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On the structure of certain Weingarten surfaces with boundary a circle(*)

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RÉSUMÉ. — Nous donnons une caractérisation d'un type particulier de surfaces de Weingarten paramétrées par un disque, en supposant celles-ci bordées par un cercle rond. Dans ce travail, nous généralisons plusieurs résultats établits par W. Meeks, H. Rosenberg et les auteurs concernant les surfaces de courbure moyenne constante.

ABSTRACT. — A characterization of a special type of Weingarten disk-type surfaces is provided when they have a round circle as boundary. The results in this paper extend previous ones established by W. Meeks, H. Rosenberg and authors where the considered surfaces were assumed to have constant mean curvature.

1. Introduction

We study in this paper a certain class of surfaces M in $\mathbb{R}^3$ satisfying a Weingarten relation of the form

$$H = f(H^2 - K) \tag{1}$$

where H is the mean curvature, K is the Gaussian curvature and f is a real smooth function defined on a interval $[-\varepsilon\,,\,\infty)$, $\varepsilon > 0$.

Furthermore, we require that f satisfies the inequality

$$4t\big(f(t)\big)^2 < 1\,. \tag{2}$$

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We call such a function f, elliptic, when it satisfies (2). The reason for this denomination is that equation (1) and inequality (2) give rise to a fully nonlinear elliptic equation. We call M a special surface when M satisfies $H = f(H^2 - K)$ for f elliptic. They have been studied by Hopf [8], Hartman and Wintner [7], Chern [5] and Bryant [3]. Here, we extend some results for constant mean curvature surfaces obtained in [2] and [6], when M is topologically a disk. Precisely we prove the following theorems.

THEOREM 1.— *Let M be a disk type special surface immersed in $\mathbb{R}^3$. Assume ∂M is a circle S^1 of radius 1. Suppose f is analytic with $f(0) > 0$. Then:*

a) $f(0) \leq 1$,

b) if $f(0) = 1$, M is a halfsphere.

THEOREM 2.— *Let M be a disk type special surface embedded in $\mathbb{R}^3$ Assume ∂M is a circle S^1 of radius 1 contained in the horizontal plane $\mathcal{H} = \{z = 0\}$. Suppose $f > 0$, $f(0) > 0$ and M cuts transversely $\mathcal{H}$ along ∂M. Then M is a spherical cap.*

We remark that the ellipticity condition (2) on M allow us to apply maximum principle (for special surfaces) and Alexandrov reflection principle techniques as it was applied in [6] and [10], for constant mean curvature surfaces (see Hopf's book [8] for further details). Futhermore, we notice that R. Bryant constructed a global quadratic form Q on a surface M satisfying (1) such that the zeros of Q are the umbilical points of M (see [3]). These facts emphasize the analogy between special surfaces and constant mean curvature surfaces. Now we state and prove the *maximum principle for special Weingarten surfaces in* $\mathbb{R}^3$ satisfying (1) and (2) in the form we shall need: if M_1, M_2 are tangent at p, M_1, on one side of M_2 near p, both M_1, M_2 satisfying (1) and (2) with respect to the same normal N at p then $M_1 = M_2$ near p. By a standard argument $M_1 = M_2$ everywhere.

1.1 Interior maximum principle

Suppose M_1, M_2 are C^2 surfaces in $\mathbb{R}^3$ which are given as graphs of C^2 functions u, $v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$.

Suppose the tangent planes of both M_1, M_2 agree at a point (x, y, z); i.e. $T_{(x,y,z)}M_1 = T_{(x,y,z)}M_2$ for $z = u(x, y) = v(x, y)$, $(x, y) \in \Omega$.

Let $H(N_1)$ and $H(N_2)$ be the mean curvature functions of u and v with respect to unit normals N_1 and N_2 that agree at (x, y, z). Let K_i, be the Gaussian curvature of M_i, $i = 1, 2$.

Suppose M_i satisfy

$$H(N_i) = f(H_i^2 - K_i)\,, \quad i = 1, 2\,,$$

for f satisfying (2).

If $u \leq v$ near (x, y) then $M_1 = M_2$ near (x, y, z), i.e. $u = v$ in a neighbourhood of (x, y).

1.2 Boundary maximum principle

Suppose M_1, M_2 as in the statement of the interior maximum principle with C^2 boundaries B_1, B_2 given by restrictions of u and v to part of the boundary $\partial\Omega$.

Suppose $T_{(x,y,z)}M_1 = T_{(x,y,z)}M_2$ and $T_{(x,y,z)}B_1 = T_{(x,y,z)}B_2$ for $z = u(x, y) = v(x, y)$, with (x, y, z) in the interior of both B_1 and B_2.

Suppose M_1, M_2 satisfy (1) and (2) with respect the same normal N at (x, y, z).

If $u \leq v$ near (x, y) then $M_1 = M_2$ near (x, y, z), i.e. $u = v$ in a neighbourhood of (x, y).

2. Proof of the interior and boundary maximum principle

Clearly, by applying a rigid motion of $\mathbb{R}^3$ which does not change the geometry of the statements, we may suppose the tangent planes of both M_1, M_2 at (x, y, z) are the horizontal xy plane $P = \{z = 0\}$, and the unit normals N_1, N_2 at (x, y, z) are equal to $N = (0, 0, 1)$.

First, we fix some notations. We denote

$$p_1 = \frac{\partial u}{\partial x}\,, \quad q_1 = \frac{\partial u}{\partial y}$$

$$p_2 = \frac{\partial v}{\partial x}\,, \quad q_2 = \frac{\partial v}{\partial y}\,,$$

$$r_1 = \frac{\partial^2 u}{\partial x^2}\,, \quad \tau_1 = \frac{\partial^2 u}{\partial y^2}\,, \quad s_1 = \frac{\partial^2 u}{\partial x\, \partial y}$$

$$r_2 = \frac{\partial^2 v}{\partial x^2}\,, \quad \tau_2 = \frac{\partial^2 v}{\partial y^2}\,, \quad s_2 = \frac{\partial^2 v}{\partial x\, \partial y}\,.$$

With this convention the normals N_1 and N_2 are given by

$$N_i = \frac{1}{(1+p_i^2+q_i^2)^{1/2}}(-p_i, -q_i, 1), \quad i = 1, 2\,.$$

The mean curvature H_i and the Gaussian curvature K_i are given by

$$\begin{aligned} 2H_i &= \frac{1}{(1+p_i^2+q_i^2)^{3/2}}\left((1+p_i^2)\tau_i - 2p_iq_is_i + (1+q_i^2)r_i\right) \\ K_i &= \frac{1}{(1+p_i^2+q_i^2)^{2}}(r_i\tau_i - s_i^2) \end{aligned}$$

for $i = 1$, 2.

We may write equation (1) for M_1 and M_2 in the following way

$$F(p_i, q_i, r_i, s_i, \tau_i) = H_i - f(H_i^2 - K_i) = 0 \tag{3}$$

for $i = 1$, 2, where F is a C^1 function in the p, q, r, s, τ variables. We fix $(x, y) \in \Omega$ and we define for $t \in [\,0\,,\,1\,]$:

$$\begin{aligned} \alpha(t) = F\big(tp_1 + (1-t)p_2\,,\, tq_1 + (1-t)q_2\,,\, tr_1 + (1-t)r_2\,,& \\ ts_1 + 1(1-t)s_2\,,\, t\tau_1 + (1-t)\tau_2\big)\,.& \end{aligned} \tag{4}$$

Let $w = u - v$.

By applying the mean value theorem, using equation (3) and differentiating equation (4) we are led to the linearized operator on Ω defined by

$$\begin{aligned} Lw := \frac{\partial F}{\partial r}(\xi)\frac{\partial^2 w}{\partial x^2} + \frac{\partial F}{\partial s}(\xi)\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial F}{\partial \tau}(\xi)\frac{\partial^2 w}{\partial y^2} + \\ + \frac{\partial F}{\partial p}(\xi)\frac{\partial w}{\partial x} + \frac{\partial F}{\partial q}(\xi)\frac{\partial w}{\partial y} = 0 \end{aligned} \tag{5}$$

where

$$\xi = (p, q, r, s, \tau)$$

$$p = cp_1 + (1-c)p_2\,, \quad q = cq_1 + (1-c)q_2$$

$$r = cr_1 + (1-c)r_2\,, \quad s = cs_1 + (1-c)s_2\,, \quad \tau = c\tau_1 + (1-c)\tau_2$$

for $0 < c(x,y) < 1$. Notice that the principal part of L is given by the symmetric matrix

$$A = A(p,q,r,s,\tau) = \begin{bmatrix} \dfrac{\partial F}{\partial r} & \dfrac{1}{2}\dfrac{\partial F}{\partial s} \\ \dfrac{1}{2}\dfrac{\partial F}{\partial s} & \dfrac{\partial F}{\partial \tau} \end{bmatrix}$$

Computations show that if $p = q = 0$ then trace $A = 1$ and

$$\det A = \frac{1}{4}\left(1 - 4t\big(f'(t)\big)^2\right),$$

where

$$t = \left[\frac{(1+p^2)\tau - 2pqs + (1+q^2)r}{2(1+p^2+q^2)^{3/2}}\right]^2 - \frac{1}{(1+p^2+q^2)^2}\,(r\tau - s^2)\,. \qquad (6)$$

Now, consider in formula (6)

$$p = cp_1 + (1-c)p_2\,, \quad q = cq_1 + (1-c)q_2$$
$$r = cr_1 + (1-c)r_2\,, \quad s = cs_1 + (1-c)s_2\,, \quad \tau = c\tau_1 + (1-c)\tau_2$$

where p_i, q_i, r_i, s_i and τ_i are varying in a neighbourhood of (x,y) and c is varying in the interval $[\,0\,,\,1\,]$. We see easily that the non negative quantity $t = t(p,q,r,s,\tau)$ is bounded from above. Hence $1 - 4t\big(f'(t)\big)^2 \geq \mu > 0$ in this neighbourhood (c is varying between 0 and 1), for some positive real number μ. As $p_i = q_i = 0$ at (x,y), $i = 1$, 2, by continuity we have that in a neighbourhood V of (x,y) the matrix $A(\xi)$ is positive definite. Furthermore, there is a positive real number λ_0 such that

$$\frac{\partial F}{\partial r}(\xi)\eta_1^2 + \frac{\partial F}{\partial s}(\xi)\eta_1\eta_2 + \frac{\partial F}{\partial \tau}(\xi)\eta_2^2 \geq \lambda_0(\eta_1^2 + \eta_1^2)$$

for any (x,y) in V and any real numbers η_1, η_2. Consequently, L is a linear second order uniformly elliptic operator with bounded coefficients in a neighbourhood of (x,y). The same conclusion holds if (x,y) is a boundary point as in the hypothesis of the boundary maximum principle statement.

Finally we have in a neighbourhood of (x,y)

$$Lw = 0\,, \quad w \leq 0\,, \quad w(x,y) = 0\,.$$

If (x, y) is an interior point then $w = u - v = 0$ in a neighbourhood of (x, y), by applying the interior maximum principle of Hopf.

If (x, y) is a boundary point lying in the interior of a C^2 portion contained in Ω, then w attains again a local maximum at (x, y) with $(\partial w/\partial \nu)(x, y) = 0$, where ν is the exterior unit normal to Ω at (x, y). This implies by using the boundary maximum principle of Hopf that $w = 0$ in a neighbourhood of (x, y), as desired. We conclude the proof of the maximum principle for special Weingarten surfaces in $\mathbb{R}^3$.

We remark that the maximum principle above leads to an *Alexandrov theorem for special Weingarten surfaces*. That is, a closed embedded special Weingarten surface M given by equation (1) with respect to a unit global normal N, for f elliptic, is a sphere. Hence, $f(0) \neq 0$ and M is a sphere of radius $R = 1/\left|f(0)\right|$.

3. Proof of Theorem 1

We consider M an immersed smooth special surface in $\mathbb{R}^3$ and N an unit normal vector field. We denote by $\langle \cdot\,, \cdot \rangle$ the inner product in $\mathbb{R}^3$ and by ∇ the standard covariant derivative in $\mathbb{R}^3$. The mean curvature vector $\vec{H}$ of M at p is given by

$$\vec{H}(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2} N(p)$$

where $\lambda_1(p)$, $\lambda_2(p)$ are the principal curvatures of M at p (respecting to N).

3.1 Proof of assertion a)

Suppose first that there is an umbilical boundary point $p \in \partial M$. Denote by v a unit tangent field along $\partial M = S^1$. Then,

$$f(0) = H(p) = \left\langle \nabla_v v, N \right\rangle_p \leqq 1\,. \tag{3.1}$$

Suppose now there are no umbilical points on the boundary. Notice that the set U of umbilical points of M is finite. Otherwise M is a spherical cap and $f(0) \leq 1$. This follows from the proof of theorem 3.2 of H. Hopf's book [8, p. 142], and from the fact that M is compact.

Let $\lambda_1, \lambda_2 : M \backslash U \to \mathbb{R}$ be the principal curvature functions with $\lambda_1 < \lambda_2$ on $M \setminus U$. Let us prove first that ellipticity condition yields

$$\lambda_2 > f(0) \quad \text{on} \quad M \setminus U\,. \tag{3.2}$$

Indeed,

$$\lambda_2 = H + \sqrt{H^2 - K} = f(H^2 - K) + \sqrt{H^2 - K}$$

and the ellipticity condition

$$4t\big(f'(t)\big)^2 < 1$$

assures

$$g(t) = f(t) + \sqrt{t}$$

is a monotonic increasing function for $t \geq 0$.

Denote by $\mathcal{F}_2$ the principal line distribution on $M \setminus U$ associated to the principal curvature λ_2. Clearly, there is a point $p \in \partial M$ where $\mathcal{F}_2$ is tangent to ∂M at p, i.e. $T_p\,\partial M = \mathcal{F}_2(p)$. If not we would obtain a line foliation of M transverse to ∂M and finite number (possibly none) of singularities of negative indices (see [8]); this is impossible since M has disk topological type. Choose then $p \in \partial M$ such that $T_p\,\partial M = \mathcal{F}_2(p)$.

Clearly

$$\lambda_2(p) = \langle \nabla_v v, N \rangle_p \leq 1 \tag{3.3}$$

by inequalities (3.1), (3.2), (3.3)

$$f(0) \leq 1\,.$$

This proves assertion a). □

3.2 Proof of assertion b)

Notice first that there is an extension for M beyond ∂M satisfying $H = f(H^2 - K)$, f elliptic and analytic. This is so, because of the boundary regularity for the underlying analytic elliptic partial differential equation ([4], [11]). If $f(0) = 1$ we will show that there are infinitely many umbilical points in ∂M. The resulting non-discreteness of U will so imply M is totally umbilical [8].

Suppose by absurd ∂M has finitely many umbilical points. Observe that the foliation $\mathcal{F}_2$ defined on $M \setminus U$ is transverse to $\partial M \setminus U$. To prove this, suppose $p \in \partial M \setminus U$ is such that $\mathcal{F}_2(p)$ is tangent to $\partial M \setminus U$. By equations (3.2) and (3.3), we derive a contradiction because $f(0) < \lambda_2(p) \leq 1$.

Suppose now, there are no umbilical points on the boundary ∂M. This means (by what we have just proved) that $\mathcal{F}_2$ is transverse to ∂M. In this case $\mathcal{F}_2$ may be seen as a foliation of M with finite number of singularities with negative index [8]. This is a contradiction since by our hypothesis M is a topological disk.

For the case where ∂M has a non zero finite number of umbilical points, consider a umbilical point $p \in \partial M$, and let $\widetilde{M}$ to be an extension of M beyond the boundary ∂M.

We first see that p is a singularity of $\mathcal{F}_2$ with negative index and finite number of separatrices, all of them smooth at p. Moreover, there is at least one separatrix going from p to the interior of M. In other words there is at least one separatrix such that, its interior tangent vector at p, say u, satisfies $\langle u, \eta\rangle > 0$, where η is the interior co-normal of M at p. This is a consequence of a straightforward computation using Bryant holomorphic quadratic form [3] such that, in a neighbourhood of p, the foliation is diffeomorphically equivalent to the standard foliation

$$\operatorname{Im} z^n (\mathrm{d}z)^2 = 0$$

on the complex z-plane.

Observe now that the foliation $\mathcal{F}_2$ on $M \setminus U$ is topologically equivalent to a foliation with finite number of singularities on M. Some of them are interior singularities on M. Others are in the boundary ∂M. Those which are in the boundary have separatrices (at least one) coming tranversally to ∂M (fig. 1). In order to see this situation is topologically impossible, we just recall M is a topological disk and use double construction to obtain a foliation of a topological sphere S^2 with finite number of singularities, all of them with negative index.

This conclude the proof of Theorem 1. □

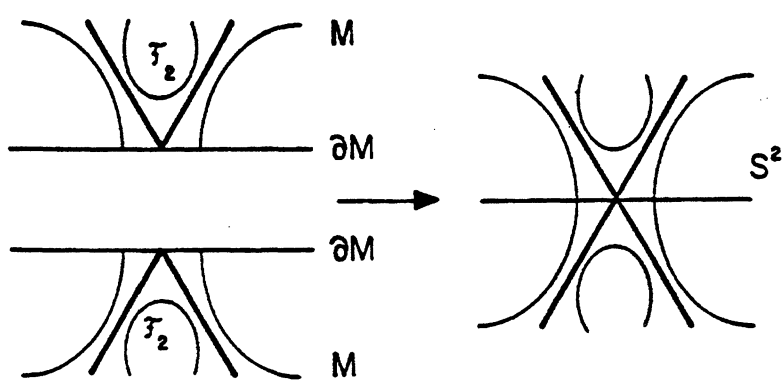

Fig. 1

4. Proof of Theorem 2

Suppose without loss of generality that M is locally contained in the upper halfspace $\mathcal{H}^+ = \{z \geq 0\}$ in a neighbourhood of ∂M. We also identify ∂M with the unit circle S^1 centered at the origin of $\mathcal{H}$.

We first show that boundary roundness determines the behavior of the mean curvature vector $\overrightarrow{H}$ along the boundary (in fact, only convexity of ∂M is required). Precisely we state the follows result.

CLAIM 1.— *Let $p \in \partial M$. Then $\langle \overrightarrow{H}(p)\,,\, p\rangle < 0$.*

Proof of Claim 1

Suppose first that there is a umbilical point $p \in \partial M$. Take a unit vector field v tangent to ∂M. Then umbilicity yields

$$H(N) = \langle \nabla_v v\,,\, N\rangle_p$$

If $N = \overrightarrow{H}/|H|$ then the mean curvature H is positive and $\langle \nabla_v v\,,\, N\rangle = |H| > 0$. So $\langle -p\,,\, M\rangle > 0$, as desired, for $\nabla_v v = -p$ is the acceleration vector of S^1.

For the case where there is no umbilical points on ∂M we recall that the foliation $\mathcal{F}_2$, parallel to the line field associated to the bigger principal curvature λ_2 defined over $M \setminus U$, has to be tangent to $\partial M = S^1$ in some point p. Let $p \in \partial M$ be such that $\mathcal{F}_2(p)$ is tangent to ∂M. Clearly

$$\lambda_2(p) = \left\langle \nabla_v v\,,\, \frac{\overrightarrow{H}}{|H|}\right\rangle_p > 0\,.$$

Notice that Claim 1 means the following: the orthogonal projection of the mean curvature vector $\overrightarrow{H}$ on $\mathcal{H}$ points into the interior of the planar domain D contained in $\mathcal{H}$ bounded by ∂M. We will denote D by $\operatorname{int}\partial M$.

We now define $M_1 \subset M$ to be the connected component of $M \cap \mathcal{H}^+$ which contains ∂M.

CLAIM 2.— *$M_1 \cap \mathcal{H} \subset \operatorname{int}\partial M$.*

This follows from Claim 1 and from Alexandrov Reflection Principle techniques used exactly in the same way it was used in the proof of Theorem 1 of [6, p. 337].

Let us denote $C_{f(0)}$ the vertical cylinder on $\mathcal{H}$ over the circle $S_{f(0)}$ of radius $1/f(0)$ centered at the origin.

CLAIM 3.— *There is a point $p \in \partial M$ such that*

$$\langle N\,,\,-p\rangle_p \geq f(0) \quad \textit{for } N = \frac{\overrightarrow{H}}{|H|}\,.$$

This means there is a point $p \in \partial M$ where the surface M has bigger (or equal) inclination respect to xy plane than the small spherical cap of radius $1/f(0)$ bounding ∂M.

Proof of Claim 3

Let $p \in \partial M$ be a point of ∂M where $\mathcal{F}_2(p)$ is tangent to ∂M at p (proof of Claim 1). Then, at this point p we have

$$\langle -p\,,\,N\rangle_p = \langle \nabla_v v\,,\,N\rangle_p = \lambda_2(p) \geq f(0)\,.$$

CLAIM 4.— *If $\operatorname{ext} C_{f(0)}$ denotes the exterior of the cylinder $C_{f(0)}$ (i.e. it is the connected region of $\mathbb{R}^3 - C_{f(0)}$ not containing the origin of $\mathcal{H}$), if $M \cap \operatorname{ext} C_{f(0)} = \emptyset$, then M is a spherical cap.*

Proof of Claim 4

The proof follows by using Claim 3 and the maximum principle (for special surfaces), comparing M_1 with a half sphere of radius $1/f(0)$ (see for instance [1]).

CLAIM 5.— *If $M_1 \cap \operatorname{int} \partial M = \emptyset$, then M is a spherical cap.*

Proof of Claim 5

First notice, if $M_1 \cap \operatorname{int} \partial M = \emptyset$ then, by Claim 2 it follows $M_1 \cap \mathcal{H} = \partial \mathcal{M}$ and M is globally contained in $\mathcal{H}^+$. Now, using Alexandrov Reflection Principle for planes normal to $\mathcal{H}$, we conclude M is rotationally symmetric (see, for instance [10]). Therefore, the round boundary is everywhere parallel to one of the principal curvature directions for M. Now because M is a topological closed disk, we conclude, by the same index reasons as before, that M is totally umbilical. This shows that M is a spherical cap (of radius $1/f(0)$).

We finish the proof of Theorem 2 supposing, by contradiction, that

$$M_1 \cap \left(\operatorname{ext} C_{f(0)}\right) \neq \emptyset \quad \text{and} \quad M_1 \cap \operatorname{int} \partial M \neq \emptyset\,.$$

At this point we may suppose M to be globally transverse to $\mathcal{H}$ without loss of generality. Therefore $M \cap \mathcal{H}$ is a finite collection of closed simple curves of $\mathcal{H}$.

Notice first that under the contradiction hypothesis there should be a curve in $\gamma \in M \cap \mathcal{H} \setminus \partial\mathcal{M}$ which is homotopically non trivial in $\mathcal{H} \setminus \partial\mathcal{M}$. This follows directely from the extended Graph Lemma for special surfaces (Lemma 3, Remark and final Remarks in [2, pp. 12, 14]).

Let $\gamma_L \in M \cap \mathcal{H}$ be the outermost homotopically non trivial curve in $\mathcal{H} \setminus \partial\mathcal{M}$. Observe that γ_L bounds a topological disk $D_L \subset M$. Moreover, D_L is locally contained in the upper half-space $\mathcal{H}^+$ along its boundary γ_L. In fact, if the disk D_L were locally contained in the lower halfspace $\mathcal{H}^-$ we would have a connected component, say C, of $M \setminus (M \cap Int\partial M)$ such that $C \cap \mathcal{H}$ contains at least two distinct closed curves both of them homotopically non trivial in $\mathcal{H} \setminus \partial\mathcal{M}$. This is a consequence of the fact that M_1 is locally contained in $\mathcal{H}^+$ along its boundary together with the hypothesis that the mean curvature vector $\vec{H}$ never vanishes and the maximum principle. This would lead to a contradiction by applying Alexandrov Reflection Principle by vertical planes as in [6].

Notice that $D_L \cap \mathcal{H}$ is the union of γ_L with null homotopic closed curves on $\mathcal{H} \setminus \gamma_L$, and as a consequence of the Graph Lemma proved in [2, Lemma 3, pp. 12-14, Remark, p. 14] each curve on $D_L \cap \mathcal{H} \setminus \gamma_L$ other than γ_L bounds a graph over its Jordan interior. We denote the Jordan interior of γ_L in $\mathcal{H}$ by $\operatorname{int}\gamma_L$. Now a standard orientation argument yields (since $H \neq 0$ on M):

$$D_L \cap (\operatorname{int}\gamma_L) = \emptyset\,.$$

So $D_L \cup \operatorname{int}\gamma_L$ is embedded (non smooth over γ_L) compact surface without boundary. Moreover M_1 is clearly contained in the closed compact solid S determined by $D_L \cup \operatorname{int}\gamma_L = \partial S$ (fig. 2).

Let $M_1(\theta)$, $0 \leq \theta \leq 2\pi$, be the 1-parameter family of surfaces obtained by rotating $M_1 = M_1(0)$ around an axis z normal to $\mathcal{H}$ and passing by the center of the round circle S_1 bounding M. Clearly $M_1(\theta) \cap D_L = \emptyset$, for every $\theta \in [0, 2\pi]$. Otherwise there would be a first parameter $\theta_0 > 0$ such that $M_1(\theta_0)$ would be tangent to $D_L \setminus \gamma_L$, and contained inside S, contradicting the maximum principle for special surfaces.

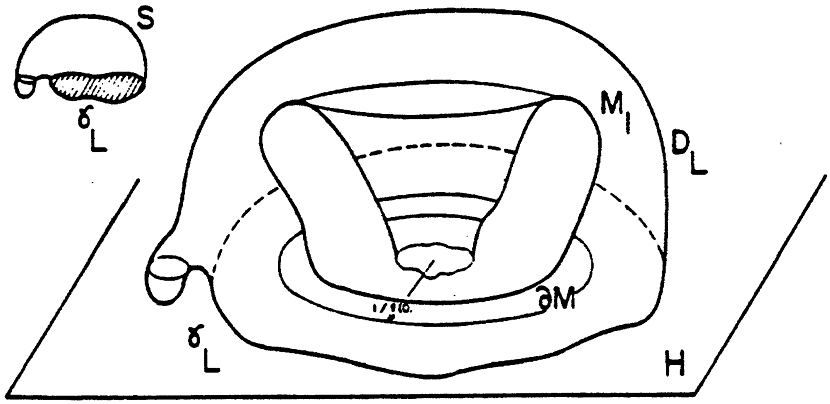

Fig. 2

Now, let $p \in M_1$ be a point of maximum distance of M_1 to the z-axis, contained in the interior of the solid S. The radius of this circle C_1 is bigger than $1/f(0)$ because of the hypothesis of contradiction. Also $D_L \cap D_1 = \emptyset$, where D_1 is the horizontal disk bounding C_1. This is again a consequence of mean curvature orientation and maximum principle.

We now finish the contradiction argument by comparing D_L with a sphere of radius $1/f(0)$ which we can actually introduce through the barrier disk D_1. This proves Theorem 2. □

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