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Persistence of Homoclinic Tangencies for Area-Preserving Maps(*)

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RÉSUMÉ. — Nous prouvons que dans une variété symplectique bidimensionnelle M, l'existence de courbes lisses invariantes dans le monde des applications symplectiques de M est un mécanisme pour créer des ouverts contenant un ensemble dense d'applications exhibant des tangences homocliniques.

ABSTRACT. — In a 2-dimensional symplectic manifold M we show that the presence of smooth invariant curves in the world of symplectic maps of M is a mechanism to create open sets containing a dense set of maps exhibiting homoclinic tangencies.

1. Introduction

In 1970, S. Newhouse [N2] proved the existence of an open set $\mathcal{U} \subset \text{Diff}^s(M)$, $s \geq 2$, where $M$ is a 2-dimensional compact manifold, with the following property: there exists a dense subset of $\mathcal{U}$ such that each $g : M \mapsto$ in this subset exhibits homoclinic tangencies (tangential intersections between the stable set and unstable set, $W^s(p)$ and $W^u(p)$ respectively, of a hyperbolic periodic point $p$). We call such a set $\mathcal{U} \subset \text{Diff}^s(M)$, with the last property, an open set of “persistence of homoclinic tangencies”, from now on OSPHT.

Later, in 1979 [N3], he proved that a mechanism to create this kind of sets is the unfolding of a dissipative homoclinic tangency. More precisely, for every $f \in \text{Diff}^s(M)$, with a homoclinic tangency associated to a dissipative hyperbolic periodic point $p$ ($|\det Df^n(p)| < 1$, where $n$ is the minimal period of $p$), there exists $\mathcal{U}$ an OSPHT such that $f \in \overline{\mathcal{U}}$.

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Here we present a mechanism to generate OSPHT's in the world of symplectic diffeomorphisms; we show that the presence of a smooth invariant curve generates, for nearby maps, this kind of open sets. To be more precise, let $M$ be a 2-dimensional compact manifold with $\omega$ a symplectic 2-form on $M$ and denote by $Diff^s_\omega$ the space of $C^s$ diffeomorphisms that preserve $\omega$, then we have the following result.

**Theorem 1.** Let $f \in Diff^\infty_\omega(M)$ admit a $C^\infty$ closed invariant curve $\gamma$ such that the rotation number $\omega = p(f|_\gamma)$ is irrational. Then for every $s \geq 4$ there exists $U \subset Diff^s_\omega(M)$ an OSPHT such that $f \in \overline{U}$. Moreover, there is a residual subset $V$ of $U$ such that every $f \in V$ has an invariant smooth curve which is accumulated by elliptic points.

The method to prove Theorem 1 is different from the dissipative case. The wild hyperbolic sets mechanism used to produce persistence of homoclinic tangencies is replaced by the rich structure around a smooth invariant curve, obtained from KAM theory [Bo], combined with the following two propositions.

**Proposition 2.** For $f \in Diff^\infty_\omega(M)$ and $\gamma$ a $C^\infty$ invariant curve assume that:

(i) $\omega$ satisfies a diophantine condition: there exist $\beta \geq 0$ and $C > 0$ such that for every $p/q \in \mathbb{Q}$ then

$$|\omega - \frac{p}{q}| > \frac{C}{q^{2+\beta}};$$

(ii) $f$ satisfies a twist condition along $\gamma$ (see Sect. 2),

(iii) there exist $\tilde{U} \subset Diff^s_\omega(M)$, such that for each $g \in \tilde{U}$ there is a continuation curve $\gamma_g$ of $\gamma$ which is invariant by $g$ and with the same rotation number $\omega$.

Then there exists $U \subset \tilde{U}$ an OSPHT and for a residual set in $U$, the continuation curve $\gamma_g$ is the limit of elliptic periodic orbits.

**Remark.** The same conclusion can be obtained in Proposition 2 if we replace the invariant curve $\gamma$ by a collection of disjoint curves $\{\gamma_i\}_{i=0}^{n-1}$ such that $f(\gamma_i) = \gamma_{i+1}$ and $f(\gamma_{n-1}) = \gamma_0$. Just take $f^n$, apply Proposition 2 and pull back $U$ by the map $f \rightarrow f^n$. 

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Proposition 3. — Let \( f \in \text{Diff}_{w}^{\infty}(M) \). Then for each \( s \geq 1 \), we have:

(i) if \( f \) exhibits a \( C^{\infty} \) invariant curve with an irrational rotation number, then for each \( \varepsilon > 0 \) there exists \( \tilde{f} \) \( C^{s}\varepsilon \)-near to \( f \) such that \( \tilde{f} \) exhibits homoclinic tangencies;

(ii) if \( f \) exhibits a homoclinic tangency associated to a hyperbolic periodic orbits, then for each \( \varepsilon > 0 \) there exists \( \tilde{f} \) \( C^{s}\varepsilon \)-near to \( f \) such that \( \tilde{f} \) has a generic (in the KAM sense) elliptic periodic point; in particular \( \tilde{f} \) exhibits \( C^{\infty} \) invariant curves.

The same conclusion of Theorem 1 holds if we replace the assumption of the presence of an invariant curve by the presence of some homoclinic tangency associated to a hyperbolic periodic point \( p \).

Corollary 4. — Assume that \( f \in \text{Diff}_{w}^{s}(M) \), \( s \geq 4 \), has a hyperbolic periodic point \( p \) and that \( f \) exhibits a homoclinic tangency associated to \( p \), then there exists \( U \subset \text{Diff}_{w}^{s}(M) \) an OSPHT such that \( f \in U \). Moreover, there is a residual subset \( V \) of \( U \) such that every \( f \in V \) has an invariant smooth curve which is accumulated by elliptic points.

A consequence of Corollary 4 is the creation of infinitely many elliptic islands accumulating KAM curves. However, these elliptic points do not accumulate at the hyperbolic point which unfolds the homoclinic tangency. A related question in the unfolding of a homoclinic tangency is whether the OSPHT’s can be constructed generating elliptic islands which accumulate at the hyperbolic periodic point. Some partial results concerning the previous question were obtained in [D]. Moreover, it seems possible to answer the question above by using [MR] and the methods of proof in the dissipative case.

This paper is organized as follows: In Section 2, Birkhoff’s normal form and KAM theorem are recalled. The proof of Proposition 3, using some tools of [Z], is presented in Section 3. Finally, in Section 4 we prove Proposition 2 and Theorem 1.

2. Birkhoff’s normal form theorem and KAM theorem

Let \( f \) be an area-preserving \( C^{r} \) diffeomorphism of the annulus \( \mathbb{A} = S^{1} \times \mathbb{R} \), with \( r \geq 4k + 4 \) and \( k \geq 0 \); here and in what follows we identify \( S^{1} \) with
\( S^1 \times \{0\} \). Assume that \( f(S^1) = S^1 \) and that \( f|_{S^1} = R_\omega \) the rotation with angle \( \omega \). So we can write

\[
f(\theta, r) = (\theta + \omega + ra(\theta, r), rb(\theta, r)).
\]  

We say that \( \omega \in \mathbb{R} \) satisfies a diophantine condition if there exist \( \beta > 0 \) and \( C > 0 \) such that for every \( p/q \in \mathbb{Q} \) then \( |\omega - p/q| > C/q^{2+\beta} \). Let \( D(C, \beta) \) be the set of these numbers with \( C \) and \( \beta \) fixed. We recall that the set \( D(\beta) = \bigcup_{C \geq 0} D(C, \beta) \) has total Lebesgue measure, i.e., \( m(D(\beta) \cap [0, 1]) = 1 \) when \( \beta > 0 \).

The following version of Birkhoff’s normal form theorem says that if \( \omega \) satisfies a diophantine condition then after an area-preserving change of coordinates the term \( ra(\theta, r) \) in (1) can be written as a polynomial function in \( r \) plus higher order terms in \( r \). More precisely, letting

\[
A_\delta = \{ (\theta, r) \mid \theta \in S^1, |r| < \delta \},
\]

we have the following result.

**Theorem 5.** — For each \( n \leq k \) there exists \( h_n : A_\delta \to A \) a \( C^{n-4n} \) area-preserving map letting \( S^1 \) invariant and such that \( \hat{f}_n = h_n^{-1} \circ f \circ h_n \) has the following form

\[
\hat{f}_n(\theta, r) = (\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1})).
\]

**Proof.** — For a proof in the \( C^\infty \) case see appendices 1 and 2 of [Do]. The finite-differentiability case follows the same lines as the \( C^\infty \) case but it is necessary to use lemma 8.1 of [H].

**Remark.** — In the case that \( f \) is \( C^\infty \) all the changes of coordinates are also \( C^\infty \), and we can choose \( n \) as large as we want.

Now consider a \( C^\infty \) symplectic diffeomorphism \( \tilde{f} \in \text{Diff}_{\omega}^\infty \) with an invariant \( C^\infty \) curve \( \gamma \). We define the twist condition along \( \gamma \) as follows: we say that \( \tilde{f} \) satisfies a twist condition along \( \gamma \) if there exists a transversal unit vector field \( X \) on \( \gamma \) such that \( \omega(D\tilde{f}X(p), X(\tilde{f}(p))) > 0 \) for all \( p \in \gamma \). When \( \rho(\tilde{f}|_\gamma) \) satisfies a diophantine condition it is well known that after a symplectic change of coordinates, \( \tilde{f} \) restricted to a neighborhood \( V \) of \( \gamma \) has the form (1) with \( X(\theta, 0) = (0, 1) \). In this case a symplectic diffeomorphism of the annulus \( \tilde{f} \) satisfies a twist condition along \( \gamma \) if and only if

\[
a_1 = \int a(\theta, 0) \, d\theta \neq 0.
\]
This number does not depend on the symplectic change of coordinates used to put $\tilde{f}$ in the form (1) and it is called the first Birkhoff coefficient.

Now we recall the KAM theorem and remark some facts that we will use in the sequel. Let $f : \mathbb{A}_\delta \rightarrow \mathbb{A}$ be a $C^\infty$ map of the annulus. We say that $f$ has the intersection property if for each curve $\gamma$ in $\mathbb{A}_\delta$ non homotopically trivial we have that $f(\gamma) \cap \gamma \neq \emptyset$. If $f$ admits an invariant curve which is non homotopically trivial and preserves a symplectic form $w$ then it is easy to see that $f$ has the intersection property. Let $s \geq 4$ and $t \in C^\infty((-\delta, \delta), \mathbb{R})$. For each $(\nu, \mu) \in C^s(\mathbb{A}_\delta, \mathbb{R})^2$ let $T_{\nu, \mu} : \mathbb{A}_\delta \rightarrow \mathbb{A}$ be the map

$$(\theta, r) \mapsto (\theta + t(r) + \nu(\theta, r), r + \mu(\theta, r)).$$

**Theorem 6.** Let $r_0 \in (-\delta, \delta)$ and assume:

(a) $t > 0$, $T_{\nu, \mu}$ satisfies a twist condition;
(b) $\alpha = t(r_0) \in D(c, \beta)$, $\alpha = t(r_0)$ satisfies a diophantine condition;
(c) $T_{\nu, \mu}$ satisfies the intersection property for every $(\nu, \mu)$ in a neighborhood of $(0, 0)$.

Let $s > 2\beta + 3$, then there exists a neighborhood $W$ in $C^s(\mathbb{A}_\delta, \mathbb{R})^2$ of $(0, 0)$ such that, for all $(\nu, \mu) \in W$, one can find $\gamma \in C^{s-2(1+\beta)}(S^1, \mathbb{R})$ and $h \in \text{Diff}^{s-2(1+\beta)}(S^1)$ with

(i) $\Gamma = \{ (\theta, \gamma(\theta)) | \theta \in S^1 \}$ is invariant under $T_{\nu, \mu}$;
(ii) $T_{\nu, \mu}|_{\Gamma}$ is $C^{s-2(1+\beta)}$ conjugated to the rotation $R_\alpha(\theta) = \theta + \alpha(\mod)1$ by the following conjugation $\theta \mapsto (h(\theta), \gamma \circ h(\theta))$.

See [Bo] and [SZ] for a proof.

**Remarks**

- The neighborhood $W$ depends a priori on $\alpha = t(r_0)$ (in fact on $(dt(r_0)/dr)^{-1}$) but it can be proved that if $r_0$ varies in a compact set $K$, such that $t(K) \subset D(\beta)$ then we can choose $W$ depending just on $K$. Because of $D(\beta)$ has total Lebesgue measure, this is what gives the rich structure (lots of other invariant curves) around an invariant curve.
- We have the following regularity statement: if $\nu, \mu$ are $C^\infty$ then $\gamma$ is $C^\infty$, see [SZ].
3. Invariant curves and homoclinic tangencies

In this section our goal is to give the proof of Proposition 3, which in turn is a consequence of the following proposition.

**Proposition 7.** Let $f : \mathbb{A}_\delta \rightarrow \mathbb{A}$ be a $C^\infty$ area-preserving map of the annulus which leaves invariant some $C^\infty$ curve $\Lambda = \{(\theta, \Psi(\theta)) \mid \theta \in S^1\}$ where $\Psi : S^1 \rightarrow \mathbb{R}$, and such that $f|_\Lambda$ has an irrational rotation number. Then for $s \geq 1$ and each $\epsilon > 0$, $f$ can be $\epsilon$-approximated in the $C^s$-topology by one $F$ which exhibits homoclinic tangencies and such that for some $\delta' < \delta$ we have $F|_{(\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'})} = f|_{(\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'})}$.

**Proof of Proposition 3**

Item (ii) follows from [N3], see also [MR], so we will only prove item (i). Because $f$ and $\gamma$ are $C^\infty$, we can find a tubular neighborhood $U$ of $\gamma$ such that there is $h : U \rightarrow \mathbb{A}_\delta$ for which $h(\gamma) : S^1 \times \{0\} \subset \mathbb{A}_\delta$ and $h^*(d\theta \wedge dr) = \omega$. So making use of Proposition 7 the result follows.

To prove Proposition 7 we need first some preliminary results presented in the following subsection.

3.1 Preliminaries

Let $f : \mathbb{A}_\delta \rightarrow \mathbb{A}$ be a $C^\infty$ area-preserving map of the annulus which leaves $S^1$ invariant, i.e., $f(S^1) = S^1$. We assume that $f|_{S^1} = R_\omega$ with $\omega = p/n$ where $p$, $n$ are relatively prime and

$$f(\theta, r) = (\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1}))$$

$$= f_n(\theta, r) + O(r^{n+1}),$$

with $a_1 > 0$. Since $f$ leaves $S^1$ invariant (see [Do]) we have that locally around $S^1$, $f(\theta, r) = (\Theta, R)$ is described by a generating function $h(\theta, R)$ in the following way

$$f(\theta, r) = (\Theta, R) \iff \begin{cases} r = \frac{\partial h}{\partial \theta}, \\ \Theta = \frac{\partial h}{\partial R}. \end{cases}$$
It is easy to check that

\[ h_n(\theta, R) = (\theta, \omega)R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} \]

is the generating function of \( f_n \). From this we get that the generating function of \( f \) has the form \( h(\theta, R) = h_n(\theta, R) + O(R^{n+2}) \).

We follow Moser and Zehnder to make a perturbation of \( f \). Consider the following two parameter family of generating functions

\[ h_{\varepsilon, \gamma}(\theta, R) = h(\theta, R) - \varepsilon R + \gamma \cos(2\pi n \theta) R^{n+1} \]

\[ = (\theta + \omega - \varepsilon)R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} + \gamma \cos(2\pi n \theta) R^{n+1} + O(R^{n+2}) \]

(2)

This family generates, for \( \varepsilon \) and \( \gamma \) small enough, the following two parameter family of diffeomorphisms \( f_{\varepsilon, \gamma} : \mathbb{A} \rightarrow \mathbb{A} \) with

\[ f_{\varepsilon, \gamma}(\theta, r) = (\theta + \omega - \varepsilon + a_1 r + a_2 r^2 + \cdots + a_n r^n + (n+1) \gamma \cos(2\pi n \theta) r^{n+1} + O(r^{n+1}), \]

\[ r + 2\pi \gamma n \sin(2\pi n \theta)(r^{n+1}) + O(r^{n+2}) \].

(3)

Observe that by the way we made the perturbation, \( S^1 \) continues to be invariant for the family \( f_{\varepsilon, \gamma} \).

**Proposition 8.** Assume that \( a_1 \neq 0 \), then for \( \varepsilon \) and \( \gamma \) small enough, \( f_{\varepsilon, \gamma} \) has two \( n \)-periodic orbits \( \{ h_i(\varepsilon, \gamma) \}_{i=0}^{n-1}, \{ e_i(\varepsilon, \gamma) \}_{i=0}^{n-1} \), which satisfy:

(a) \( \{ h_i(\varepsilon, \gamma) \}_{i=0}^{n-1} \) is a hyperbolic \( n \)-periodic orbit with

\[ h_i(\varepsilon, \gamma) \rightarrow \left( \frac{i}{n}, 0 \right) \]

for \( \gamma \) fixed and \( \varepsilon \to 0 \);

(b) \( \{ e_i(\varepsilon, \gamma) \}_{i=0}^{n-1} \) is an elliptic \( n \)-periodic orbit with

\[ e_i(\varepsilon, \gamma) \rightarrow \left( \frac{2i + 1}{2n}, 0 \right) \]

for \( \gamma \) fixed and \( \varepsilon \to 0 \);
(c) there exist $\delta > 0$ and
\[ \psi_{h_i}(\epsilon, \gamma) \in W^s_{\text{loc}}(h_i(\epsilon, \gamma)) \cap W^u_{\text{loc}}(h_{i+1}(\epsilon, \gamma)) \]
for which we have
\[ \psi_{h_i}(\epsilon, \gamma) \longrightarrow \psi_{h_i}(0, \gamma) \in \left( \frac{2i + 1}{2n} - \delta, \frac{2i + 1}{2n} + \delta \right) \times \{0\}, \]
where $\delta$ does not depend on $\epsilon$ when this is small enough;
(d) the angle
\[ \mathcal{L} \left( T_{\psi_{h_i}} W^s_{\text{loc}}(h_i(\epsilon, \gamma)), T_{\psi_{h_i}} W^u_{\text{loc}}(h_{i+1}(\epsilon, \gamma)) \right) \longrightarrow 0 \]
for $\gamma$ fixed and $\epsilon \to 0$.

The proof of this proposition is contained in [Z], so we only present the construction of the periodic points and shows how the homoclinic points are found and refer to [Z] for the rest of the details.

Proof. — We begin by making the following change of coordinates $\ell(\theta, \rho) = (\theta, \epsilon \rho) = (\theta, r)$ which allows us to see what happens in a microscopic neighborhood of $S^1$. In terms of $\theta$ and $\rho$, $\tilde{f} = \ell^{-1} \circ f \circ \ell$ is written as
\[ \tilde{f}_{\epsilon, \gamma}(\theta, \rho) = \left( \theta + \frac{p}{n} - \epsilon + a_1 \epsilon \rho + \cdots + a_n(\epsilon \rho)^n + \right. \]
\[ \left. + (n + 1) \gamma \cos(2\pi n) (\epsilon \rho)^n + O((\epsilon \rho)^{n+1}), \right. \]
\[ \rho + 2\pi \gamma n \sin(2\pi n) \epsilon^n \rho^{n+1} + O(\epsilon^{n+1}) \right) \]
\[ = \left( \theta + \frac{p}{n} - \epsilon + a_1 \epsilon \rho + \cdots + a_n(\epsilon \rho)^n + \right. \]
\[ \left. + (n + 1) \gamma \cos(2\pi n) (\epsilon \rho)^n + O(\epsilon^{n+1}), \right. \]
\[ \rho + 2\pi \gamma n \sin(2\pi n) \epsilon^n \rho^{n+1} + O(\epsilon^{n+1}) \right). \]

We get for the $n$-th iterate of $\tilde{f}_{\epsilon, \gamma}$ the following expression
\[ \tilde{f}_{\epsilon, \gamma}^n(\theta, \rho) = \left( \theta + p - n \epsilon + na_1 \epsilon \rho + O(\epsilon^2), \right. \]
\[ \left. \rho + 2\pi \gamma n^2 \sin(2\pi n) \epsilon^n \rho^{n+1} + O(\epsilon^{n+1}) \right). \]
where

\[ O(\varepsilon^2) = n a_2 \varepsilon^2 \rho^2 + \cdots + n a_n (\varepsilon \rho)^n + \]
\[ + n(n + 1) \gamma \cos(2\pi \theta \rho n)(\varepsilon \rho)^n + O(\varepsilon^{n+1}). \]

The fixed points of \( \tilde{f}^n_{\varepsilon, \gamma} \) are the solutions of the equations

\[ \theta = \theta + p - n \varepsilon + n a_1 \varepsilon \rho + O(\varepsilon^2) \quad (6) \]
\[ \rho = \rho + 2\pi \gamma n^2 \sin(2\pi \theta n)\varepsilon^n \rho^{n+1} + O(\varepsilon^{n+1}). \quad (7) \]

The fact that \( a_1 \neq 0 \) and the implicit function theorem imply that there exists \( \rho(\varepsilon) \) a solution of \( (6) \) which equals \( 1/a_1 \) when \( \varepsilon = 0 \). Using this solution in \( (7) \) we get \( 2n \) solutions \( \{ h_i(\varepsilon), e_i(\varepsilon) \} \) with \( i = 1, \ldots, n \) which equal

\[ \left( \frac{i}{n}, \frac{1}{a_1} \right), \left( \frac{2i + 1}{2n}, \frac{1}{a_1} \right) \]

respectively when \( \varepsilon = 0 \). Since \( p, n \) are relative primes, the uniqueness part of the implicit function theorem gives that

\[ \{ e_i(\varepsilon) = (e_i(\varepsilon), \rho(\varepsilon)) \} \quad \text{and} \quad \{ h_i(\varepsilon) = (h_i(\varepsilon), \rho(\varepsilon)) \} \]

are actually part of a \( n \)-periodic orbit. To determine the nature of these orbits we make another change of coordinates. Let \( \phi \) be any of the points \( \{ e_i, h_i \}_{i=0}^{n-1} \) and let \( \tilde{\ell}(\psi, x) = (\psi + \phi, \rho(\varepsilon) + \varepsilon^{(n-1)/2} x) \) then

\[ \tilde{f}^n_{\varepsilon, \gamma} = \tilde{\ell}^{-1} \circ \tilde{f}^n_{\varepsilon, \gamma} \circ \tilde{\ell} \]

takes the following form

\[ \tilde{f}(\psi, x) = (\psi + \tilde{f}_1(\varepsilon, \psi, \varepsilon^{(n-1)/2} x), x + \tilde{f}_2(\varepsilon, \psi, \varepsilon^{(n-1)/2} x)), \quad (8) \]

where

\[ \tilde{f}_1(\varepsilon, \psi, y) = -n \varepsilon + n a_1 (\rho(\varepsilon) + y) + \cdots + n a_n \varepsilon^n (\rho(\varepsilon) + y)^n + \]
\[ + n(n + 1) \gamma \cos(2\pi \psi n)\varepsilon^n (\rho(\varepsilon) + y)^n + O(\varepsilon^{n+1}) \quad (9) \]

and

\[ \tilde{f}_2(\varepsilon, \psi, y) = (-1)^{\sigma} 2\pi \gamma n^2 \sin(2\pi \psi n)\varepsilon^{n-(n-1)/2} (\rho(\varepsilon) + y)^{n+1} + \]
\[ + O(\varepsilon^{n-(n-1)/2+1}), \quad (10) \]

with \( \sigma = \pm 1 \) depending on the value of \( \phi \). From \( (9) \) and the fact \( \tilde{f}_1(\varepsilon, 0, 0) = (0, 0) \) we get

\[ \tilde{f}_1(\varepsilon, \psi, y) = \varphi_0(\varepsilon, \psi) + \varphi_1(\varepsilon, \psi) y + \varphi_2(\varepsilon, \psi) y^2 \quad (11) \]
with

\[ \varphi_0(\epsilon, \psi) = \hat{f}_1(\epsilon, \psi, 0) = O(\epsilon^{n+1}), \]
\[ \varphi_1(\epsilon, \psi) = \frac{\partial \hat{f}_1}{\partial y}(\epsilon, \psi, 0) = a_1 \epsilon + O(\epsilon^2), \]
\[ \varphi_2(\epsilon, \psi) = \frac{\partial^2 \hat{f}_1}{\partial^2 y}(\epsilon, \psi, \hat{y}) = O(1) \]

and \(0 \leq \hat{y} \leq y\). All of these together imply that we can write

\[
\hat{f}(\psi, x) = \left( \psi + na_1 \epsilon^{(n+1)/2} x + O(\epsilon^{n/2+1}), \right.
\]
\[
\left. x + (-1)^{\sigma} 2\pi \left( \frac{1}{a_1} \right)^{n+1} \gamma n^2 \sin(2\pi \psi n) \epsilon^{(n+1)/2} + O(\epsilon^{n/2+1}) \right) .
\]

Now from (12) the Jacobian matrix at \((0, 0)\) equals to

\[
J(\epsilon) = \begin{pmatrix}
1 + O(\epsilon^{n/2+1}) & \quad na_1 \epsilon^{(n+1)/2} + O(\epsilon^{n/2+1}) \\
R + O(\epsilon^{n/2+1}) & \quad 1 + O(\epsilon^{n/2+1})
\end{pmatrix}
\]

where

\[
R = (-1)^{\sigma} 4\pi^2 \left( \frac{1}{a_1} \right)^{n+1} \gamma n^3 \epsilon^{(n+1)/2}, \quad \sigma = \begin{cases} 1 & \text{at } \epsilon_i \\ 0 & \text{at } h_i. \end{cases}
\]

From here it follows that

\[
\text{tr } J(\epsilon) = 2 + O(\epsilon^{n/2+1})
\]
\[
\det J(\epsilon) = 1 - (-1)^{\sigma} 4\pi^2 \left( \frac{1}{a_1} \right)^n \gamma n^4 \epsilon^{n+1} + O(\epsilon^{n/2+1}). \quad \text{(13)}
\]

So we conclude from (13) that we have an elliptic orbit at \(\{\epsilon_i\}_{i=0}^{n-1}\) and a hyperbolic orbit at \(\{h_i\}_{i=0}^{n-1}\) with eigenvalues given by

\[
\lambda_s = 1 - \pi \sqrt{\left( \frac{1}{a_1} \right)^n \gamma n^4 \epsilon^{(n+1)/2} + O(\epsilon^{n/2+1})}
\]
\[
\lambda_u = 1 + \pi \sqrt{\left( \frac{1}{a_1} \right)^n \gamma n^4 \epsilon^{(n+1)/2} + O(\epsilon^{n/2+1})}. \quad \text{(14)}
\]
The local stable (unstable) manifold $W_{\text{loc}}^{s(u)}(0)$ of $(0,0)$ of $f_{\varepsilon,\gamma}$ is described by the following proposition, whose proof follows immediately from Proposition 1 and 2 of $[Z]$. 

**Proposition 9.** There exist $C_1$, $C_2$ and $\varepsilon_0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the local stable (unstable) manifolds $W_{\text{loc}}^{s(u)}(0)$ are given in $|\psi| \leq 3/4n$ by

$$W_{\text{loc}}^{s(u)}(0) = \text{graph } g^{s(u)}$$

$$g^s(\varepsilon, \psi) = -\frac{2}{n} \frac{\gamma}{a_1^{n+2}} \sin(\pi n \psi) + u^s(\varepsilon, \psi), \quad u^s(\varepsilon, 0) = 0 \quad (15)$$

$$g^u(\varepsilon, \psi) = \frac{2}{n} \frac{\gamma}{a_1^{n+2}} \sin(\pi n \psi) + u^u(\varepsilon, \psi), \quad u^u(\varepsilon, 0) = 0$$

where $|u^{s(u)}(\varepsilon, \psi)| < C_1 \varepsilon$, $\text{Lip}(u^{s(u)}) < C_2 \varepsilon$ and $u^{s(u)}(\varepsilon, 0) = 0$.

From this proposition it follows that the $W_{\text{loc}}^{s(u)}(h_i)$ are the graphs of functions $g_1^{s(u)}$ defined on an interval with center at $h_i$ and length equals $3\pi/4n$. To prove part (c) of Proposition 8, let us show that $W_{\text{loc}}^{u}(h_i(\varepsilon)) \cap W_{\text{loc}}^{s}(h_{i+1}(\varepsilon)) \neq \emptyset$. We argue by contradiction. Observe that, since $S^1$ is left invariant by $f_{\varepsilon,\gamma}$, the annulus is decomposed in two regions and the periodic orbit $\{h_i\}_{i=0}^{n-1}$ lies in one of these sides (fig. 1).

![Fig. 1](image_url)

Now following $[Z]$, we build a curve $C_0$ in the annulus in the following way: the vertical line $(1/2n, x)$ intersects $W_{\text{loc}}^{u}(h_0(\varepsilon))$ and $W_{\text{loc}}^{s}(h_1(\varepsilon))$ in the points $P$ and $Q$ respectively (fig. 1); let $C_0$ be the path that goes from $h_0$ until $P$ through $W_{\text{loc}}^{u}(h_0(\varepsilon))$, then follows by the vertical segment from $P$ until $Q$ and then continues from this point until $h_1$ through $W_{\text{loc}}^{s}(h_1(\varepsilon))$. Define now the curve $C$ as being $\bigcup_{0}^{n-1} f_{\varepsilon,\gamma}(C_0)$. This curve is a non
homotopically trivial Jordan curve. Let $G$ be the region bounded by $\mathbb{S}^1$ and this curve. It is easy to see, using the properties of the stable and unstable manifolds described in Proposition 8, that $m(G) > m(f_{\varepsilon, \gamma}(G))$ therefore contradicting the area-preserving property.

By (15) the angle between these manifolds at the intersection point goes to zero when $\varepsilon$ goes to zero. □

3.2 Proof of Proposition 7

The proof of the proposition will be made through a sequence of steps that consist in making some reductions and perturbations. We dedicate one item to each one.

• We change coordinates with $h(\theta, r) = (\theta, r - \Psi(\theta)) = (\bar{\theta}, \bar{r})$ so that $\overline{f}(\bar{\theta}, \bar{r}) = h \circ f \circ h^{-1}$ has $h(\Lambda) = \mathbb{S}^1$ as an invariant curve. Observe that $\overline{f}$ is $C^\infty$ and

$$\|h(\theta, r)\|_{C^s} \leq 1 + \|\Psi\|_{C^s},$$

$$\|h^{-1}(\theta, r)\|_{C^s} \leq 1 + \|\Psi^{-1}\|_{C^s},$$

so if we prove the proposition for $\overline{f}$ then we will also have it proved for $f$.

• Thus we assume that $f(\mathbb{S}^1) = \mathbb{S}^1$ and $f|\mathbb{S}^1$ is conjugated to $R_\omega$ with $\omega$ an irrational number. Consider $f_\beta(\theta, r) = f(\theta, r) + (\beta, 0)$ then by [H] we can find $\beta_n \to 0$ with $n \to \infty$ such that $f_{\beta_n}(\mathbb{S}^1) = \mathbb{S}^1$ and $f_{\beta_n}|_{\mathbb{S}^1}$ has a rotation number $\omega_n = \omega + \beta_n$ satisfying a diophantine condition, and once more by [H] we know that there exists $h_n : \mathbb{S}^1 \to \mathbb{S}^1$ a $C^\infty$ diffeomorphism, conjugating $f_{\beta_n}|_{\mathbb{S}^1}$ with $R_{\omega_n}$. Consider $H_n(\theta, r) = (h_n(\theta), r/h_n'(\theta))$, then $H_n^{-1} \circ f_{\beta_n} \circ H_n = \widehat{f}$ satisfies $\widehat{f}(\mathbb{S}^1) = \mathbb{S}^1$ and $\widehat{f}|_{\mathbb{S}^1} = R_{\omega_n}$. Also these changes of coordinates can be made uniformly in the sense that there is some constant $M_n > 0$ such that

$$\max \left\{ \|H_n(\theta, r)\|_{C^s}, \|H_n^{-1}(\theta, r)\|_{C^s} \right\} < M_n.$$

So once more, it is enough to prove the proposition for this map.

• We assume there that $f(\mathbb{S}^1) = \mathbb{S}^1$ and $f|_{\mathbb{S}^1} = R_\omega$ with $\omega$ satisfying a diophantine condition. By Theorem 5, we can write after a change of coordinates

$$f(\theta, r) = (0 + \omega + a_1r + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1})),$$
we may assume that $a_1 \neq 0$ unless we perturb $f$ in such a way that the new $f$ has $a_1 \neq 0$, even more we choose $a_1 > 0$ (in the case $a_1 < 0$ we take $f^{-1}$). After this we perturb once again so the rotation number of $f|_{S_1}$ becomes rational. We apply now the Proposition 8 to get a sequence of maps $f_k \to f$ such that $f_k$ has a hyperbolic periodic orbit $\{h_i(k)\}_{i=1}^n$ with $\psi_{h_i}(k) \subset W^s_{loc}(h_i(k)) \cap W^u_{loc}(h_{i+1}(k))$, and the angle at point $x$ goes to zero as $k \to \infty$. Moreover, $h_i(k) = i/n$ and

$$\psi_{h_i}(k) \to \psi'_{h_i} \in \left(\frac{2i+1}{2n} - \delta, \frac{2i+1}{2n} + \delta\right) \times \{0\}.$$ 

So we can use the following lemma (see [N1]).

**Lemma.** — Let $\varepsilon > 0$ and $s \in \mathbb{N}$. There exists $C(s) > 0$ such that given $\delta$ and a linear subspace $H \subset \{v = (v_1, v_2) \mid |v_2| \leq C(s)\delta^{s-1}\varepsilon|v_1|\}$ : there exists a $C^s$ area-preserving diffeomorphism $\varphi : A \leftrightarrow H$ such that $\varphi(0) = 0$, $D\varphi\{v_2 = 0\} = H$ and $\varphi(\theta, r) = (\theta, r)$ for $\text{dist}((\theta, r), (0, 0)) \geq \delta$ and $\|\varphi - \text{id}\|_{C^s} \leq \varepsilon$.

So, we can get perturbations $\tilde{f}_k$ of $f_n$ with the property that $\tilde{f}_k$ exhibits homoclinic tangencies and $\tilde{f}_k \to f$. If the tangency it not quadratic, with a new perturbation, we make it quadratic. □

4. Proof of Theorem 1

**Proof of Proposition 2**

Let $\tilde{U}$ be an open neighborhood of $f$ where the continuation of $\gamma$ exists, i.e., for each $g \in \tilde{U}$ there exists an invariant curve $\gamma_g$ such that the rotation number of $g \mid \gamma_g$ equals that of $f \mid \gamma$; this neighborhood is provided by KAM theory. Since $f$ and $\gamma$ are $C^\infty$ we apply Theorem 6 and the remark which follows to conclude the existence of a subset $\mathcal{U}$ of $\tilde{U}$ for which the following property holds : for each $g \in \mathcal{U}$ such that $g$ is a $C^\infty$ map, the invariant curve $\gamma_g$ prolongation of $\gamma$ is also $C^\infty$. Now Proposition 3 allows us to conclude that this neighborhood is an OSPHT. To see the existence of the residual set we observe first that, by the remark following Theorem 6, for each $g \in \tilde{U}$ there are lots of invariant curves, in particular $\gamma_g$ is the limit of other invariant curves satisfying the twist condition and whose rotation numbers satisfy diophantine conditions. We also notice that each $C^\infty$ map
Proof of Theorem 1

We approximate \( f \) by \( \tilde{f} \), a \( C^\infty \) map having a generic elliptic periodic orbit; it is a consequence of the proof of Proposition 3. Let \( U_1 \) be a set containing \( f \) and for which this elliptic periodic point survives. Choose an invariant \( C^\infty \) curve of \( \tilde{f} \) associated to this elliptic periodic point. Observe that this curve is invariant by \( f^n \) where \( n \) is the period of the elliptic periodic point. By KAM theorem we have a subset \( U \) of \( U_1 \), in which the curve survives. Now the remark after Proposition 2 allows us to conclude the proof. \( \square \)

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References


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