= -\int_{B(z_0,r_0/2)} \Delta\psi_n\Big(\frac{z-z_0}{r_0}\Big)\,\eta(z)\,\mathrm{d}z\\
&= \int_\Omega (a_{nx_1}b_{nx_2} - a_{nx_2}b_{nx_1})\Big(\frac{z-z_0}{r_0}\Big)\,\eta(z)\,\mathrm{d}z\\
&= \int_\Omega (\widehat{a}_{nx_1}\widehat{b}_{nx_2} - \widehat{a}_{nx_2}\widehat{b}_{nx_1})(z)\,\eta(z)\,\mathrm{d}z\,.
\end{aligned}$$

Hence

$$\widehat{\psi}_n(z) = \psi_n\Big(\frac{z-z_0}{r_0}\Big)\,.$$

Returning to the proof of Theorem 2, since

$$\big|\widehat{\psi}_n\big|_{L^\infty} = \big|\psi_n\big|_{L^\infty}\,,\quad \big|\nabla\widehat{a}_n\big|_{L^2} = \big|\nabla a_n\big|_{L^2}\quad\text{and}\quad \big|\nabla\widehat{b}_n\big|_{L^2} = \big|\nabla b_n\big|_{L^2}\,,$$

we obtain by (28)

$$C_\infty(\Omega) \geq \frac{1}{2\pi}\,\frac{\big|\widehat{\psi}_n\big|_{L^\infty}}{\big|\nabla\widehat{a}_n\big|_{L^2}\big|\nabla\widehat{b}_n\big|_{L^2}} = \frac{1}{2\pi}\,L(h_n) \longrightarrow \frac{1}{2\pi}$$

and finally

$$C_\infty(\Omega) \geq \frac{1}{2\pi}$$

which proves (8).

We now return to the special case where Ω is a simply connected domain. First we prove the following result.

LEMMA 3.— *Let φ be the solution of the Dirichlet problem*

$$\begin{cases} -\Delta\varphi = a_{x_1}b_{x_2} - a_{x_2}b_{x_1} & \text{in } B_1 \\ \varphi = 0 & \text{on } S^1 \end{cases} \tag{34}$$

then we have

$$C_\infty(B_1) \le \frac{1}{2\pi} \quad \text{where } B_1 = \left\{x \in \mathbb{R}^2 \mid |x| < 1\right\} \quad \text{and} \quad S^1 = \partial B_1 \,.$$

Proof of Lemma 3.— By the representation of the Green formula

$$\varphi(y) = \int_{S^1}\left(\varphi\frac{\partial E(x-y)}{\partial v} - E(x-y)\frac{\partial\varphi}{\partial v}\right)\mathrm{d}s + \int_{B_1} E(x-y)\Delta\varphi\,\mathrm{d}x \tag{35}$$

where $y \in B_1$, $E(x-y) = (1/2\pi)\log|x-y|$ and ν is the exterior normal vector. Choosing $y = 0$ in (35), we have

$$\varphi(0) = \int_{B_1} E(x)\Delta\varphi\,\mathrm{d}x \,.$$

It is easy to see from (2) that

$$\Delta\varphi = \frac{1}{r}\left(a_r b_\theta - a_\theta b_r\right),$$

so we deduce

$$\begin{aligned}
\varphi(0) &= \frac{1}{2\pi}\int_0^1 \mathrm{d}r\int_0^{2\pi}\left(\log\frac{1}{r}\right)\left(a_r b_\theta - a_\theta b_r\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^1 \mathrm{d}r\int_0^{2\pi}\left(\log\frac{1}{r}\right)\left(\left(ab_\theta\right)_r - \left(ab_r\right)_\theta\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^1 \mathrm{d}r\int_0^{2\pi}\left(\log\frac{1}{r}\right)\left(ab_\theta\right)_r\mathrm{d}\theta = \frac{1}{2\pi}\int_0^1\frac{\mathrm{d}r}{r}\int_0^{2\pi} ab_\theta\,\mathrm{d}\theta \,.
\end{aligned}$$

Using

$$\int_0^{2\pi} ab_\theta\,\mathrm{d}\theta = \int_0^{2\pi}(a-\overline{a})b_\theta\,\mathrm{d}\theta \quad \text{where } \overline{a}(r) = \int_0^{2\pi} a(r,\sigma)\,\mathrm{d}\sigma \,,$$

we have

$$\left|\int_0^{2\pi} ab_\theta \,\mathrm{d}\theta\right| \leq |a-\overline{a}|_{L^2(0,2\pi)}\, |b_\theta|_{L^2(0,2\pi)} \leq |a_\theta|_{L^2(0,2\pi)}\, |b_\theta|_{L^2(0,2\pi)}\,,$$

then

$$|\varphi(0)| \leq \frac{1}{2\pi}\int_0^1 |a_\theta|_{L^2(0,2\pi)}\, |b_\theta|_{L^2(0,2\pi)} \frac{\mathrm{d}r}{r}$$

and using the Cauchy–Schwarz inequality, we deduce that

$$|\varphi(0)| \leq \frac{1}{2\pi}\, |\nabla a|_{L^2(B_1)} |\nabla b|_{L^2(B_1)}\,. \tag{36}$$

We consider a function T which is defined from D^2 to D^2 by: $T(z) = (z_0+z)/(1-\overline{z}_0 z)$ where $z_0 \in D^2$. We remark that T is a conformal transformation and $T(0)=z_0$.

Denote by $\widetilde{a} = a\circ T$, $\widetilde{b} = b\circ T$ and $\widetilde{\varphi} = \varphi \circ T$, an easy computation gives

$$\begin{cases} -\Delta\widetilde{\varphi} = \widetilde{a}_{x_1}\widetilde{b}_{x_2} - \widetilde{a}_{x_2}\widetilde{b}_{x_1} & \text{in } B_1 \\ \quad \widetilde{\varphi} = 0 & \text{on } S^1, \end{cases}$$

$$|\nabla \widetilde{a}|_{L^2(B_1)} = |\nabla a|_{L^2(B_1)} \quad \text{and} \quad |\nabla \widetilde{b}|_{L^2(B_1)} = |\nabla b|_{L^2(B_1)}$$

Applying (5) for $\widetilde{\varphi}$ we obtain

$$|\varphi(z_0)| \leq \frac{1}{2\pi} |\nabla a|_{L^2(B_1)} |\nabla b|_{L^2(B_1)} \quad \forall\ z_0 \in B_1\,,$$

then

$$\|\varphi\|_{L^\infty(B_1)} \leq \frac{1}{2\pi} |\nabla a|_{L^2(B_1)} |\nabla b|_{L^2(B_1)}\,, \tag{37}$$

thus

$$C_\infty(B_1) \leq \frac{1}{2\pi}\,.$$

Returning to the proof of Theorem 2, since Ω is simply connected, there exists a conformal transformation $\widetilde{T}$ such that $\widetilde{T}(\Omega) = B_1$, let

$$\widehat{a} = a\circ \widetilde{T}\,, \quad \widehat{b} = b\circ\widetilde{T} \quad \text{and} \quad \widehat{\varphi} = \varphi\circ\widetilde{T}\,, \tag{38}$$

by a similar argument as in the proof of (37) and by (38) we find

$$\|\widehat{\varphi}\|_{L^\infty(\Omega)} \leq \frac{1}{2\pi} |\nabla \widehat{a}|_{L^2(\Omega)} |\nabla \widehat{b}|_{L^2(\Omega)} \,. \tag{39}$$

Finally

$$C_\infty(\Omega) \leq \frac{1}{2\pi}$$

which together with the first step of the proof yields (9).

Remark 2.— We do not know whether $C_\infty(\Omega) \leq 1/2\pi$ for every bounded domain Ω of $\mathbb{R}^2$, not necessarily simply connected. But the following suggests that this may be true.

Let $\Omega = B(0, R_2) \setminus B(0, R_1)$ and let

$$C_{\infty,\mathrm{rad}}(\Omega) = \mathrm{Sup} \frac{\|\varphi\|_{L^\infty(\Omega)}}{\|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)}}$$

where the above maximum is taken over map $\varphi \in H^1_0(\Omega)$, a, $b \in H^1(\Omega)$ such that $(a, b)(x_1, x_2) = (x_1, x_2) g(r)$.

PROPOSITION 1.— *With the above hypothesis we have*

$$C_{\infty,\mathrm{rad}}(\Omega) \leq \frac{1}{2\pi} \,.$$

Proof of Proposition 1.— Using (2) and (3) we deduce that

$$\varphi(r) = \frac{1}{2} \int_r^{R_2} \sigma\, g^2(\sigma)\, \mathrm{d}\sigma - \frac{1}{2} \frac{\log(r/R_2)}{\log(R_1/R_2)} \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\, \mathrm{d}\sigma$$

then

$$\|\varphi\|_\infty \leq \frac{1}{2} \mathrm{Max} \left\{ \left| \int_r^{R_2} \sigma\, g^2(\sigma)\, \mathrm{d}\sigma \right| , \left| \frac{\log(r/R_2)}{\log(R_1/R_2)} \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\, \mathrm{d}\sigma \right| \right\}$$

where $R_1 \leq r \leq R_2$. Hence,

$$\|\varphi\|_\infty \leq \frac{1}{2} \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\, \mathrm{d}\sigma \,. \tag{40}$$

Now we claim that

$$\int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma \le \int_{R_1}^{R_2} \sigma^3 g'^2(\sigma)\,\mathrm{d}\sigma + R_2^2\, g^2(R_2) - R_1^2\, g^2(R_1)\,. \qquad (41)$$

Indeed, computation yields

$$\int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma = \frac{R_2^2}{2}\, g^2(R_2) - \frac{R_1^2}{2}\, g^2(R_1) - \int_{R_1}^{R_2} \sigma^2\, g'(\sigma) g(\sigma)\,\mathrm{d}\sigma$$

and

$$\left| \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma - \frac{R_2^2}{2}\, g^2(R_2) + \frac{R_1^2}{2}\, g^2(R_1) \right| \le \int_{R_1}^{R_2} \sigma^2 \left|g'(\sigma)\right| \left|g(\sigma)\right| \mathrm{d}\sigma\,.$$

Using the inequality of Cauchy–Schwarz, we obtain

$$\begin{aligned} \left| \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma - \frac{R_2^2}{2}\, g^2(R_2) + \frac{R_1^2}{2}\, g^2(R_1) \right| &\le \\ &\le \left(\int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma \right)^{\frac{1}{2}} \left(\int_{R_1}^{R_2} \sigma^3\, g'^2(\sigma)\,\mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &\le \frac{1}{2} \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma + \frac{1}{2} \int_{R_1}^{R_2} \sigma^3 g'^2(\sigma)\,\mathrm{d}\sigma\,. \end{aligned}$$

Finally we have

$$\frac{1}{2} \int_{R_1}^{R_2} \sigma\, g^2(\sigma)\,\mathrm{d}\sigma \le \frac{1}{2} \int_{R_1}^{R_2} \sigma^3\, g'^2(\sigma)\,\mathrm{d}\sigma + \frac{R_2^2}{2}\, g^2(R_2) - \frac{R_1^2}{2}\, g^2(R_1)$$

hence we proved (41).

By a similar computation as in the proof of Theorem 2, we have

$$\|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)} = \pi \left(\int_{R_1}^{R_2} \sigma^3\, g'^2(\sigma)\,\mathrm{d}\sigma + R_2^2 g^2(R_2) - R_1^2 g^2(R_1) \right),$$

using (40) and (41), we obtain

$$\|\varphi\|_\infty \le \frac{1}{2\pi} \|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)}$$

and finally

$$C_{\infty,\mathbf{rad}}(\Omega) \le \frac{1}{2\pi}\,.$$

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