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On the Impossibility of a Generalization of the HOMFLY – Polynomial to Labelled Oriented Graphs(*)

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RÉSUMÉ. — Nous montrons que toute généralisation du polynôme HOMFLY à des graphes nommés orientés (LOGs) est presque triviale. Ainsi la géométrie des entrelacs est indispensable pour le calcul du polynôme. De plus cela nous montre que ce polynôme ne peut pas être utilisé comme invariant pour les groupes LOG.

ABSTRACT. — We show that any generalization of the HOMFLY polynomial to labelled oriented graphs (LOGs) is almost trivial. This means on the one hand that the geometry of links is essential for computing this polynomial. On the other hand it shows that this polynomial cannot serve as an invariant for LOG groups.

1. Motivation, Definitions and Main Results

Consider a group presentation: $D = \langle x_1, \dots, x_n \mid R_1, \dots, R_n \rangle$, where each relator is of the form $x_i x_j = x_j x_k$. We can assign a *labelled oriented graph* (LOG) $\Gamma(D)$ to D by defining a vertex i for every generator x_i and an oriented edge from i to k labelled by j for the relator $x_i x_j = x_j x_k$. We also call D a labelled oriented graph if there is no danger of ambiguity.

From a regular projection of an oriented tame link L in $\mathbb{R}^3$, you can read off a Wirtinger presentation D of the knot group $\pi_1(\mathbb{R}^3 \setminus L)$ (see for instance [8]). This presentation has the form of a labelled oriented graph $\Gamma(D)$, where each component of $\Gamma(D)$ is a circle or a single vertex. The

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latter shows up, whenever there is a component without selfcrossings in L which only overcrosses other components. Let $\mathcal{P}$ be the set of those labelled oriented graphs, where each component is a circle or a single vertex. The orientation on the link components induces an orientation on the circles of the corresponding LOG. We define $\mathcal{C}$ to be the set of the LOGs in $\mathcal{P}$ where each circle has an orientation. Not every LOG $\Gamma \in \mathcal{C}$ may be realized as a link projection of a tame link L in $\mathbb{R}^3$ (see [10] for a discussion of the realization of such groups as knot groups). There are many statements which are true for link (or knot) complements in 3-space, but it is not known whether they hold for the standard 2-complexes of some more general class of LOGs (for example asphericity of a knot complement or residually finiteness of knot groups).

LOGs also show up in connection with Whitehead's asphericity conjecture (see for example [3], [4] or [9]). In this case the LOG Γ has the form of a tree and is called a *labelled oriented tree* (LOT). Spines of complements of ribbon-disks in the 4-ball are homotopy equivalent to LOTs [4]. It would be most interesting to find new invariants for LOT and LOG groups.

Let $\mathcal{K}$ be the set of all regular projections of oriented, tame links in the 3-sphere under the equivalence relation of ambient isotopy. Two regular projections of ambient isotope links differ only by a finite sequence of the so-called *Reidemeister moves* (see for example [1]).

Let K_+, K_-, $K_0 \in \mathcal{K}$ be three projections which differ only in one crossing as shown in figure 2. Let $T \in \mathcal{K}$ be the trivial knot without any crossing. The following theorem was shown in 1985 by several different authors (see for example [7]):

THEOREM 1.1.— *There is exactly one map* $P : \mathcal{K} \to \mathbf{Z}[\ell^{\pm 1}, m^{\pm 1}]$ *which satisfies:*

(1) $P(T) = 1$,

(2) $\ell \cdot P(K_+) + \ell^{-1} \cdot P(K_-) + m \cdot P(K_0) = 0$ *and*

(3) P *is invariant under Reidemeister moves.*

It is natural to ask, whether there is a purely algebraic version of this theorem which does not use the geometry. The algebraic analogues of link projections are the labelled oriented graphs of the set $\mathcal{C}$. For these, the Reidemeister moves and the triples K_+, K_-, K_0 have algebraic analogues:

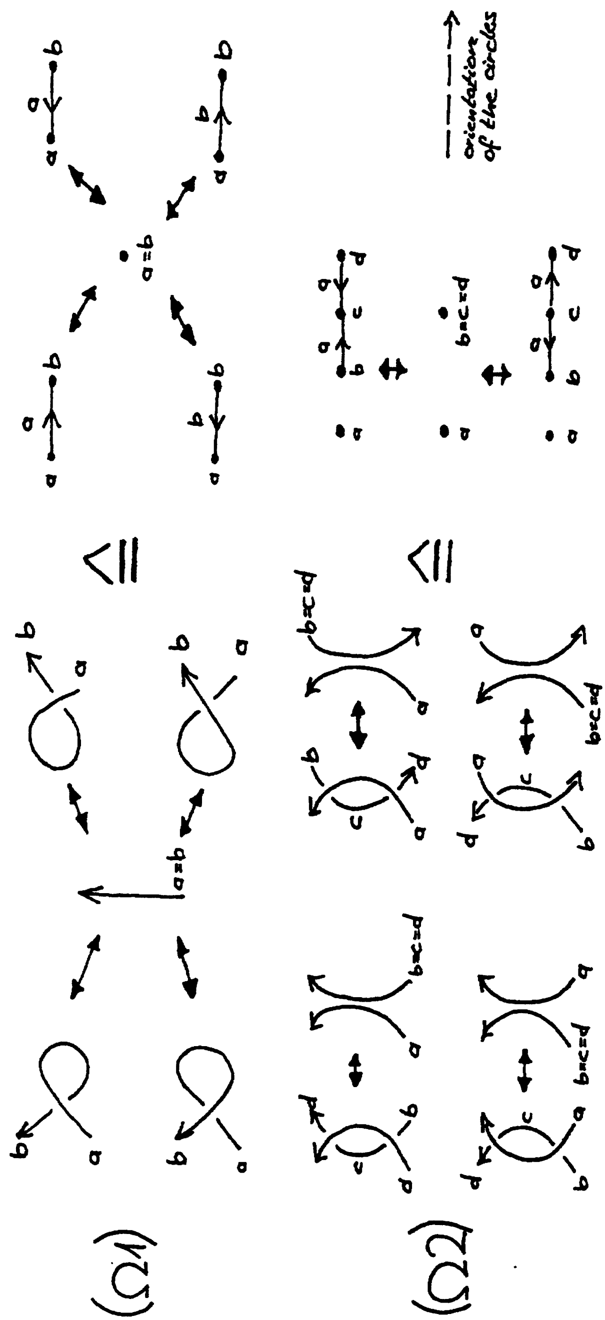

Fig. 1 Two of the Reidemeister moves for link projections in $\mathcal{K}$ and for LOGs in $\mathcal{C}$

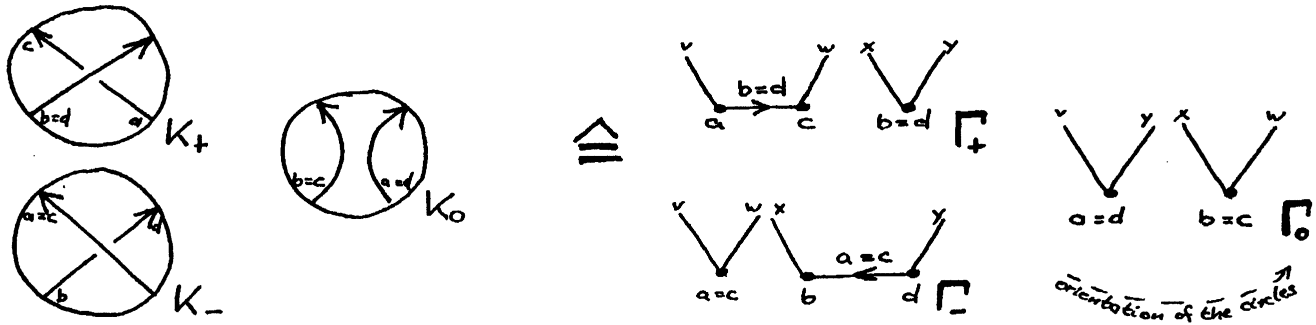

Fig. 2 The difference between K_+, K_- and K_0 of $\mathcal{K}$ and between Γ_+, Γ_- and Γ_0 of $\mathcal{C}$

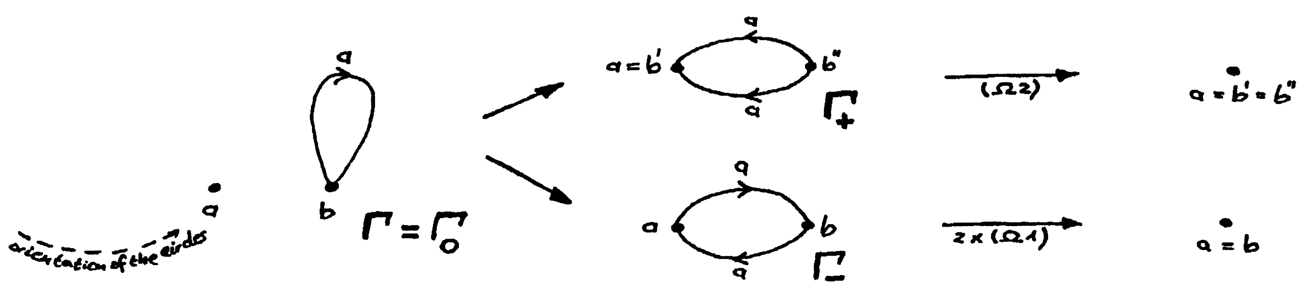

Fig. 3 The switch relation determining $Q(\Gamma)$

($\Omega 1$) If Γ contains an edge whose label is also the label of one of its bounding vertices, cancel this edge and identify the two bounding vertices.

$(\Omega 1)^{-1}$ Separate an arbitrary vertex a into two vertices a', a'' and insert between them an edge with arbitrary orientation whose label is either a' or a''.

($\Omega 2$) If Γ contains two edges with the same label and opposite orientations which have a common boundary vertex, cancel these edges and identify the three bordering vertices.

$(\Omega 2)^{-1}$ Separate an arbitrary vertex into three vertices and join them by a pair of edges with opposite orientations such that the resulting component of the graph is again a circle. Label the edges by the same arbitrary label from the set of vertex labels of Γ.

There is also an algebraic analogue of ($\Omega 3$). But remarkably enough we can prove our theorem without making use of this move. Hence we omit its description, as it is rather lengthy because we have to distinguish between the different combinations of orientations of the arcs.

Let Γ be a LOG two of whose components are oriented intervals with starting vertices c, d and end vertices a, b. Γ is not in $\mathcal{C}$, it is just a helping device for constructing a triple $(\Gamma_+, \Gamma_-, \Gamma_0)$ of LOGs in $\mathcal{C}$ which are analogous to (K_+, K_-, K_0) as follows:

Γ_+ Identify b and d and insert between a and c an edge labelled $b = d$ which is oriented *with* the orientation of the obtained circle.

Γ_- Identify a and c and insert between b and d an edge labelled $a = c$ which is oriented *against* the orientation of the obtained cirle.

Γ_0 Identify a and d on the one hand and b and c on the other hand.

The extremal case may occur where one (or both) of the intervals does not contain any edges, hence its starting and end vertex are the same. To insert an edge between them means here to adjoin a loop to the vertex.

The relation between Γ_+, Γ_- and Γ_0 is called a *switch relation.* Exchanging one of these three LOGs by any of the other two is called a *switch.*

Let $\Gamma(T) \in \mathcal{C}$ be the graph corresponding to the projection of the trivial knot without any crossing. It consists of a single vertex.

Now we have the tools to generalize theorem 1.1 to LOGs in $\mathcal{C}$. Unfortunately, the HOMFLY polynomial for LOGs in $\mathcal{C}$ is no longer highly distinctive as it is for links in $\mathbb{R}$. It only detects the parity of the number of components:

THEOREM 1.2.— *Let Q be a map from $\mathcal{C}$ to a commutative ring R of Laurent polynomials in the variables ℓ and m which satisfies:*

(1) $Q(\Gamma(T)) = 1$,

(2) $\ell \cdot Q(\Gamma_+) + \ell^{-1} \cdot Q(\Gamma_-) + m \cdot Q(\Gamma_0) = 0$

(3) Q is invariant under Reidemeister moves.

Then $m^2 = (\ell + \ell^{-1})^2$ in R, and Q is of the following form:

$$Q(\Gamma) = \begin{cases} 1 & \text{if } \Gamma \in \mathcal{C} \text{ has an odd number of components} \\ -\dfrac{\ell + \ell^{-1}}{m} & \text{if } \Gamma \in \mathcal{C} \text{ has an even number of components.} \end{cases}$$

This means that the HOMFLY polynomial cannot be generalized to all LOGs without getting (almost) trivial. Hence a purely algebraic version of theorem 1.1 is not possible. The HOMFLY polynomial is not even an interesting invariant of the class $\mathcal{C}$ of LOGs. This implies that it is neither an interesting invariant of LOTs by the following argument:

Assume that $S : \mathcal{A} \to \mathbf{Z}[\ell^{\pm 1}, m^{\pm 1}]$ is a map which generalizes the HOMFLY-polynomial to some set $\mathcal{A}$ of LOGs which contains the set of LOTs. Let $\Gamma_+ \in \mathcal{A}$ be a LOT. Then Γ_0 in many cases has two components where one component is homotopic to a circle, and every circle appears in this way. Hence the class $\mathcal{P}$ has to be contained in $\mathcal{A}$. Let $\{\Gamma_0^i \mid i \in I\}$, I an index set, be the set of all graphs in $\mathcal{C}$ obtained from Γ_0 by orienting the components. Then $S(\Gamma_0)$ must be a quotient of $Q(\Gamma_0^i)$ for all $i \in I$. But then theorem 1.2 says that S is (almost) trivial.

Here is an overview of the proof of theorem 1.2. We start by establishing the relation between the variables:

LEMMA 1.3.— *Suppose there is a map Q from $\mathcal{C}$ to some commutative ring R of Laurent polynomials in the variables ℓ and m which satisfies (1), (2) and (3) of theorem 1.2. Then $m^2 = \left(\ell + \ell^{-1}\right)^2$ in R.*

Next we show the existence of such a polynomial:

LEMMA 1.4.— *Any* LOG $\Gamma \in \mathcal{C}$ *can be reduced by switches and Reidemeister moves to the trivial* LOG $\Gamma(T) \in \mathcal{C}$.

For link diagrams, this is obvious: Any link can be trivialized by switching crossings. Hence the assertion is true for those LOGs in $\mathcal{C}$ which can be realized as links in $\mathbb{R}^3$. We have to show that the other LOGs in $\mathcal{C}$, too, can be reduced to the trivial LOG $\Gamma(T)$. The proof is by induction on the number of edges.

We then prove the main theorem by calculating the actual values of Q. For a given LOG Γ, lemma 1.4 provides us with a sequence of switches which reduces Γ to the trivial LOG $\Gamma(T)$. We go through this sequence in the inverse direction, starting with $\Gamma(T)$ whose polynomial is given by condition (1). At every step, we calculate Q using formula (2) and replacing m^2 by $\left(\ell+\ell^{-1}\right)^2$, as we may because of lemma 1.3.

In the last section we mention briefly some results about geometric analogues of the LOGs of $\mathcal{C}$.

2. Proofs

Proof of lemma 1.3

The stated equality arises from the following example.

Let Γ be the LOG depicted in figure 3. Take it as Γ_0. Then the associated Γ_+ and Γ_- both can be reduced to the trivial graph by Reidemeister moves. By condition (1) we know that Q takes the value 1 on these. Hence, by condition (2), we get

$$Q(\Gamma) = -m^{-1}\left(\ell\cdot 1+\ell^{-1}\cdot 1\right) = -\frac{\ell+\ell^{-1}}{m}\,.$$

Now consider Γ' depicted in figure 4. By the same argument as for Γ, we get

$$Q(\Gamma') = -\frac{\ell+\ell^{-1}}{m}\,.$$

Now consider Γ as Γ_- and Γ' as Γ_+, see figure 4. The associated Γ_0 is trivial. Hence condition (2) yields

$$\begin{aligned} 0 &= \ell Q(\Gamma_+) + \ell^{-1}Q(\Gamma_-) + mQ(\Gamma_0) \\ &= -\ell\cdot\frac{\ell+\ell^{-1}}{m} - \ell^{-1}\cdot\frac{\ell+\ell^{-1}}{m} + m\cdot 1 = -m^{-1}\left(\ell+\ell^{-1}\right)^2 + m \end{aligned}$$

$$\iff m^2 = \left(\ell+\ell^{-1}\right)^2.$$

Proof of lemma 1.4

Let n be the number of edges in Γ. We use induction on n. For $n = 0$, Γ consists of k vertices. If $k = 1$, then $Q(\Gamma) = 1$ because of condition (1). For $k > 1$ regard Γ as Γ_0. The associated Γ_+ and Γ_- can be reduced by (Ω1) to the graph which only consists of $k - 1$ vertices, see figure 5. Continueing this process on Γ_+ and Γ_-, we finally reach the trivial graph $\Gamma(T)$.

Now assume all graphs with at most n edges can be reduced to $\Gamma(T)$. Let Γ contain $n + 1$ edges. We have to distinguish several cases:

Case 1

A component of Γ contains a vertex and an edge which have the same label. Then there is an innermost of such pairs, that is one that bounds a part Γ' of Γ which does not contain such a pair. Let a be the label of the chosen pair. Choose an arbitrary edge b in Γ'. Depending on the orientation of b, regard Γ as Γ_+ or as Γ_-. The associated Γ_0 has only n edges, hence by assumption it can be reduced to $\Gamma(T)$. Γ_- or Γ_+, respectively, has still $n + 1$ edges, but the part corresponding to Γ' has one edge less than Γ' itself. We repeat this operation until there are no edges left in the the part corresponding to Γ'. Then the vertex and the edge a lie next to each other and hence can be cancelled by (Ω1). We are left with only n edges, and the assumption holds.

Case 2

There is no such pair as in the first case. Choose an arbitrary component $\overline{\Gamma}$ of Γ. Then all the vertices carrying the same labels as the edges in $\overline{\Gamma}$ lie in other components of Γ. Hence all the edges of $\overline{\Gamma}$ can be transported to these other components by switches. In each switch relation, Γ_0 has only n edges so that the assumption holds for it. We are left with $\overline{\Gamma}$ only containing a single vertex which we call a.

Case 2.1.— Γ contains an edge labelled a. Then regard Γ as Γ_0. Figure 6 shows the switch relation in the case where a is oriented against the orientation of the circle in which it lies. For the other orientation, the switch relation is analogous. Both Γ_+ and Γ_- obtain a new edge by this operation, so that they both have $n + 2$ edges. But now we can cancel two edges by Reidemeister moves, hence we come down to n edges per graph, and the assumption holds.

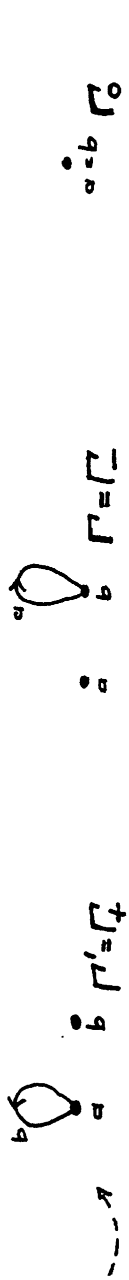

Fig. 4 The switch relation between Γ and Γ' yielding the equality stated in lemma 1.3

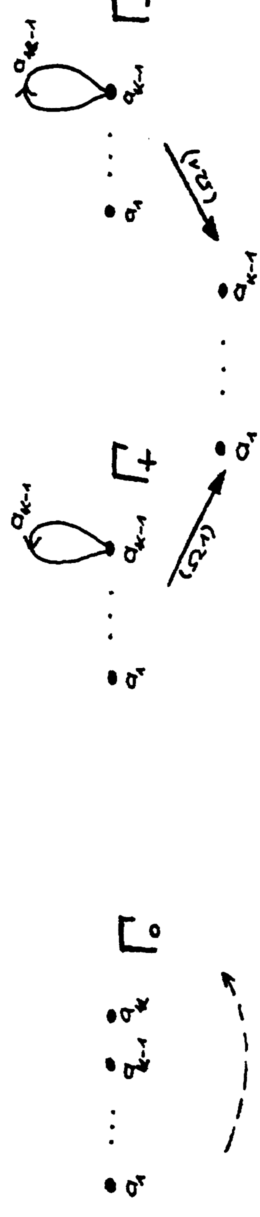

Fig. 5 Reduction of a $\Gamma \in C$ without edges

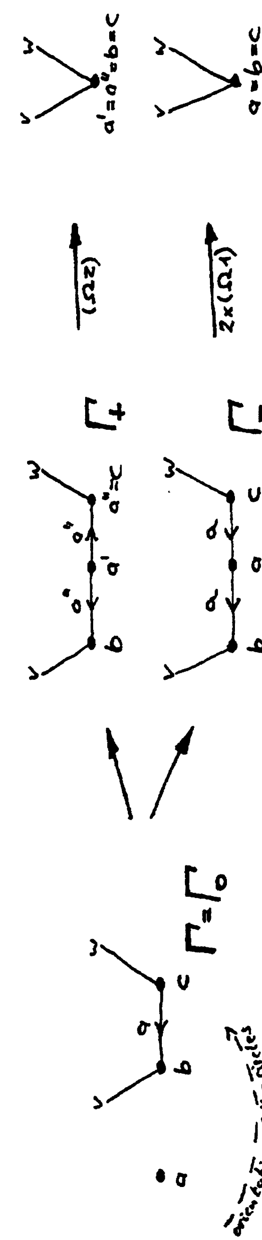

Fig. 6 Reduction of a $\Gamma \in C$ which fits case 2.1

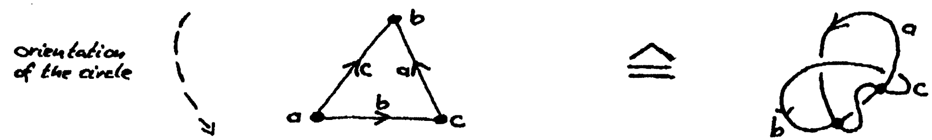

Fig. 7 An example of a $\sigma\tau$ diagram to a non-link LOG

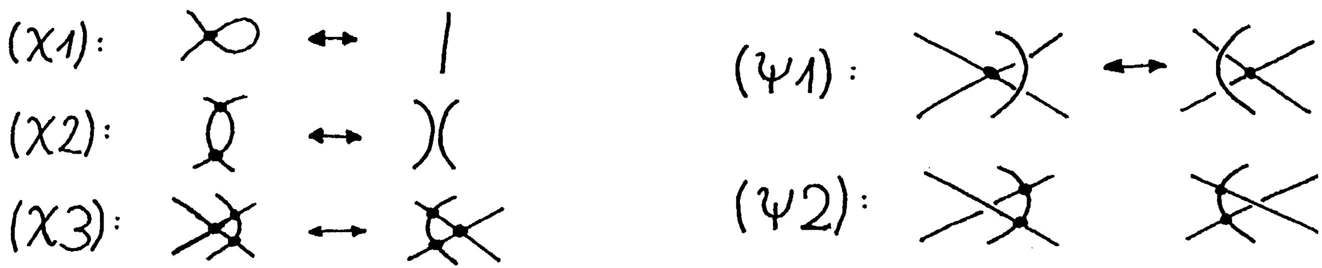

Fig. 8 Analogues of the Reidemeister moves for $\sigma\tau$ diagrams

Case 2.2.— Γ does not contain an edge labelled a. Then choose a second component and transport all its edges to other components by switches. Doing this, the first component does not obtain any edges, as there is no edge a in the second component. If the second component fits in case 2.1, we continue our series of operations there. If not, we treat a third component and so on. If none of the components fits in case 2.1, we are left with a graph which does not contain any edges at all. In fact, we do not have to go that far: We can stop after the first reduction of the number of edges, as the assumption then holds.

Proof of theorem 1.2

The proof is by induction on (n, r), where $n(\Gamma)$ is the number of edges of a given LOG $\Gamma \in \mathcal{C}$, and $r(\Gamma)$ is the number of switches needed to get the trivial LOG $\Gamma(T)$ by the algorithm described in the proof of lemma 1.4. If we follow the proof of lemma 1.4, we see that after any switch the two resulting LOGs have either fewer edges than the one we started with, or they have the same number of edges while the number of switches needed to get to the trivial LOG is reduced by one. In case 2.1 this is only achieved after one or two Reidemeister moves.

For $(n, r) = (0, 0)$ we have $\Gamma = \Gamma(T)$ and $Q(\Gamma) = 1$ by condition (1), hence the assumption holds. Now let $\Gamma \in \mathcal{C}$ be a LOG with n edges, and r switches are needed to get the trivial LOG by the algorithm described in the proof of lemma 1.4. Assume that any LOG of $\mathcal{C}$ with fewer edges or with the same number of edges but fewer switches needed satisfies the assertion of theorem 1.2. Then we know from the discussion above that the two LOGs achieved from Γ by the switch required by the algorithm fulfil the assumption. Hence we can calculate $Q(\Gamma)$ from their polynomials. If for example $\Gamma = \Gamma_+$ with an odd number of components, then $Q(\Gamma_-) \equiv 1$, $Q(\Gamma_0) \equiv -(\ell + \ell^{-1})/m$, and (2) implies:

$$Q(\Gamma_+) \equiv -\ell^{-1}(\ell^{-1} - (\ell + \ell^{-1})) = 1\,.$$

There are another five cases to check: Γ can be Γ_+, Γ_- or Γ_0, where each time we have to distinguish between Γ having an odd or even number of components.

Appendix: Geometric Realization of non-link LOGs

In [2] Fenn, Rimányi and Rourke generalized the notion of the classical braid groups and introduced the *permutation braid groups.* They are generated by the elementary braids σ_i which generate the braid groups, and additional elements τ_i which we get from σ_i by replacing the crossing by a black point (fig. 8). Defining relations are the braid forms of the moves (Ω2) and (Ω3) and the moves (X2), (X3), (Ψ1) and (Ψ2) depicted in figure 8.

LOGs which cannot be realized as link projections can be realized as a closed permutation braid: try to draw a link projection which realizes a presentation of a non-link group. You will find yourself having to get on the other side of an arc without any relation left that tells you to cross it. In this case introduce a crossing of the τ_i type. We call such a diagram a *$\sigma\tau$-diagram.* For an example see figure 7.

Using the same argument as Alexander used in order to show that every tame link in $\mathbb{R}^3$ can be represented as a closed braid (see for instance [6]), one can show that any $\sigma\tau$-diagram can be drawn in the form of a closed permutation braid. When reading off the LOG from a $\sigma\tau$-diagram, the crossings of the τ_i type are simply ignored. It is easy to show that the moves depicted in figure 8 do not change the associated LOG.

In [5], it was shown that this list is in fact complete:

THEOREM .— *Two $\sigma\tau$ diagrams represent the same* LOG *if and only if they differ by a finite sequence of moves* (X1), (X2), (X3), (Ψ1) *and* (Ψ2).

Hence the classes of LOGs modulo the classical Reidemeister moves which we treat in this paper are in one-to-one correspondence with closed permutation braids, and our result is as well valid for these.

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