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Realization of Hölder Complexes(*)

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RÉSUMÉ. — Un complexe de Hölder est un graphe fini tel qu'à chaque arête est associé un nombre rationnel positif et on sait que c'est un invariant bi-lipschitzien des ensembles semi-algébriques singuliers de dimension 2. On montre dans cet article que tout complexe de Hölder peut être réalisé comme un ensemble semi-algébrique de dimension 2. Pour ce faire on plonge le graphe dans un tore de dimension n qu'on fait contracter sur un point singulier de telle sorte que les générateurs s'évanouissent avec les vitesses rationnelles et différentes.

ABSTRACT. — Hölder Complex, a graph and a rationaly-valued function on the set of the edges of the graph, is a bi-Lipschitz invariant of 2-dimensional semialgebraic singular sets. Here we prove that each Hölder Complex can be realized as a 2-dimensional semialgebraic set. For this purpose we embed the graph to an n-dimensional torus. The torus is vanishing in a singular point such that the generators are vanishing with different rational rates.

1. Introduction

The paper is devoted to the local geometry of 2-dimensional semialgebraic sets. The local bi-Lipschitz classification theorem is proved in [1]. The main notion of the classification is a so-called Geometric Hölder Complex. It is a local version of a simplicial complex with some additional geometric information (see the definition below). A Hölder Complex can be considered as a combinatorial object – a finite graph with a rational-valued function defined on the set of edges.

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The following question is natural. Let us define a Hölder Complex in a combinatorial way. Does it correspond to some semialgebraic set?

The answer is positive. To prove the Realization theorem we define a semialgebraic set $T(\beta_1, \ldots, \beta_k)$. It is a generalization of the real algebraic set which gives an example of the noncoincidence of L_p-cohomology and Intersection Homology [2]. The set $T(\beta_1, \ldots, \beta_k)$ has a toric link at the singular point and all generators of the torus have different vanishing rates in this point. It gives us a possibility to separate vanishing rates of all edges of a Hölder Complex.

2. Definitions and notations

Let us recall some definitions from [1]. Let Γ be a connected graph without loops, $V_\Gamma = \{a_1, a_2, \ldots, a_k\}$ be the set of vertices and $E_\Gamma = \{g_1, g_2, \ldots, g_r\}$ be the set of edges of the graph.

DEFINITION 2.1.— *A Hölder Complex* (Γ, β) *is a graph* Γ *with an associated function* $\beta\colon E_\Gamma \to [1, \infty[\cap Q$ *(here* Q *is the ring of rational numbers).*

DEFINITION 2.2.— *A Curvilinear triangle* T *is a subset of* $\mathbb{R}^n$ *homeomorphic to a 2-dimensional simplex satisfying the following properties.*

1) *Each internal (in the induced topology) point* $t \in T$ *has an open neighbourhood* $U_t \subset T$ *such that* U_t *is a smooth 2-dimensional submanifold of* $\mathbb{R}^n$ *at each point* $t' \in U_t$.
2) *The boundary of* T *is a union of three analytic curves* $\gamma_1, \gamma_2, \gamma_3$ *such that* γ_i *(for* $i = 1, 2, 3$*) has a neighbourhood at each internal (in the induced from* $\mathbb{R}$ *topology on* γ_i*) point which is a smooth 1-dimensional submanifold of* $\mathbb{R}^n$.
3) *Locally* T *is a smooth manifold with a boundary at each smooth point of the boundary.*

Boundary points of γ_i *we call vertices of* T.

DEFINITION 2.3.— *A standard* β*-Hölder triangle* ST_β *is a subset of the plane* $\mathbb{R}^2$ *bounded by the following curves:*

$$\{y = 0\}, \quad \{y = x^\beta\}, \quad \{x = 1\}.$$

Let us consider a cone $C\Gamma$ over Γ. Let A_0 be the vertex of $C\Gamma$. We can consider $C\Gamma$ as a topological space with the standard topology of a simplicial complex.

DEFINITION 2.4.— *A subset $H(\Gamma,\beta) \subset \mathbb{R}^n$ is called a Geometric Hölder Complex corresponding to (Γ,β) if it satisfies the following conditions.*

1) $H(\Gamma,\beta)$ is a subanalytic subset of $\mathbb{R}^n$.

2) There exists a homeomorphism $F: C\Gamma \to H(\Gamma,\beta)$.

3) The set $H(\Gamma,\beta) \cap S_{F(A_0),r}$ is empty or homeomorphic to Γ, for every r. (We use the notation $S_{F(A_0),r}$ for the sphere centered at the point $F(A_0)$ with the radius r.)

4) The image of the triangle (A_0, a_i, a_j, g) (where a_i and a_j are vertices of Γ, g is the edge connecting a_i and a_j, (A_0, a_i, a_j, g) is the subcone of $C\Gamma$ over g) has the following properties :

(a) $F(A_0, a_i, a_j, g)$ is a subanalytic subset of $\mathbb{R}^n$;

(b) $F(A_0, a_i, a_j, g)$ is subanalytically bi-Lipschitz equivalent to the standard $\beta(g)$-Hölder triangle $\mathrm{ST}_{\beta(g)}$;

(c) let $L: \mathrm{ST}_{\beta(g)} \to F(A_0, a_i, a_j, g)$ be this subanalytic bi-Lipschitz map; then

$$L(0,0) = F(A_0)\,, \quad L(1,0) = F(a_i)\,, \quad L(1,1) = F(a_j)\,.$$

DEFINITION 2.5.— *A β-Hölder triangle HT_β is a subset of $\mathbb{R}^n$ satisfying the following conditions.*

1) HT_β is a curvilinear triangle.

2) HT_β is bi-Lipschitz equivalent to some standard β-Hölder triangle ST_β.

3) The bi-Lipschitz map $L: \mathrm{ST}_\beta \to \mathrm{HT}_\beta$ is subanalytic. (The image of the point $(0,0)$ is called a Hölder vertex of HT_β.)

DEFINITION 2.6.— *A standard β-horn SH_β (here $\beta \in \mathbb{Q} \cap [\,1\,, +\infty\,[$) is a semialgebraic set in $\mathbb{R}^3$ defined by the following conditions:*

$$(x_1^2 + x_2^2)^q = y^{2p}\,, \quad 0 \le y \le 1\,,$$

$(x_1,\, x_2,\, y)$ are coordinates of a point in $\mathbb{R}^3$ and $\beta = p/q$ with $\mathrm{GCD}(p,q) = 1$.

We proved in [1] that every 2-dimensional semialgebraic (as well as semianalytic and subanalytic) set X is a Geometric Hölder Complex in a neighbourhood of a given point $a_0 \in X$ corresponding to some Hölder Complex. Here we are going to prove the following result.

REALIZATION THEOREM. — *Let (Γ, β) be a Hölder Complex. Then there exist a semialgebraic 2-dimensional set $X \subset \mathbb{R}^n$, a point $a_0 \in X$ and $\varepsilon > 0$ such that $X \cap B_{a_0,\varepsilon}$ is a Geometric Hölder Complex corresponding to the Hölder Complex (Γ, β) (here $B_{a_0,\varepsilon}$ is a closed ball in $\mathbb{R}^n$ centered at the point a_0 with the radius ε).*

3. The set $T(\beta_1, \ldots, \beta_k)$. Polar maps

We consider the space $\mathbb{R}^{2k+1}$ with coordinates $(x_1, y_1, x_2, y_2, \ldots, x_k, y_k, z)$. Let $D(\beta_1, \ldots, \beta_k)$ (here $\beta_i = p_i/q_i$ with p_i, $q_i \in \mathbb{Z}$ and $\mathrm{GCD}(p_i, q_i) = 1$) be a subvariety of $\mathbb{R}^{2k+1}$ given by the following equations:

$$\begin{aligned} z^{2p_1} &= \left(x_1^2 + y_1^2\right)^{q_1} \\ &\vdots \\ z^{2p_i} &= \left(x_i^2 + y_i^2\right)^{q_i} \\ &\vdots \\ z^{2p_k} &= \left(x_k^2 + y_k^2\right)^{q_k}. \end{aligned} \tag{1}$$

(The set described in the paper [2] is a special 3-dimensional example of $D(\beta_1, \beta_2)$.)

Let

$$T(\beta_1, \ldots, \beta_k) = D(\beta_1, \ldots, \beta_k) \cap \{z \geq 0\}. \tag{2}$$

LEMMA 3.1

1) $\dim T(\beta_1, \ldots, \beta_k) = k + 1$.

2) The link of $T(\beta_1, \ldots, \beta_k)$ at the point $(0, \ldots, 0)$ is homeomorphic to T^k (a k-dimensional torus).

(Remind that the link of $T(\beta_1, \ldots, \beta_k)$ is the intersection of $T(\beta_1, \ldots, \beta_k)$ with a small sphere centered at $(0, \ldots, 0)$.)

Proof

1) Consider a section of $T(\beta_1, \ldots, \beta_k)$ by the plane $z = c$. We obtain the equations

$$x_i^2 + y_i^2 = c_i \,,$$

where $c_i = c^{2p_i/q_i}$. Clearly, these equations define a k-dimensional torus. The variety $T(\beta_1, \ldots, \beta_k)$ we obtain as a suspension of it. So, (1) is proved.

2) Let $r(z)$ be a function defined in the following way:

$$r(z) = \sqrt{z^2 + \sum_{i=1}^{k} z^{\beta_i}} \,.$$

This function $r(z)$ is a one-to-one function, for small z. Thus, for sufficiently small $\varepsilon > 0$, the link $T(\beta_1, \ldots, \beta_k) \cap S_{0,\varepsilon}$ is equal to the torus $T(\beta_1, \ldots, \beta_k) \cap \{(x_1, y_1, \ldots, x_k, y_k, z) \in \mathbb{R}^{2k+1} \mid z = r^{-1}(\varepsilon)\}$. □

Each point of $T(\beta_1, \ldots, \beta_k)$ has uniquely defined polar coordinates $(\psi_1, \psi_2, \ldots, \psi_k, z)$: ψ_i is the angle coordinate of the corresponding point of the circle $x_i^2 + y_i^2 = c_i$ and z is a z-coordinate in $\mathbb{R}^{2k+1}$. Let $x^0 = (\psi^0, z^0) = (\psi_1^0, \ldots, \psi_k^0, z^0)$ be a point of $T(\beta_1, \ldots, \beta_k)$. Let L_{x^0} be a curve on $T(\beta_1, \ldots, \beta_k)$ defined as follows:

$$L_{x^0} = \{(\psi_1, \psi_2, \ldots, \psi_k, z) \mid \psi_1 = \psi_1^0, \ldots, \psi_k = \psi_k^0\} \,.$$

We call L_{x^0} *a polar line generated by* x^0. Now we can define a polar map in the following way.

Denote, for $\varepsilon > 0$, the set

$$T(\beta_1, \ldots, \beta_k) \cap \{(x_1, y_1, \ldots, x_k, y_k, z) \in \mathbb{R}^{2k+1} \mid z \leq \varepsilon\}$$

by $T^\varepsilon(\beta_1, \ldots, \beta_k)$. Let $P_{\varepsilon_1,\varepsilon_2}: T^{\varepsilon_1}(\beta_1, \ldots, \beta_k) \to T^{\varepsilon_2}(\beta_1, \ldots, \beta_k)$ be a map defined as follows:

$$P_{\varepsilon_1,\varepsilon_2}(\psi_1, \ldots, \psi_k, z) = \left(\psi_1, \ldots, \psi_k, \frac{\varepsilon_1}{\varepsilon_2} z\right) .$$

We call $P_{\varepsilon_1,\varepsilon_2}$ *a polar map*. Observe that $P_{\varepsilon_1,\varepsilon_2}$ is a bi-Lipschitz map.

Remark 3.1.— $T(\beta_1)$ is an usual β_1-horn.

Remark 3.2.— $T(\beta_1, \dots, \beta_k)$ is included to $T(\beta_1, \dots, \beta_k, \dots, \beta_n)$ (here $n \geq k+1$) as a semialgebraic subset defined by the following equations $\psi_{k+1} = b_1$, $\psi_{k+2} = b_2$, ..., $\psi_n = b_{n-k}$, $b_1, \dots, b_{n-k} \in \mathbb{R}$.

4. Proof of the Realization theorem

We use the induction on the number of edges. Suppose that each Hölder Complex (Γ, β) whose graph Γ has less or equal than k edges is realized as a semialgebraic subset of $T(\beta_1, \dots, \beta_k)$ such that all vertices of Γ belong to the section by the plane $z = 1$ and, for each vertex a, we have $\psi_i(a) = 0$ or $\psi_i(a) = \pi$. (We can identify the graph Γ and its image by the map F; see Definition 2.4.)

For $k = 1$, the assertion is trivial: Γ has two vertices a_1 and a_2. Set $\psi(a_1) = 0$, $\psi(a_2) = \pi$ and the edge connecting a_1 and a_2 be a half-circle. So, (Γ, β) is realized as a half of the standard β-horn.

Now consider a Hölder Complex (Γ, β) such that Γ has $(k+1)$ edges. Let g be an edge such that $\beta(g) = \min_{\tilde{g} \in E_\Gamma} \beta(\tilde{g})$. Let us consider a graph $\widetilde{\Gamma} = \Gamma - g$. We have two possibilities: $\widetilde{\Gamma}$ is a connected graph or $\widetilde{\Gamma}$ is not connected.

Suppose that $\widetilde{\Gamma}$ is not connected. Then it is a union of two connected components $\widetilde{\Gamma} = \widetilde{\Gamma}^1 \cup \widetilde{\Gamma}^2$ (we include also a case when one of these components is just a vertex). We can suppose that $g_1, \dots, g_\ell \in E_{\tilde{\Gamma}^1}$, $g_{\ell+1}, \dots, g_k \in E_{\tilde{\Gamma}_2}$, $g_{k+1} = g$. Now consider a set $T\big(\beta_1, \dots, \beta_k, \beta(g)\big)$ and a section of that by the plane $z = 1$. This section is a $(k+1)$-dimensional torus (see the proof of the Lemma 3.1). By the induction hypotheses, the subcomplex $(\widetilde{\Gamma}^1, \widetilde{\beta}^1)$, where $\widetilde{\beta}^1 = \beta|_{\tilde{\Gamma}^1}$, can be realized as a semialgebraic subset of $T(\beta_1, \dots, \beta_k)$ which can be considered as a semialgebraic subset of $T\big(\beta_1, \dots, \beta_k, \beta(g)\big)$ given by the equation $\psi_{k+1} = 0$ (see the Remark 3.2). By the same way, $(\widetilde{\Gamma}^2, \widetilde{\beta}^2)$, where $\widetilde{\beta}^2 = \beta|_{\tilde{\Gamma}^2}$, can be realized as a semialgebraic subset of $T(\beta_1, \dots, \beta_k)$ which can be considered as a semialgebraic subset of $T\big(\beta_1, \dots, \beta_k, \beta(g)\big)$ given by the equation $\psi_{k+1} = \pi$. Suppose that g connects vertices $a_1 \in \widetilde{\Gamma}^1$ and $a_2 \in \widetilde{\Gamma}^2$;

let a_1 has polar coordinates $(\psi_1(a_1), \ldots, \psi_k(a_1), 0)$ and let a_2 has polar coordinates $(\psi_1(a_2), \ldots, \psi_k(a_2), \pi)$. We connect these two vertices by the following curve $\Psi(\theta) = \{\psi_1(\theta), \psi_2(\theta), \ldots, \psi_{k+1}(\theta), 1\}$ where

$$\psi_{k+1}(\theta) = \theta\,, \quad \psi_i(\theta) = \begin{cases} \psi_i(a_1) & \text{if } \psi_i(a_1) = \psi_i(a_2) \\ \theta & \text{if } \psi_i(a_1) = 0 \text{ and } \psi_i(a_2) = \pi \\ \pi + \theta & \text{if } \psi_i(a_1) = \pi \text{ and } \psi_i(a_2) = 0, \end{cases} \tag{3}$$

$1 \le i \le k$, $\theta \in [0\,,\,\pi]$. Clearly, $\Psi(0) = a_1$ and $\Psi(\pi) = a_2$. Define

$$H_{\beta(g)} := \bigcup_\theta L_{\Psi(\theta)}\,,$$

the union of the polar lines generated by $\Psi(\theta)$.

LEMMA 4.1.— *The set $H_{\beta(g)}$ is a $\beta(g)$-Hölder triangle.*

Proof.— $H_{\beta(g)}$ is a semialgebraic set because it is defined by the system (3) which can be written as a system of algebraic equations and inequalities in terms of variables x_i, y_i, for $1 \le i \le k+1$, and by the inequalities $0 \le z \le 1$. Hence, $H_{\beta(g)} \cap B_{0,\varepsilon}$ (here $B_{0,\varepsilon}$ is a closed ball in $\mathbb{R}^{2k+3}$ centered at 0 with the radius ε) is a Geometric Hölder Complex $H(\overline{\Gamma}, \alpha)$ corresponding to some graph $\overline{\Gamma}$ with some rational-valued function α defined on its edges [1]. Since $H_{\beta(g)}$ is a curvilinear triangle (by the construction), $H_{\beta(g)} \cap B_{0,\varepsilon_0}$, for sufficiently small $\varepsilon_0 \le \varepsilon$, is bi-Lipschitz equivalent to the standard α_0-Hölder triangle where $\alpha_0 = \min_{\overline{g} \in E_{\overline{\Gamma}}} \alpha(\overline{g})$ [1, Second Structural Lemma]. But $H_{\beta(g)} \cap B_{0,\varepsilon_0}$ is bi-Lipschitz equivalent to $H_{\beta(g)}$ (the bi-Lipschitz equivalence is given by the polar map $P_{\varepsilon_0,1}$).

To complete the proof of the lemma we must show that $\alpha_0 = \beta(g)$. Let γ_ε be the equidistant line in $H_{\beta(g)}$, namely $\gamma_\varepsilon = H_{\beta(g)} \cap S_{0,\varepsilon}$. By [1], there exists a subanalytic bi-Lipschitz map $\Upsilon\colon H_{\beta(g)} \to \mathrm{ST}_{\alpha_0}$ such that $\Upsilon(\gamma_\varepsilon) = \mathrm{ST}_{\alpha_0} \cap \{(x,y) \in \mathbb{R}^2 \mid x = \varepsilon\}$. Denote by $\ell(\gamma_\varepsilon)$ the length of γ_ε. Since Υ is a bi-Lipschitz map, we have

$$c_1 \varepsilon^{\alpha_0} \le \ell(\gamma_\varepsilon) \le c_2 \varepsilon^{\alpha_0}\,, \tag{4}$$

for some positive constants c_1 and c_2. To prove that $\alpha_0 = \beta(g)$ we will compute the length of γ_ε from another side. Consider the function

$$r(z) = \sqrt{z^2 + \sum_{i=1}^{k+1} z^{p_i/q_i}}$$

which is a one-to-one function, for small z. So, $r^{-1}(\varepsilon)$ is a well-defined function, for small ε. By the Lemma 3.1,

$$\gamma_\varepsilon = H_{\beta(g)} \cap \{(x_1, y_1, \ldots, x_{k+1}, y_{k+1}, z) \in \mathbb{R}^{2k+3} \mid z = r^{-1}(\varepsilon)\}.$$

Consider the following set

$$\begin{aligned} T^\varepsilon &= T(\beta_1, \ldots, \beta_k, \beta(g)) \\ &\quad \cap \{(x_1, y_1, \ldots, x_{k+1}, y_{k+1}, z) \in \mathbb{R}^{2k+1} \mid z = r^{-1}(\varepsilon)\}. \end{aligned}$$

It is a smooth manifold homeomorphic to a $(k+1)$-dimensional torus. The equidistant line γ_ε belongs to this set. There are $(k+1)$ differencial 1-forms $\mathrm{d}\psi_1^\varepsilon, \ldots, \mathrm{d}\psi_k^\varepsilon$ and $\mathrm{d}\psi_{k+1}^\varepsilon$ on T^ε corresponding to the coordinate system $\{\psi_1, \ldots, \psi_k, \psi_{k+1}\}$. By (3), we have

$$\ell(\gamma_\varepsilon) = \int_{\gamma_\varepsilon} \sum_{i=1}^{k+1} m_i \,\mathrm{d}\psi_i^\varepsilon \qquad \text{where } m_i = \begin{cases} 1 & \text{if } \psi_i(a_1) \neq \psi_i(a_2) \\ 0 & \text{if } \psi_i(a_1) = \psi_i(a_2), \end{cases}$$

$$\int_{\gamma_\varepsilon} \sum_{i=1}^{k+1} m_i \,\mathrm{d}\psi_i^\varepsilon \leq \sum_{i=1}^{k+1} \int_{\gamma_\varepsilon} m_i \,\mathrm{d}\psi_i^\varepsilon .$$

By the definition of the equidistant line γ_ε,

$$\int_{\gamma_\varepsilon} m_i \,\mathrm{d}\psi_i^\varepsilon = m_i \pi z^{\beta_i} .$$

Using the above formula we obtain

$$\ell(\gamma_\varepsilon) \leq \sum_{i=1}^{k+1} m_i \pi z^{\beta_i} .$$

If z sufficiently small ($z < 1$) there exists $\widetilde{C}_2 > 0$ such that

$$\ell(\gamma_\varepsilon) \leq \sum_{i=1}^{k+1} m_i \pi z^{\beta_i} \leq \widetilde{C}_2 z^{\beta(g)} ,$$

because $\beta(g) = \min_{1 \leq i \leq k+1} \beta_i$.

By the definition of the function $r(\varepsilon)$, we have $r(\varepsilon) = a\varepsilon + \mathrm{o}(\varepsilon)$, with $a > 0$.

Hence, $\ell(\gamma_\varepsilon) \leq C_2' \varepsilon^{\beta(g)}$, where $C_2' = a\widetilde{C}_2$. To obtain an estimate of $\ell(\gamma_\varepsilon)$ from below let us go back to the formulas (3)

$$\ell(\gamma_\varepsilon) = \int_{\gamma_\varepsilon} \sum_{i=1}^{k+1} m_i \, \mathrm{d}\psi_i^\varepsilon \geq \int_{\gamma_\varepsilon} m_{k+1} \, \mathrm{d}\psi_{k+1}^\varepsilon \, .$$

By (3), $m_{k+1} = 1$. Thus,

$$\ell(\gamma_\varepsilon) \geq \int_{\gamma_\varepsilon} \mathrm{d}\psi_{k+1}^\varepsilon = \pi z^{\beta(g)} \geq C_1' \varepsilon^{\beta(g)} \, ,$$

for some positive constant C_1'. So,

$$C_1' \varepsilon^{\beta(g)} \leq \ell(\gamma_\varepsilon) \leq C_2' \varepsilon^{\beta(g)} \, . \tag{5}$$

From (4) and (5) we obtain that $\beta(g) = \alpha_0$.

Lemma 4.1 is proved. □

Thus, the realization of (Γ, β) is given by the union of the realizations of $(\widetilde{\Gamma}^1, \widetilde{\beta}^1)$, $(\widetilde{\Gamma}^2, \widetilde{\beta}^2)$ and $H_{\beta(g)}$. It is a semialgebraic set because it is a finite union of semialgebraic sets.

Now consider the second case: $\widetilde{\Gamma}$ is a connected graph. In this case, by the induction hypotheses, $(\widetilde{\Gamma}, \widetilde{\beta})$ (where $\widetilde{\beta} = \beta|_{\widetilde{\Gamma}}$) can be realized as a semialgebraic subset of $T(\beta_1, \ldots, \beta_k)$ which can be considered as a semialgebraic subset of $T\big(\beta_1, \ldots, \beta_k, \beta(g)\big)$ defined by the equation $\psi_{k+1} = 0$. The edge g connects two vertices a_1 and a_2. Now we can glue the realization of $(\widetilde{\Gamma}, \widetilde{\beta})$ and the curvilinear triangle $H_{\beta(g)}$ generated by the curve $\Psi(\theta) = \big\{\psi_1(\theta), \psi_2(\theta), \ldots, \psi_{k+1}(\theta)\big\}$:

$$\psi_{k+1}(\theta) = \theta \quad \text{and} \quad \psi_i(\theta) = \begin{cases} \psi_i(a_1) & \text{if } \psi_i(a_1) = \psi_i(a_2) \\ \dfrac{\theta}{2} & \text{if } \psi_i(a_1) = 0 \text{ and } \psi_i(a_2) = \pi \\ \pi + \dfrac{\theta}{2} & \text{if } \psi_i(a_1) = \pi \text{ and } \psi_i(a_2) = 0, \end{cases} \tag{6}$$

for $1 \leq i \leq k$, $\theta \in [0, 2\pi]$, $a_1 = \big(\psi_1(a_1), \ldots, \psi_k(a_1), 0\big)$ and $a_2 = \big(\psi_1(a_2), \ldots, \psi_k(a_2), \pi)\big)$.

Set $H_{\beta(g)} := \bigcup_\theta L_{\Psi(\theta)}$. By the same arguments as in the Lemma 4.1, we can prove that $H_{\beta(g)}$ is a $\beta(g)$-Hölder triangle.

The union of the realization of $(\widetilde{\Gamma}, \widetilde{\beta})$ and $H_{\beta(g)}$ is a semialgebraic realization of (Γ, β).

The Realization theorem is proved. $\square$

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