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Pleating coordinates for the Earle embedding (*)

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Résumé. — On étudie les *coordonnées de plissage* pour la *section d'Earle* de l'espace $\mathcal{QF}$ des groupes quasifuchsiens pour un tore $\mathcal{T}_1$ épointé. Cette section $\mathcal{E}_\theta$ est constituée de groupes quasifuchsiens Γ pour lesquels il existe une involution conforme Θ de la sphère de Riemann qui induit la symétrie rhombique θ sur Γ, en échangant les générateurs marqués. La section d' Earle s'identifie naturellement à l'espace de Teichmüller Teich$(\mathcal{T}_1)$. On peut la considérer comme extension holomorphe de la ligne des losanges de Teich$(\mathcal{T}_1)$ dans $\mathcal{QF}$. Les *rayons de plissage* en $\mathcal{E}_\theta$ sont les ensembles sur lesquelles les classes projectives de la mesure de plissage du bord du coeur convexe sont fixes; ils héritent de la symétrie θ. On montre que ces rayons sont des lignes qui rencontrent la ligne rhombique dans $\mathcal{E}_\theta$ en les points critiques des fonctions de longeur correspondantes, ainsi on analyse les rayons de plissage rationnels dans $\mathcal{E}_\theta$ d'après [10, 11]. On montre qu'ils ne contiennent pas de singularités et qu'ils forment un feuilletage dense de $\mathcal{E}_\theta$, ce qui permet de calculer la position exacte de $\mathcal{E}_\theta$ dans l'espace ambient, voir la Figure 1. En étendant les résultats aux rayons irrationnels, on obtient les coordonnées de plissage sur $\mathcal{E}_\theta$.

Abstract. — We study *pleating coordinates* for the *Earle slice* of quasifuchsian space $\mathcal{QF}$ for the once punctured torus $\mathcal{T}_1$. This slice consists of quasifuchsian groups Γ for which there is a conformal involution Θ of the Riemann sphere which induces the rhombic symmetry θ on Γ which interchanges a pair of marked generators. The slice $\mathcal{E}_\theta$ is naturally identified with the Teichmüller space Teich$(\mathcal{T}_1)$. It can be thought as a holomorphic extension of the rhombus line in Teich$(\mathcal{T}_1)$ into $\mathcal{QF}$. *Pleating rays* are the loci in $\mathcal{E}_\theta$ on which the projective classes of the bending measure of

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the boundary of the convex core are fixed; they inherit the symmetry θ. We show that these rays are lines which meet the rhombus line in critical points of the corresponding length functions, and hence analyse rational pleating rays in $\mathcal{E}_\theta$ following [10, 11]. We show they are non-singular and densely foliate $\mathcal{E}_\theta$, allowing computation of the exact position of $\mathcal{E}_\theta$ in the ambient parameter space, see Figure 1. Extending our results to irrational rays gives pleating coordinates on $\mathcal{E}_\theta$.

1. Introduction

This paper is about pleating coordinates for the *Earle slice* of quasifuchsian space $\mathcal{QF}$ for the once punctured torus $\mathcal{T}_1$. A quasifuchsian once punctured torus group is a marked discrete subgroup $\Gamma \simeq \pi_1(\mathcal{T}_1)$ of $PSL_2(\mathbf{C})$ whose domain of discontinuity consists of two simply connected invariant components $\Omega^\pm$ whose quotients are punctured tori. In [5], Earle introduced certain special slices of $\mathcal{QF}$, consisting of groups for which Ω^+ and Ω^- are conformally equivalent under a map induced by a given involution of $\pi_1(\mathcal{T}_1)$. In this paper, we study the slice $\mathcal{E}_\theta$, which we call the *Earle slice*, consisting of those groups in $\mathcal{QF}$ for which there is a conformal involution Θ of the Riemann sphere which interchanges Ω^+ and Ω^-, and which induces the rhombic symmetry θ on Γ. This means that the induced map on Γ interchanges the marked generators A and B. In particular, a Fuchsian group lies in $\mathcal{E}_\theta$ if and only if the quotient torus is conformally a rhombus. Thus $\mathcal{E}_\theta$ can be thought of as a holomorphic extension of the rhombus line in the Teichmüller space $\mathrm{Teich}(\mathcal{T}_1)$ into $\mathcal{QF}$.

Earle proved (in the context of closed surfaces of arbitrary genus) that groups with such a symmetry give a holomorphic embedding of $\mathrm{Teich}(\mathcal{T}_1)$ into $\mathcal{QF}$. As is well known, the classical representation of $\mathrm{Teich}(\mathcal{T}_1)$ is the upper half plane $\mathbf{H}$. Thus in our situation, $\mathcal{E}_\theta$ is the conformal image of $\mathbf{H}$ under a Riemann map. In section 3, we write down an explicit family $\Gamma(d) = \langle A(d), B(d) : d \in \mathbf{C}\rangle$ of groups for which the matrix coefficients of the generators $A(d), B(d)$ are holomorphic functions of d on $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and such that $\mathcal{E}_\theta$ can be identified with $\{d \in \mathbf{C}^+ : \Gamma(d) \in \mathcal{QF}\}$ where $\mathbf{C}^+ = \{d \in \mathbf{C} : \Re e d > 0\}$. Thus there is exactly one group in $\mathbf{C}^+$ for each conjugacy class of quasifuchsian groups in $\mathcal{E}_\theta$. For $d \in \mathbf{R}^+ = \mathbf{R} \cap \mathbf{C}^+$ the group thus obtained is always Fuchsian, however in general, it is not at all clear for which d the group $\Gamma(d)$ is in $\mathcal{QF}$.

The method of pleating coordinates, originated in [10], can be viewed among other things as a method of computing the exact set of parameter values which correspond to a given quasiconfomal deformation class of a holomorphic family of Kleinian groups; in the present context, this means precisely, to determine for which d the group $\Gamma(d)$ is in $\mathcal{QF}$. The results of this paper allow one, among other things, to answer this question.

The method depends on locating what we call *pleating varieties* in $\mathcal{QF} \setminus \mathcal{F}$, where $\mathcal{F}$ is the space of Fuchsian groups. These may be thought of as loci in $\mathcal{QF}$ on which the shape and combinatorics of the dynamics on the limit set of Γ are of a fixed type. However it is easier to make a formal definition in terms of the action of Γ by isometries on hyperbolic 3-space $\mathbf{H}^3$, as follows.

Recall that a quasifuchsian punctured torus group acts on $\mathbf{H}^3$ with quotient $\mathbf{H}^3/\Gamma$ homeomorphic to $\mathcal{T}_1 \times (-1,1)$. The ends of $\mathbf{H}^3/\Gamma$ at infinity are the Riemann surfaces $\Omega^{\pm}/\Gamma$, each homeomorphic to $\mathcal{T}_1$. Let $\mathcal{C}$ be the hyperbolic convex hull of the limit set Λ of Γ in $\mathbf{H}^3$; equivalently $\mathcal{C}/\Gamma$ is the convex core of $\mathbf{H}^3/\Gamma$. The boundary $\partial\mathcal{C}/\Gamma$ of $\mathcal{C}/\Gamma$ has two connected components $\partial\mathcal{C}^{\pm}/\Gamma$, each homeomorphic to $\mathcal{T}_1$. These components are each pleated surfaces whose pleating or bending loci carry a transverse measure, the bending measure, whose projective classes we denote $pl^{\pm}(\Gamma)$.

Recall that the set of measured geodesic laminations on a hyperbolic surface is independent of the hyperbolic structure. Denote by $PML(\mathcal{T}_1)$ the set of projective measured laminations on $\mathcal{T}_1$. For $\xi, \eta \in PML(\mathcal{T}_1)$, define $\mathcal{P}_{\xi,\eta} = \{q \in \mathcal{QF} : pl^+(q) = \xi, pl^-(q) = \eta\}$. The (ξ,η)-*pleating variety* is defined to be the set $\mathcal{P}_{\xi,\eta} \subset \mathcal{QF}$. The philosophy of the method of pleating coordinates is that it is possible to identify and explicitly compute the exact position of a dense set of pleating varieties, namely, those for which the underlying laminations are *rational*, i.e., consist entirely of closed leaves. This programme has been carried out for the whole space $\mathcal{QF}(\mathcal{T}_1)$ in [14]; the present case, being a one complex dimensional slice, is much simpler, and this is what we examine here. Rather than use the full force of the results in [14], we introduce some techniques from complex analysis which depend on $\mathcal{E}_\theta$ being a biholomorphic image of $\mathbf{H}$. We believe the same techniques should be useful elsewhere.

The maximal number of closed leaves in a geodesic lamination on a punctured torus, is one. A rational pleating variety in $\mathcal{QF}(\mathcal{T}_1)$ is therefore specified by two simple closed curves; it is not hard to see that these curves must be distinct. Let γ be a simple closed geodesic on $\mathcal{T}_1$; it is represented in Γ by all those elements whose axes project to γ. The collection of all such elements consists of all members of a conjugacy class together with their inverses. We note that, up to ambiguity of sign, which will not affect our remarks below, the trace $\operatorname{Tr} g, g \in \Gamma$ is constant on this set. A simple closed geodesic also defines a projective measure class in $PML(\mathcal{T}_1)$; namely the class of the transverse δ-measure on its support. Thus for rational $\xi \in PML$, we may without ambiguity write $\operatorname{Tr}\xi$ for the trace of any element of $g \in \Gamma$ whose axis projects to the support of ξ.

The key point in the pleating coordinate method in the present situation is first, that for ξ, η rational, the pleating variety $\mathcal{P}_{\xi,\eta}$ is the union of connected components of the real locus $(\operatorname{Tr}\xi)^{-1}(\mathbf{R}) \cap (\operatorname{Tr}\eta)^{-1}(\mathbf{R})$ for the known holomorphic functions $\operatorname{Tr}\xi, \operatorname{Tr}\eta$; and second, that the exact position of these components can be identified and computed as a function of the parameter d.

Recall that $PML(\mathcal{T}_1)$ may be identified with the extended real line $\hat{\mathbf{R}} = \mathbf{R} \cup \infty$, in such a way that rational laminations correspond to rational numbers $\hat{\mathbf{Q}} = \mathbf{Q} \cup \infty$. With this identification understood, for $x, y \in \hat{\mathbf{R}}$, we let $\mathcal{P}_{x,y} = \{d \in \mathcal{E}_\theta : pl^+(d) = x, pl^-(d) = y\}$. Then for groups in $\mathcal{E}_\theta \setminus \mathcal{F}$, we have the further restriction that the boundary components $\partial\mathcal{C}^\pm$ are conjugate under the involution so that $pl^+(d) = x$ if and only if $pl^-(d) = 1/x$. Applying the pleating coordinate method to $\mathcal{E}_\theta$ we shall further prove that:

1. $\mathcal{P}_{x,1/x} \neq \emptyset$ provided $x \neq \pm 1$, and $\mathcal{P}_{x,y} = \emptyset$ otherwise.
2. The pleating varieties $\mathcal{P}_{x,1/x}$ and $\mathcal{P}_{1/x,x}$ are complex conjugate embedded arcs in $\mathcal{E}_\theta$. These arcs both limit on a unique point b_x representing a Fuchsian group in $\mathcal{E}_\theta$; the set $\mathcal{P}_{x,1/x} \cup \mathcal{P}_{1/x,x} \cup \{b_x\}$ is closed in $\mathcal{E}_\theta$.
3. For each $x \in \hat{\mathbf{Q}} \setminus \{\pm 1\}$, b_x is the unique critical point of the function $\operatorname{Tr} x$ on the positive real axis. This point is a minimum of $|\operatorname{Tr} x|$. Further, $\operatorname{Tr} x$ is strictly monotonic on $\mathcal{P}_{x,1/x}$ and the only other limit point of $\mathcal{P}_{x,1/x}$ in $\hat{\mathbf{C}}$ is a point $c_x \in \partial\mathcal{E}_\theta$ representing a cusp group at which $|\operatorname{Tr} x| = 2$.
4. The rational pleating varieties are dense in $\mathcal{E}_\theta$.

This allows us to draw the picture shown in Figure 1. The positive real axis $\mathbf{R}^+$ represents Fuchsian groups with the rhombic symmetry, and only the upper half of the Earle slice is shown, the picture being symmetrical under reflection in the real axis. As in [10, 14], we use normalised complex length as a substitute for the trace function to interpolate the irrational rays; on an irrational ray the point b_x is the unique critical point of this normalised length on $\mathbf{R}^+$.

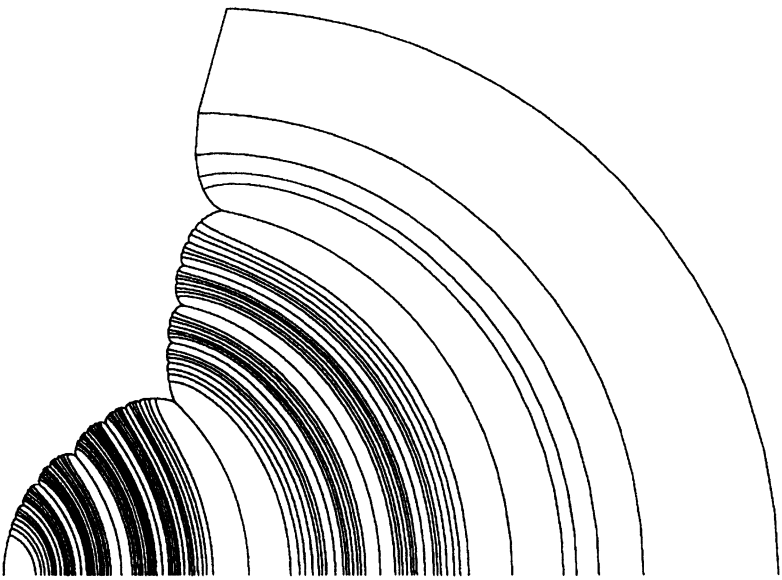

Fig. 1. The Earle Slice. *Courtesy of Peter Liepa.*

This is the upper half of the Earle slice; the complete picture is symmetrical under reflection in the real axis. The slice meets the real axis in the interval $(0, \infty)$ consisting of points representing the Fuchsian groups in $\mathcal{E}_\theta$. This interval is the image of the semicircle centre 0 radius 1 representing rhombi in Teich$(\mathcal{T}_1)$ under the Riemann map from ψ : $\mathbf{H} \to \mathcal{E}_\theta$, and $\psi(i) = 1/\sqrt{2}$.

The lines shown are rational pleating rays: the imaginary axis $\{iy : y > 1\}$ above i maps to the pleating ray $\mathcal{P}_{0,\infty}$, while $\{iy : 0 < y < 1\}$ maps to $\mathcal{P}_{\infty,0}$. Each ray ends in a cusp group, the boundary point $x \in \mathbf{Q}$ being mapped to the cusp point c_x.

In so far as possible, the methods of this paper have been kept independent of those in [14], so as not to obscure the much simpler situation in this present context. In dealing with the irrational pleating varieties, however, we need to use some rather general principles developed in [14], notably what we have stated as theorem 6.13 in section 6. We also refer to McMullen [24] who has used completely different techniques to prove the existence of pleating coordinates for Bers slices; we note however that his methods give existence only and do not allow one to locate the pleating varieties explicitly as we do here.

The paper is organized as follows. In section 2, we set up notation and prove Earle's theorem in our context. In section 3 we derive an explicit parameterisation and discuss some basic symmetries and the relation with the classical Teichmüller space of flat tori. In section 4 we explain the enumeration of simple closed curves on the punctured torus and derive some preliminary results about rational pleating varieties. The serious work begins in section 5 where we prove our main result theorem 5.1 about the structure rational pleating varieties, including most of the points listed above. Finally in section 6, we show how to interpolate the irrational rays and prove our main results theorems 6.16 and 6.17. The appendix 1 contains a summary by Peter Liepa of the method used to draw figure 1. In appendix 2, we show that pleating rays in $\mathcal{E}_\theta$ are not in general geodesics with respect to the hyperbolic metric on $\mathcal{E}_\theta$ induced by the canonical Riemann map from $\mathbf{H}$ to $\mathcal{E}_\theta$. A similar result for pleating rays in the Maskit embedding was recently proved by Matthews [23].

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2. Punctured tori and the Earle slice

Let $\mathcal{T}_1$ be an oriented once-punctured torus. An ordered pair α, β of generators of $\pi_1(\mathcal{T}_1)$ is called *canonical* if the algebraic intersection number of α and β with respect to the given orientation of $\mathcal{T}_1$ is equal to $+1$. The commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ represents a loop around the puncture.

A discrete subgroup $\Gamma \subset PSL_2(\mathbf{C})$ is called a *quasifuchsian once punctured torus group* if it is the image of a faithful representation $\rho : \pi_1(\mathcal{T}_1) \to PSL_2(\mathbf{C})$, such that $\rho([\alpha, \beta])$ is parabolic and such that the region of discontinuity Ω for the action of Γ on the Riemann sphere $\hat{\mathbf{C}}$ has exactly two simply connected invariant components $\Omega^{\pm}$. The group Γ is *marked* by the ordered pair of generators $A = \rho(\alpha), B = \rho(\beta)$.

The quotients $\Omega^{\pm}/\Gamma$ are both homeomorphic to $\mathcal{T}_1$ and inherit an orientation induced from the orientation of $\hat{\mathbf{C}}$. We choose the labelling so that Ω^{+} is the component such that the homotopy basis of Ω^{+}/Γ induced by the ordered pair of marked generators A, B of Γ is canonical. The group Γ is *Fuchsian* if the components $\Omega^{\pm}$ are round discs.

The following theorem is an adaptation of the main result of [5], see also [19, 20], to the present case. The proof is essentially the same as the original one given for the case of compact Riemann surfaces of genus greater than two. Recall that an isomorphism of Kleinian groups is called *type pre-*

serving if it maps loxodromic elements in $PSL_2(\mathbf{C})$ to loxodromics and parabolics to parabolics.

THEOREM 2.1.— *Let θ be an involution of $\pi_1(\mathcal{T}_1)$ induced by an orientation reversing diffeomorphism of a Riemann surface $\mathcal{T}_1$. Let (α, β) be a homotopy basis of $\pi_1(\mathcal{T}_1)$ canonical with respect to the orientation induced by the conformal structure on $\mathcal{T}_1$. Then, up to conjugation in $PSL_2(\mathbf{C})$, there exists a unique marked quasifuchsian group $\rho : \pi_1(\mathcal{T}_1) \to \Gamma = \langle A, B\rangle$, such that:*

1. *There is a conformal map $\mathcal{T}_1 \to \Omega^+/\Gamma$ inducing the representation ρ.*
2. *There is a Möbius transformation $\Theta \in PSL_2(\mathbf{C})$ of order two, which restricts to a conformal homeomorphism $\Omega^{\pm} \to \Omega^{\mp}$, such that $\Theta(\gamma z) = \theta(\gamma)\Theta(z)$ for all $\gamma \in \Gamma$ and $z \in \hat{\mathbf{C}}$.*

Proof. — First we show the existence of a marked quasifuchsian group $\Gamma = \langle A, B\rangle$ satisfying the above conditions. Fix a holomorphic universal covering map from the upper half plane $\mathbf{H}$ to $\mathcal{T}_1$, identifying $\pi_1(\mathcal{T}_1)$ with the group G of covering transformations. By hypothesis, there is an orientation reversing diffeomorphism of $\mathcal{T}_1$ that induces the involution θ. Choosing a particular lift of this diffeomorphism of $\mathcal{T}_1$ to $\mathbf{H}$, we get an orientation reversing diffeomorphism $f : \mathbf{H} \to \mathbf{H}$ satisfying $f(gz) = \theta(g)f(z)$ for all $g \in G$ and $z \in \mathbf{H}$. We remark that θ is a type preserving isomorphism of the Fuchsian group G. Put $h(z) = f(\bar{z})$ for z in the lower half plane $\mathbf{H}^*$. Then h is an orientation preserving diffeomorphism from $\mathbf{H}^*$ to $\mathbf{H}$ satisfying $h(gz) = \theta(g)h(z)$ for all $g \in G$ and $z \in \mathbf{H}^*$. Now we can define a Beltrami differential μ with respect to G by

$$\mu = \begin{cases} 0 & \textit{in } \mathbf{H} \\ h_{\bar{z}}/h_z & \textit{in } \mathbf{H}^*. \end{cases}$$

It should be remarked that we can choose the diffeomorphism of $\mathcal{T}_1$ to be quasiconformal, and that lifting this diffeomorphism, one automatically gets $|h_{\bar{z}}/h_z| < 1$. By the Measurable Riemann Mapping Theorem [1], there exists a quasiconformal map $w : \mathbf{C} \to \mathbf{C}$, unique up to conjugation in $PSL_2(\mathbf{C})$, which satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$. Hence w and $w \circ h^{-1}$ are conformal on $\mathbf{H}$. Put $\Gamma = wGw^{-1}$, $A = w\alpha w^{-1}$ and $B = w\beta w^{-1}$. Then $\Gamma = \langle A, B\rangle$ is a marked quasifuchsian group with invariant regions of discontinuity $\Omega^+ = w(\mathbf{H})$ and $\Omega^- = w(\mathbf{H}^*)$. Since the conformal map $w : \mathbf{H} \to \Omega^+$ induces the conjugacy between G and Γ, it projects to a conformal map $\mathcal{T}_1 \to \Omega^+/\Gamma$ satisfying condition (1). Moreover $M = whw^{-1} : \Omega^- \to \Omega^+$ is conformal. We claim that M is in fact a Möbius transformation. Put $\Theta = M$ in Ω^- and $\Theta = M^{-1}$ in Ω^+. Then for all $\gamma \in \Gamma$, we have

$\Theta\gamma\Theta^{-1} = \theta(\gamma)$ in the region of discontinuity Ω, so that Θ induces the type preserving isomorphism θ from Γ to itself. The Marden Isomorphism Theorem [21] states that if Γ is a geometrically finite Kleinian group of the second kind, then a conformal map from Ω to itself which induces a type preserving automorphism of Γ is a Möbius transformation. Thus Θ is Möbius; moreover by construction, $\Theta^2 = \mathrm{id}$ on Ω, which means that Θ is elliptic of order two and satisfies condition 2.

Next we show the uniqueness of $\Gamma = \langle A, B\rangle$ up to conjugacy in $PSL_2(\mathbf{C})$. For $i = 1, 2$ assume that $\Gamma_i = \langle A_i, B_i\rangle$ are marked quasifuchsian groups $\Gamma_i = \langle A_i, B_i\rangle$ with invariant regions of discontinuity $\Omega_i^{\pm}$ satisfying the conditions of the theorem with the Möbius transformations $\Theta_i : \Omega_i^- \to \Omega_i^+$. Then condition 1 gives a conformal map $H : \Omega_1^+ \to \Omega_2^+$ so that in Ω_2^+ we have $HA_1H^{-1} = A_2$ and $HB_1H^{-1} = B_2$. Put $F = H$ in Ω_1^+ and $F = (\Theta_2)^{-1} \circ H \circ \Theta_1$ in Ω_1^-. Then F maps Ω_1 to Ω_2 inducing a type preserving isomorphism from Γ_1 to Γ_2. The Marden Isomorphism Theorem again shows that F is a Möbius transformation, which gives the result. $\square$

Theorem 2.1 shows that the map sending $(\mathcal{T}_1; \alpha, \beta)$ to $(\Omega^+/\Gamma; A, B)$ defines a holomorphic embedding of the Teichmüller space Teich$(\mathcal{T}_1)$ of $\mathcal{T}_1$ into the space $\mathcal{QF} = \mathcal{QF}(\mathcal{T}_1)$ of marked quasifuchsian punctured torus groups modulo conjugation in $PSL_2(\mathbf{C})$. The idea is that the quasi-conformal deformation space $Def(G)$ of the Kleinian group $G = \langle \Gamma, \Theta\rangle$ is a holomorphic submanifold of $Def(\Gamma) = \mathcal{QF}$ naturally isomorphic to Teich$(\Omega/G) =$ Teich$(\mathcal{T}_1)$. The embedding depends only on the choice of the involution θ of $\pi_1(\mathcal{T}_1)$. We call the image, an *Earle slice* of $\mathcal{QF}$, and denote it $\mathcal{E}_\theta$. In the next section, we make an explicit choice of θ, and show how to realise the corresponding slice as a domain in $\mathbf{C}$.

3. Parametrisation of the rhombic Earle slice

3.1. Parametrisation

Let $\theta : \pi_1(\mathcal{T}_1) \to \pi_1(\mathcal{T}_1)$ be the involution $\theta(\alpha) = \beta, \theta(\beta) = \alpha$. Clearly, θ satisfies the condition of theorem 2.1. We begin by finding an explicit parametrisation of the groups in the corresponding Earle slice which we call *rhombic* because this slice can be thought of as a holomorphic extension of the rhombus line in Teich$(\mathcal{T}_1)$ into $\mathcal{QF}$, c.f. proposition 3.8. The parametrisation turns out to be essentially the restriction of Jørgensen's parametrisation [7] of $\mathcal{QF}(\mathcal{T}_1)$.

Suppose that θ is induced by the elliptic transformation Θ and denote the commutator $[A, B]$ by P. By assumption, P is parabolic; denote its fixed point by x_P. Since $\Theta A\Theta^{-1} = B$, we have $\Theta P\Theta^{-1} = P^{-1}$ and it follows that x_P is also a fixed point of Θ. From now on, θ will always denote this explicit involution and $\mathcal{E}_\theta$ will denote the corresponding Earle slice.

THEOREM 3.1.— *Let α, β be a canonical pair of generators for $\pi_1(\mathcal{T}_1)$ and let θ be the involution defined above. Let $\rho : \pi_1(\mathcal{T}_1) \to PSL_2(\mathbf{C})$ be a marked quasifuchsian punctured torus group in the Earle slice $\mathcal{E}_\theta$. Then, after conjugation by Möbius transformations if necessary, we can take representatives of $A = \rho(\alpha), B = \rho(\beta)$ in $SL(2, \mathbf{C})$ of the form $A = A(d)$, $B = B(d)$, $d \in \mathbf{C}^*$, where*

$$A(d) = \begin{pmatrix} \frac{d^2+1}{d} & \frac{d^3}{2d^2+1} \\ \frac{2d^2+1}{d} & d \end{pmatrix}, B(d) = \begin{pmatrix} \frac{d^2+1}{d} & -\frac{d^3}{2d^2+1} \\ -\frac{2d^2+1}{d} & d \end{pmatrix}.$$

The parameter d^2 is uniquely determined by ρ. The pairs of matrices $A(d), B(d)$ and $A(-d), B(-d)$ are uniquely determined by ρ and the normalisation $P(z) = z + 2$ and $\Theta(z) = -z$.

Proof. — Writing $P = [A, B]$ and using the remark about fixed points above, we can normalise so that $P(z) = z + 2$ and $\Theta(z) = -z$. Because Γ is a discrete subgroup of $PSL_2(\mathbf{C})$ and P is a commutator, when we lift Γ to $SL_2(\mathbf{C})$ the representative of P in $SL_2(\mathbf{C})$ is

$$P = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

We remark that $\operatorname{Tr} P = -2$, because if $\operatorname{Tr} P = 2$, then A and B have a common fixed point (see theorem 4.3.5 (i) in [2]), which implies that Γ must be elementary, a contradiction.

The point $B^{-1}A^{-1}(\infty)$ is a fixed point of Θ, and we deduce that $B^{-1}A^{-1}(\infty) = 0$. Combining this with $\Theta A\Theta^{-1} = B$ we find $\operatorname{Tr} AB = 2 + \frac{1}{d^2}$. Now writing $P = AB\Theta B^{-1}A^{-1}\Theta^{-1}$ we find expressions for A, B and AB which have the stated form. □

Remark 3.2. — We can also characterize the Earle slice $\mathcal{E}_\theta$ in terms of trace functions on $\mathcal{QF}$. Setting $x = \operatorname{Tr} A$, $y = \operatorname{Tr} B$ and $z = \operatorname{Tr} AB$, where A, B are the generator pair of the marked group $\Gamma = \langle A, B\rangle$ in $\mathcal{QF}$, gives an embedding of $\mathcal{QF}$ into $\{(x, y, z) \in \mathbf{C}^3 : x^2 + y^2 + z^2 = xyz\}$. In [7], Jørgensen gives the following explicit formula for the generators A, B of Γ in terms of the traces x, y, z, with the normalisation $[A, B] : z \to z + 2$:

$$A = \begin{pmatrix} x - \frac{y}{z} & \frac{x}{z^2} \\ x & \frac{y}{z} \end{pmatrix}, B = \begin{pmatrix} y - \frac{x}{z} & -\frac{y}{z^2} \\ -y & \frac{x}{z} \end{pmatrix}.$$

With $\Theta(z) = -z$ as above, $\Theta A \Theta^{-1} = B$ implies $\operatorname{Tr} A = \operatorname{Tr} B$. Conversely if $\operatorname{Tr} A = \operatorname{Tr} B$, one checks that $\Theta A \Theta^{-1} = B$. One concludes that $\mathcal{E}_\theta = \mathcal{QF} \cap \{(x, y, z) \in \mathbf{C}^3 : x = y\}$.

We have not been able to ascribe an obvious geometrical meaning to our parameter d. However one can see it determines the group as follows. The parameters $x = \operatorname{Tr} A, y = \operatorname{Tr} B$ and $z = \operatorname{Tr} AB$ determine the marked group $\langle A, B \rangle$ up to conjugacy in $PSL_2(\mathbf{C})$. Assuming that $x = y$, then the Markov equation $x^2 + y^2 + z^2 = xyz$ implies that y/z determines x. In our notation, $y/z = d$.

We write $\Gamma(d) = \langle A(d), B(d) \rangle \subset SL_2(\mathbf{C})$ for the marked group corresponding to the parameter d. The trace $2 + \frac{1}{d^2}$ of $A(d)B(d)$ is an invariant up to conjugation of $\Gamma(d)$. We note also that $\operatorname{Tr} A(d)B(d)^{-1} = 2(d^2 + 2)$. The choice of sign $\pm d$ corresponds to the ambiguity in lifting $\Gamma(d)$ to $SL_2(\mathbf{C})$. Thus d^2 distinguishes groups $\Gamma(d)$ up to conjugation, and in particular is a holomorphic global coordinate for $\mathcal{E}_\theta$, see [19, 5].

Set

$$\tilde{\mathcal{E}}_\theta = \{d \in \mathbf{C} : \Gamma(d) \in \mathcal{E}_\theta\}.$$

PROPOSITION 3.3.— *The group $\Gamma(d)$ is Fuchsian if and only if $d \in \mathbf{R}^* = \mathbf{R} - \{0\}$. In addition, $\mathbf{R}^* \subset \tilde{\mathcal{E}}_\theta$.*

Proof. — If $d \in \mathbf{R}^*$ then $\Gamma(d) \subset PSL_2(\mathbf{R})$ and $A(d), B(d)$ and $A(d)B(d)$ are all hyperbolic since their traces equal $\frac{2d^2+1}{d}, \frac{2d^2+1}{d}$ and $2 + \frac{1}{d^2}$ respectively. One can easily verify that the region outside the isometric circles of $A(d)^{\pm 1}, B(d)^{\pm 1}$ and $(A(d)B(d))^{\pm 1}$ (if $d \geqslant \frac{1}{\sqrt{2}}$), or of $A(d)^{\pm 1}, B(d)^{\pm 1}$ and $(A(d)B(d)^{-1})^{\pm 1}$, (if $d \leqslant \frac{1}{\sqrt{2}}$), and between the lines $\Re z = \pm 1$, satisfies all the conditions for Poincaré's theorem and hence that $\Gamma(d)$ is discrete and free, see also theorem 2.1 [8].

Conversely if $\Gamma(d) = \langle A(d), B(d) \rangle$ is Fuchsian, then the traces $\operatorname{Tr} A(d) = \frac{2d^2+1}{d}$ and $\operatorname{Tr} A(d)B(d) = 2 + \frac{1}{d^2}$ are both real, hence $d \in \mathbf{R}^*$. $\square$

We note in passing that by recent powerful results of Minsky [25], $\Gamma(d)$ is a punctured torus group if and only if $d \in \overline{\tilde{\mathcal{E}}_\theta}$. On the other hand, there are certainly discrete but not torsion free groups $\Gamma(d)$ outside $\overline{\tilde{\mathcal{E}}_\theta}$, see [28].

Let $\iota : \mathbf{C} \to \mathbf{C}$ denote complex conjugation. This induces a symmetry of $\tilde{\mathcal{E}}_\theta$, as follows.

PROPOSITION 3.4.— *The set $\tilde{\mathcal{E}}_\theta$ is invariant under complex conjugation. We have $\Omega(\bar{d})^+ = \iota(\Omega(d)^-)$, and the natural action of the marked* *group* $\Gamma(\bar{d}) = \langle A(\bar{d}), B(\bar{d})\rangle$ *on* $\Omega(\bar{d})^+$ *is the same as the action induced by conjugating the action of* $\Gamma(d) = \langle A(d), B(d)\rangle$ *as a marked group on* $\Omega(d)^-$ *by* ι.

Proof. — The group $\Gamma(\bar{d}) = \langle A(\bar{d}), B(\bar{d})\rangle$ is the conjugate of $\Gamma(d)$ by ι. Clearly, $\Gamma(\bar{d})$ is also a quasifuchsian once punctured torus group and, since $P(z) = z+2$ and $\Theta(z) = -z$ commute with ι, it belongs to $\tilde{\mathcal{E}}_\theta$. By considering fixed points, we see that $\iota(\Lambda(d)) = \Lambda(\bar{d})$, and hence $\iota(\Omega(d)) = \Omega(\bar{d})$. The generators $A(\bar{d}), B(\bar{d})$ are a canonical pair in $\iota(\Omega(d)^-)$ and the result follows. □

PROPOSITION 3.5.— *The imaginary axis* $\{d \in \mathbf{C} : Re(d) = 0\}$ *is outside* $\tilde{\mathcal{E}}_\theta$.

Proof. — From the trace equations $\mathrm{Tr}\, A(d)B(d) = 2 + \frac{1}{d^2}$ and $\mathrm{Tr}\, A(d)$ $B(d)^{-1} = 2(d^2+1)$, $A(d)B(d)$ and $A(d)B(d)^{-1}$ are elliptic on $\{d = iy \in \mathbf{C}^* : |y| \leqslant 1\}$ and $\{d = iy \in \mathbf{C}^* : |y| \geqslant 1\}$ respectively. On the other hand, any group $\Gamma(d)$ for $d \in \mathcal{E}_\theta$ is free and discrete, hence cannot contain elliptic elements. The result follows. □

As a consequence of proposition 3.5, we can choose the parameter d for parametrising $\mathcal{E}_\theta$ in the right half plane $\mathbf{C}^+ = \{d \in \mathbf{C} : \mathrm{Re} d > 0\}$, giving an embedding of $\mathcal{E}_\theta$ into $\mathbf{C}^+$. In other words, d is a holomorphic global coordinate for $\mathcal{E}_\theta$. From now on, we shall identify points in $\mathcal{E}_\theta$ with their image in this embedding. We sometimes refer to the positive real axis as the *Earle line* and denote it $\mathcal{F}_\theta$; from proposition 3.3, we have $\mathcal{F}_\theta = \mathcal{E}_\theta \cap \mathcal{F}$ where $\mathcal{F}$ is the space of Fuchsian punctured torus groups.

3.2. Symmetries of $\mathcal{E}_\theta$

We have already seen in lemma 3.4, that complex conjugation defines an anti-holomorphic involution of $\mathcal{E}_\theta$. There is also a holomorphic involution σ.

PROPOSITION 3.6.— *The map* $\sigma(d) = \frac{1}{2d}$ *defines a holomorphic involution of* $\mathcal{E}_\theta$. *The action of the marked group* $\Gamma(\sigma(d)) = \langle A(\sigma(d)), B(\sigma(d))\rangle$ *on* $\Omega(\sigma(d))^+$ *is conformally equivalent to the action of the marked group* $\Gamma(d) = \langle B(d), A(d)^{-1}\rangle$ *on* $\Omega(d)^+$.

Proof. — Let $\Gamma(d) = \langle A(d), B(d)\rangle$ be a marked quasifuchsian group in $\mathcal{E}_\theta$. The pair $B(d), A(d)^{-1}$ is also a canonical set of generators for $\Gamma(d)$, with the same components $\Omega(d)^\pm$ as $\Gamma(d) = \langle A(d), B(d)\rangle$. Thus using the same conformal involution Θ, we verify the conditions of theorem 2.1 for the group $\langle B(d), A(d)^{-1}\rangle$. In other words, $\langle B(d), A(d)^{-1}\rangle$ is also in $\mathcal{E}_\theta$ and so there exists $\sigma(d) \in \mathcal{E}_\theta$ such that $\Gamma(\sigma(d)) = \langle A(\sigma(d)), B(\sigma(d))\rangle$ is conjugate as a marked group to $\Gamma(d) = \langle B(d), A(d)^{-1}\rangle$. We have

$$2 + \frac{1}{\sigma^2(d)} = \operatorname{Tr} A(\sigma(d))B(\sigma(d)) = \operatorname{Tr} B(d)A(d)^{-1} = 2(1+2d^2)$$

so that $\sigma(d) = \frac{1}{2d}$, where we choose the sign to ensure $\Re(\sigma(d)) > 0$. □

Remark 3.7.— In fact, one can verify directly that $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ $B(\sigma(d))$ *conjugates* $\Gamma(\sigma(d)) = \langle A(\sigma(d)), B(\sigma(d))\rangle$ *to* $\Gamma(d) = \langle B(d), A(d)^{-1}\rangle$.

3.3. The Earle slice and the classical upper half plane

The Teichmüller space Teich($\mathcal{T}_1$) of once punctured tori can be naturally identified with the upper half plane $\mathbf{H}$. Briefly, for any $\tau \in \mathbf{H}$, let $G(\tau)$ denote the marked group generated by $\hat{A} : z \mapsto z+1$ and $\hat{B}(\tau) : z \mapsto z+\tau$. We consider $G(\tau)$ acting on $\mathbf{C}(\tau) = \{z \in \mathbf{C} | z \neq n + m\tau \;\; for \;\; m, n \in \mathbf{Z}\}$. The generators $\hat{A}$ and $\hat{B}(\tau)$ define a canonical homotopy basis of the marked Riemann surface $\mathbf{C}(\tau)/G(\tau)$. This correspondence defines the conformal map from $\mathbf{H}$ to Teich($\mathcal{T}_1$). By composing the natural conformal map from Teich($\mathcal{T}_1$) to $\mathcal{E}_\theta$ defined in theorem 3.1, we have a conformal homeomorphism $\psi : \mathbf{H} \to \mathcal{E}_\theta$, which we again call the *Earle embedding*. The following result relates the symmetries of $\mathbf{H}$ and $\mathcal{E}_\theta$.

PROPOSITION 3.8.— *1.* $\psi^{-1} \circ \sigma \circ \psi(\tau) = -1/\tau$.

2. Under the Earle embedding ψ, *the semicircle* $\{\tau \in \mathbf{H} : |\tau| = 1\}$ *corresponds to the Earle line* $\mathcal{F}_\theta$.

3. $\psi^{-1} \circ \iota \circ \psi(\tau) = 1/\bar{\tau}$.

Proof.

1. The proof of proposition 3.6 shows that σ is the involutive element of the Teichmüller modular group which replaces the canonical generators $A(d), B(d)$ with the pair $B(d), A(d)^{-1}$. The corresponding map on the τ plane is induced by $\tau \to -1/\tau$.

2. Following [8], a marked punctured torus $(S; \alpha, \beta)$ is called *rhombus* if S admits an anticonformal involution which induces the involution of $\pi_1(S)$ sending α to β. Then $\{\tau \in \mathbf{H} : |\tau| = 1\}$ in $\mathbf{H}$ and $\mathcal{F}_\theta$ in $\mathcal{E}_\theta$ are the rhombus line in their respective embeddings of Teich($\mathcal{T}_1$). Therefore the Earle embedding ψ maps $\{\tau \in \mathbf{H} : |\tau| = 1\}$ to $\mathcal{F}_\theta$.

3. Assertion 3 follows from Assertion 2, because $\mathcal{F}_\theta$ and $\{\tau \in \mathbf{H} | |\tau| = 1\}$ are the fixed point sets of ι and $\tau \to 1/\bar{\tau}$ respectively. $\square$

From this proposition, we can deduce that $\psi(i) = 1/\sqrt{2}$, and that $\psi(\{iy \in \mathbf{C} : y > 0\})$ is the intersection of the circle centre 0 and radius $1/\sqrt{2}$ with $\mathcal{E}_\theta$.

4. Simple closed curves and the pleating locus

4.1. Simple closed curves

Denote by $\mathcal{S}$, the set of free unoriented homotopy classes of simple closed non-boundary parallel curves on $\mathcal{T}_1$. As is well known, this set may be naturally identified with $\hat{\mathbf{Q}} = \mathbf{Q} \cup \infty$. One way to see this is described in [30], see also [31, 3]. Let $\mathcal{L}$ denote the integer lattice $m + in, m, n \in \mathbf{Z} \subset \mathbf{C}$. Topologically $\mathcal{T}_1$ is the quotient of the punctured plane $\mathbf{C}(i) = \mathbf{C} - \mathcal{L}$ by the natural action of $G(i) = \langle \hat{A}, \hat{B}(i) \rangle \equiv \mathbf{Z}^2$ by horizontal and vertical translations. A straight line of rational slope in $\mathbf{C} - \mathcal{L}$ projects onto a simple closed curve on the marked punctured torus $S(i) = \mathbf{C}(i)/G(i)$, and the projection of all lines of the same rational slope and the same orientation are homotopic. We denote the unoriented homotopy class obtained by projecting the line of slope $-q/p$ by $[L_{p/q}]$. Relative to our choice of marking, $[L_{p/q}]$ is in the homology class of $\alpha^{-p}\beta^{q}$ or $\alpha^{p}\beta^{-q}$ on $\mathcal{T}_1$, where α, β are projections of horizontal and vertical lines corresponding to $\hat{A}, \hat{B}(i)$ respectively. Setting $1/0 = \infty$, we obtain:

PROPOSITION 4.1.— *The map $\hat{\mathbf{Q}} \to \mathcal{S}$ defined by $p/q \mapsto [L_{p/q}]$ is well-defined and bijective.*

Proof. — See [30, 3] or [31]. $\square$

Remark 4.2.— The reason for the choice of convention that $[L_{p/q}]$ corresponds to $\alpha^{-p}\beta^{q}$, is that if we identify the Teichmüller space $\mathrm{Teich}(\mathcal{T}_1)$ of once punctured tori with the upper half plane $\mathbf{H}$, then one can easily compute that the boundary point $p/q \in \hat{\mathbf{R}}$ is the point where the extremal length of curves in the class $[L_{p/q}]$ has shrunk to zero, see also lemmas 5.3, 5.4, and 5.5.

Suppose that $\rho : \pi_1(\mathcal{T}_1) \to \Gamma \subset PSL_2(\mathbf{C})$ is a quasifuchsian punctured torus group, marked as usual by generators $A = \rho(\alpha), B = \rho(\beta)$. We denote the unique geodesic in the homotopy class of $\rho([L_{p/q}])$ in $\mathbf{H}^3/\Gamma$ by $\gamma_{p/q}$. In particular, for $d \in \mathcal{E}_\theta$, we denote by $\gamma_{p/q}(d)$ the corresponding geodesic in $\mathbf{H}^3/\Gamma(d)$.

In [32], D. Wright gave a beautiful recursive scheme for finding, for each $p/q \in \hat{\mathbf{Q}}$, an explicit word $W_{p/q}$ in the marked generators $\langle \alpha, \beta \rangle$ of $\pi_1(\mathcal{T}_1)$ representing $[L_{p/q}]$. Although this scheme is not logically necessary in what follows, since we deal entirely with traces which depend only on conjugacy classes in $\Gamma(d)$, it is essential for carrying out computations. We record it here, see also [10] for details.

The words are generated from the initial data

$$W_{0/1} = \beta,\ W_{1/0} = \alpha^{-1}$$

by the formula

$$W_{(p+r)/(q+s)} = W_{r/s} W_{p/q},$$

whenever $p/q < r/s$ and $ps - qr = -1$.

It is easy to see that $W_{p/q}$ is in the correct homology class; to see that it represents a simple curve, observe that the curves represented by the generator pair $W_{0/1} = \beta,\ W_{1/0} = \alpha^{-1}$ are both simple, and that for any $p/q < r/s$ with $ps - qr = -1$, both $(W_{p/q}, W_{(p+r)/(q+s)})$ and $(W_{(p+r)/(q+s)}, W_{r/s})$ are marked pairs of generators obtained from the previous pair $(W_{p/q}, W_{r/s})$ by a Nielsen move (equivalently a Dehn twist) about one or other of $W_{p/q}$ or $W_{r/s}$. We also note that the recursion is set up in such a way that $[W_{p/q}^{-1}, W_{r/s}] = [B^{-1}, A^{-1}]$ whenever $p/q < r/s$ and $ps - qr = -1$.

The canonical isomorphism $\pi_1(S) \to \Gamma_d,\ d \in \mathcal{E}_\theta$, sending α, β to $A(d), B(d)$, allows us to identify $W_{p/q}$ with a specific element of $\Gamma(d)$.

4.2. The pleating locus

We are now ready to discuss the convex hull boundary and the pleating locus. Let $d \in \mathcal{E}_\theta$ and let $\Gamma = \langle A(d), B(d) \rangle$ be the corresponding marked quasifuchsian group with regular set and limit set Ω, Λ respectively. The 3-manifold $\mathbf{H}^3/\Gamma$ is homeomorphic to $\mathcal{T}_1 \times (0,1)$. The surfaces Ω/Γ at infinity form the boundary $\mathcal{T}_1 \times \{0,1\}$. As in the introduction, let $\partial\mathcal{C}$ be the boundary in $\mathbf{H}^3$ of the hyperbolic convex hull of Λ; it is clearly invariant under the action of Γ. The nearest point retraction $\Omega \to \partial\mathcal{C}$, defined as in [6] by mapping $x \in \Omega$ to the unique point of contact with $\partial\mathcal{C}$ of the largest horoball in H^3 centered at x with interior disjoint from $\partial\mathcal{C}$, can easily be modified to a Γ-equivariant homeomorphism [12]. We denote two connected components of $\partial\mathcal{C}$ corresponding to $\Omega^\pm$ by $\partial\mathcal{C}^\pm$ respectively. Thus each component $\partial\mathcal{C}^\pm/\Gamma$ is topologically a punctured torus. (In the special case in which Γ

is Fuchsian, $\partial\mathcal{C}$ is a flat plane whose two sides serve as a substitute for the two components $\partial\mathcal{C}^{\pm}$.)

As shown in detail in [6], $\partial\mathcal{C}^{\pm}/\Gamma$ are pleated surfaces in $\mathbf{H}^3/\Gamma$. More precisely, there are complete hyperbolic surfaces $S^{\pm}$, each homeomorphic to $\mathcal{T}_1$, and maps $f^{\pm} : S^{\pm} \to \mathbf{H}^3/\Gamma$, such that every point in $S^{\pm}$ is in the interior of some geodesic arc which is mapped by $f^{\pm}$ to a geodesic arc in $\mathbf{H}^3/\Gamma$, and such that $f^{\pm}$ induce isomorphisms $\pi_1(\mathcal{T}_1) \to \Gamma$. Further, $f^{\pm}$ are isometries onto their images with the path metric induced from $\mathbf{H}^3$. The *bending* or *pleating locus* of $\partial\mathcal{C}^{\pm}/\Gamma$ consists of those points of $S^{\pm}$ contained in the interior of one and only one geodesic arc which is mapped by $f^{\pm}$ to a geodesic arc in $\mathbf{H}^3/\Gamma$. For Γ non-Fuchsian, the pleating loci are geodesic laminations, meaning they are closed unions of pairwise disjoint simple geodesics on $S^{\pm}$. We denote these laminations by $|\,pl^{\pm}(d)|$, and usually identify such a lamination with its image under $f^{\pm}$ in $\mathbf{H}^3/\Gamma$. (See also section 6.1 below, especially for an explanation of the notation.) Motivated by the identification of laminations on $\mathcal{T}_1$ with $\hat{\mathbf{R}}$ (see the introduction and section 6.1), a geodesic lamination is called *rational* if it consists entirely of closed leaves. For the moment, we concentrate on the special case in which at least one of the pleating loci is rational. (It will follow from our results that in this case the other will be automatically rational too.) Since the maximum number of pairwise disjoint simple closed curves on a punctured torus is one, such a lamination consists of a single simple closed geodesic and is therefore of the form $\gamma_{p/q}$ for some $p/q \in \hat{\mathbf{Q}}$.

For $p/q, r/s \in \hat{\mathbf{Q}}$, define:

$$\mathcal{P}^{+}_{p/q} = \{d \in \mathcal{E}_\theta : |pl^{+}(d)| = \gamma_{p/q}\}$$
$$\mathcal{P}^{-}_{p/q} = \{d \in \mathcal{E}_\theta : |pl^{-}(d)| = \gamma_{p/q}\}$$

and

$$\mathcal{P}_{p/q,r/s} = \mathcal{P}^{+}_{p/q} \cap \mathcal{P}^{-}_{r/s}.$$

We call $\mathcal{P}_{p/q,r/s}$ the *$(p/q, r/s)$-pleating variety* or *pleating ray.* This terminology will be justified by theorem 5.1 below.

It is clear from the definitions that $\mathcal{P}_{p/q,r/s} \cap \mathcal{P}_{p'/q',r'/s'} = \emptyset$ unless $p/q = p'/q', r/s = r'/s'$. We can deduce some further easy properties from the symmetries σ, ι. This uses the following lemma which is a restatement in the present situation of lemma 2.1 in [17].

LEMMA 4.3. — *Let $j : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be an conformal or anticonformal bijection. Suppose that the pleating locus $|\,pl^{+}(d)|$ of the marked group*

$\Gamma(d) = \langle A(d), B(d)\rangle$ consists of a simple closed geodesic represented by a word $W(A(d), B(d))$. Then the pleating locus of the marked group

$$j\Gamma(d)j^{-1} = \langle jA(d)j^{-1}, jB(d)j^{-1}\rangle$$

is a closed geodesic represented by the word $W(jA(d)j^{-1}, jB(d)j^{-1})$.

Roughly, this works because the convex hull can be defined in terms of maximal round disks inscribed inside the regular set and round circles are preserved by conformal maps. The condition that a certain simple closed geodesic is the pleating locus becomes the condition that its fixed points are the intersection points of two maximal inscribed disks.

Applying this lemma to the involutions θ, ι and σ we find:

COROLLARY 4.4.— *Suppose that $d \in \mathcal{E}_\theta$ and that $d \in \mathcal{P}^+_{p/q}$. Then*

1. *$d \in \mathcal{P}^-_{q/p}$,*
2. *$\bar{d} \in \mathcal{P}^+_{q/p}$, and*
3. *$\sigma(d) \in \mathcal{P}^+_{-q/p}$.*

Proof.

1. Suppose that $|pl^+(d)| = \gamma_{p/q}$. Apply lemma 4.3 to the involution $\Theta : z \mapsto -z$ sending $\Omega^+(\Gamma(d))$ to $\Omega^-(\Gamma(d))$ and the marked generators $A(d), B(d)$ to the marked generators $B(d), A(d)$.

2. From 1, we have $|pl^-(d)| = \gamma_{q/p}$, so that the pleating locus of the marked group $\Gamma(d) = \langle A(d), B(d)\rangle$ in the component $\partial\mathcal{C}(d)^-$ is represented by the word $W_{q/p} = W_{q/p}(A(d), B(d))$. Then by the lemma, the pleating locus of the marked group $\Gamma(\bar{d}) = \langle A(\bar{d}), B(\bar{d})\rangle$ obtained by conjugating the action of $\Gamma(d) = \langle A(d), B(d)\rangle$ by ι, in the component $\iota(\partial\mathcal{C}(d)^-)$, is a closed geodesic represented by the word $W_{q/p}(A(\bar{d}), B(\bar{d}))$. On the other hand, by lemma 3.4, the action of $\Gamma(\bar{d}) = \langle A(\bar{d}), B(\bar{d})\rangle$ on $\iota(\Omega(d)^-)$ is the same as the action of $\Gamma(\bar{d}) = \langle A(\bar{d}), B(\bar{d})\rangle$ on $\Omega(\bar{d})^+$. Thus $|pl^+(\bar{d})| = \gamma_{q/p}$.

3. By construction, the action of $\Gamma(\sigma(d)) = \langle A(\sigma(d)), B(\sigma(d))\rangle$ on $\Omega(\sigma(d))^+$ is the same as the action of $\Gamma(d) = \langle B(d), A^{-1}(d)\rangle$ on $\Omega(d)^+$. Applying the lemma, we find that if $|pl^+(d)| = \gamma_{p/q}$ so that the pleating locus of $\Gamma(d) = \langle A(d), B(d)\rangle$ acting on $\Omega(d)^+$ is represented by the word $W_{p/q}(A(d), B(d))$, then the pleating locus of $\Gamma(d) = \langle B(d), A(d)^{-1}\rangle$ is represented by the word $W_{p/q}(B(d), A(d)^{-1})$. Clearly from the definitions, $W_{p/q}(B(d), A(d)^{-1}) = W(-q/p)(A(d), B(d))$, giving the result. □

Combining these results we find:

PROPOSITION 4.5.— *The sets* $\mathcal{P}_{x,y}, x, y \in \hat{\mathbf{Q}}$ *are pairwise disjoint and:*

1. $\mathcal{P}_{x,x} = \emptyset$.
2. $\mathcal{P}_{x,y} = \emptyset$ *unless* $y = 1/x$.
3. $\iota(\mathcal{P}_{x,1/x}) = \mathcal{P}_{1/x,x}$ *and* $\sigma(\mathcal{P}_{x,1/x}) = \mathcal{P}_{-1/x,-x}$

Proof. — The first part of the statement is obvious from the definitions. It is not hard to see ([13] proposition 3.3 and corollary 3.4), that, if $\Gamma(d)$ is not Fuchsian, the same geodesic cannot be simultaneously contained in both sides of $\partial\mathcal{C}$, hence $\mathcal{P}_{x,x} = \emptyset$. The remainder is just a restatement of 4.4 above. □

We remark that since $\sigma \circ \iota$ fixes the union of pleating varieties $\mathcal{P}_{0,\infty} \cup \mathcal{P}_{\infty,0}$, it follows from proposition 3.8 that $\mathcal{P}_{0,\infty} \cup \mathcal{P}_{\infty,0}$ corresponds to the imaginary axis in the upper half plane model of Teich$(\mathcal{T}_1)$.

From now on, we shall write $\mathcal{P}_{p/q}$ for $\mathcal{P}_{p/q,q/p}$. We deduce immediately that

COROLLARY 4.6.— *The pleating varieties* $\mathcal{P}_{\pm 1}$ *are empty.*

We shall prove in theorem 4.10 below, that for $x \neq \pm 1$, $\mathcal{P}_x \neq \emptyset$. The geometrical explanation why $\mathcal{P}_{\pm 1} = \emptyset$ is that the projection of the involution Θ to $\mathbf{H}^3/\Gamma(d)$ exchanges the boundaries of the convex core, whereas it fixes the two geodesics corresponding to words $A(d)B(d)$ and $A(d)B(d)^{-1}$.

4.3. Traces and pleating rays

In this section, we collect some deeper facts about the pleating varieties $\mathcal{P}_{p/q}$ for $p/q \in \hat{\mathbf{Q}}$, which will be the key to our proof of the main theorem 5.1.

For any $p/q \in \hat{\mathbf{Q}}$, consider the trace $\operatorname{Tr} W_{p/q}$ of the special word $W_{p/q}$ associated to p/q by the Wright recursion rules above. For $d \in \mathbf{C}^*$, we have a representation $\rho : \pi_1(\mathcal{T}_1) \to SL(2, \mathbf{C})$, so that we may consider the function $T_{p/q}(d) = \operatorname{Tr} W_{p/q}(d)$ as a rational function on $\mathbf{C}^*$. We define the *hyperbolic locus* of $T_{p/q}$ to be the set

$$\mathcal{H}_{p/q} = \{d \in \mathbf{C}^* : T_{p/q}(d) \in \mathbf{R},\ |T_{p/q}(d)| > 2\}.$$

The following proposition gives some basic properties.

PROPOSITION 4.7.— *For all $p/q \in \hat{\mathbf{Q}}$:*

1. *The Earle line $\mathcal{F}_\theta$ is contained in the hyperbolic locus $\mathcal{H}_{p/q}$.*

2. *$T_{p/q} = T_{q/p}$ and hence $\mathcal{H}_{p/q} = \mathcal{H}_{q/p}$.*

3. *The pleating varieties $\mathcal{P}_{p/q}$ and $\mathcal{P}_{q/p}$ are contained in $\mathcal{H}_{p/q}$.*

Proof.

1. If $\Gamma(d) = \langle A(d), B(d) \rangle$ is Fuchsian, then $W_{p/q}$ is hyperbolic.

2. $\Theta A(d) \Theta^{-1} = B(d)$ and the fact that traces depend only on conjugacy classes implies that $\mathrm{Tr}(W_{p/q}) = \mathrm{Tr}(\Theta W_{p/q} \Theta^{-1}) = \mathrm{Tr}(W_{q/p})$.

3. This is lemma 4.6 of [10]. The proof is as follows. For any $d \in \mathcal{P}_{p/q}$, the axis $\mathrm{Ax}\, W_{p/q}(d)$ is a lift to $\mathbf{H}^3$ of the pleating locus $\gamma_{p/q}$ of $\partial\mathcal{C}(d)/\Gamma(d)$. This axis is the intersection of two support planes of $\partial\mathcal{C}(d)$. Because $W_{p/q}(d)$ preserves $\partial\mathcal{C}(d)$, it leaves these planes invariant, hence is purely hyperbolic. □

By statement 3 above, for $p/q \in \hat{\mathbf{Q}}$, the pleating variety $\mathcal{P}_{p/q}$ is contained in the hyperbolic locus $\mathcal{H}_{p/q}$. However (see [10]), the converse is not always true. Theorem 4.8 below gives a much deeper relationship between rational pleating varieties and hyperbolic loci. The main point is that $\mathcal{P}_{p/q}$ is both open and closed in $\mathcal{H}_{p/q} \setminus \mathcal{F}_\theta$. Similar results are proved in proposition 5.4 of [10] and theorem 3.7 of [11]. A more general result involving irrational pleating laminations is proved in [14]; this we shall summarise and use in section 6. Here, we sketch a proof along the lines of [10] and [11].

A subgroup F of $\Gamma(d)$ is called *$F-$peripheral* if it is Fuchsian and if one of the two open discs bounded by the circle containing the limit set $\Lambda(F)$ of F contains no points of $\Lambda(\Gamma(d))$. We call this open disc *the peripheral disc of F* and denote it by $\Delta(F)$.

Peripheral discs correspond exactly to the circle chains discussed in [10]. For $d \in \mathcal{P}_{p/q}$ there are exactly two distinct conjugacy classes $\mathcal{C}^+, \mathcal{C}^-$ of $F-$peripheral discs, corresponding to the two components of $\partial\mathcal{C}(d)$. Discs in $\mathcal{C}^+$ intersect along lifts of the axis of $\gamma_{p/q}$ at an angle ϑ, say, while discs in $\mathcal{C}^-$ intersect in the same angle along the axes of lifts of $\gamma(q/p)$. No disc in $\mathcal{C}^+$ intersects any disc in $\mathcal{C}^-$. The union of the discs in $\mathcal{C}^+$ is exactly Ω^+ and the union of the discs in $\mathcal{C}^-$ is Ω^-. The limit set $\Lambda(\Gamma(d))$ is the common boundary of these two families of discs. In the special case in which $d \in \mathcal{F}_\theta$, $\vartheta = 0$ and all the discs in $\mathcal{C}^+$ degenerate into one disc whose boundary consists of the entire limit set $\Lambda(\Gamma(d))$.

Let $\mathcal{U}_{p/q}$ be the set of subgroups of $\Gamma(d)$ generated by $V = V(d)$ and $W^{-1}VW = W(d)^{-1}V(d)W(d)$ with the condition that V represents the free homotopy class of $\gamma_{p/q}$ and V and W generate $\Gamma(d)$. A key idea needed for the next theorem is the following result (c.f. proposition 3.6 in [11]): $d \in \mathcal{E}_\theta \setminus \mathcal{F}_\theta$ is contained in $\mathcal{P}_{p/q}$ if and only if $\Gamma(d)$ has an F−peripheral subgroup $F = \langle V, W^{-1}VW\rangle \in \mathcal{U}_{p/q}$ and $\Delta(F) \subset \Omega(d)^+$.

Now we can prove our theorem.

THEOREM 4.8.— *The rational pleating variety $\mathcal{P}_{p/q}$ is open and closed in $\mathcal{H}_{p/q} \setminus \mathcal{F}_\theta$.*

Proof. — First we show the openness. Let $d_0 \in \mathcal{E}_\theta \setminus \mathcal{F}_\theta$ be a point of $\mathcal{P}_{p/q}$. Then $\Gamma(d_0)$ has an F−peripheral subgroup $F(d_0) \in \mathcal{U}_{p/q}$. The peripheral disc $\Delta(F(d_0))$ of $F(d_0)$ is covered by images of a fundamental domain for $F(d_0)$ acting in $\Delta(F(d_0))$, whose sides are a finite number of geodesic arcs whose endpoints can be taken to be specific fixed points of elements of $F(d_0)$. Under a small variation of d in which $F = \langle V, W^{-1}VW\rangle$ remains Fuchsian, these arcs do not move much and still define a fundamental domain for $F(d)$ acting in the disc bounded by $\Lambda(F(d))$. As in proposition 3.1 of [11], it follows that $F = F(d)$ remains F−peripheral. Now as in lemma 3.2 in [11], $d \in \mathcal{H}_{p/q}$ implies that $F(d)$ is Fuchsian. Since $\mathcal{E}_\theta \setminus \mathcal{F}_\theta$ is open in the right half plane, we conclude that $\mathcal{P}_{p/q}$ is open in $\mathcal{H}_{p/q} \setminus \mathcal{F}_\theta$.

Now we show the closedness. Consider $d_n \in \mathcal{P}_{p/q}$ with $d_n \to d_\infty \in \mathcal{H}_{p/q} \setminus \mathcal{F}_\theta$. As in [11] theorem 3.7, the groups $\Gamma(d_n)$ have F−peripheral subgroups $F(d_n) = \langle V(d_n), W(d_n)V(d_n)W^{-1}(d_n)\rangle \in \mathcal{U}_{p/q}$. The limit group $F(d_\infty)$ is clearly Fuchsian, and the circle containing $\Lambda(F(d_\infty))$ bounds an open disc which is the limit of the peripheral discs $\Delta(F(d_n))$. It is easy to see by a limiting argument that this disc contains no limit points of $\Gamma(d_\infty)$, and hence that $F(d_\infty)$ is also F−peripheral. Hence we conclude that $d_\infty \in \mathcal{P}_{p/q}$, which shows that $\mathcal{P}_{p/q}$ is closed in $\mathcal{H}_{p/q} \setminus \mathcal{F}_\theta$. □

COROLLARY 4.9.— *The rational pleating variety $\mathcal{P}_{p/q}$ is a union of connected components of $\mathcal{H}_{p/q} \setminus \mathcal{F}_\theta$.*

We remark that since $\mathcal{H}_{p/q}$ is defined without reference to $\mathcal{E}_\theta$, this corollary gives a way of determining whether or not $d \in \mathcal{E}_\theta$, without having to test discreteness of $\Gamma(d)$ in any other way. Thus it is a vital key to finding the location of $\mathcal{E}_\theta \subset \mathbf{C}^*$.

The following result based on [13] shows that, for $p/q \in \hat{\mathbf{Q}} \setminus \{\pm 1\}$, the pleating variety $\mathcal{P}_{p/q}$ is non-empty. This is a key point in the study of pleating coordinates. A more general result is given in theorem 8.9 in [14] which we shall need in the last section.

THEOREM 4.10. — *If $d \in \mathcal{E}_\theta \setminus \mathcal{F}_\theta$ is in the hyperbolic locus $\mathcal{H}_{p/q}$ and sufficiently near the Earle line $\mathcal{F}_\theta$, then $d \in \mathcal{P}_{p/q} \cup \mathcal{P}_{q/p}$.*

Proof. — Following [13], we shall construct directly a component of $\partial\mathcal{C}(d)$ whose pleating locus is $\gamma_{p/q}$.

Let $V = V_{p/q}(d) \in \Gamma(d)$ represent the free homotopy class of $\gamma_{p/q}$ and choose $W \in \Gamma(d)$ such that V, W are a generator pair. Since $d \in \mathcal{H}_{p/q}$ and the commutator $[V, W]$ is parabolic, the group $F = \langle V, W^{-1}VW \rangle$ is Fuchsian. Let δ be the common perpendicular to the axis $\mathrm{Ax}\, V$ of V and $\mathrm{Ax}\, W^{-1}VW$, and let W' be the unique element of $PSL_2(\mathbf{C})$ which sends $\mathrm{Ax}\, W^{-1}VW$ to $\mathrm{Ax}\, V$ in such a way that the complex distance between δ and $W'(\delta)$ is real and equal to the real part of the complex distance between δ and $W(\delta)$. Clearly, the group $G(d) = \langle V, W' \rangle$ is Fuchsian. The map $\chi : V, W' \mapsto V, W$ defines a natural isomorphism of $G(d)$ with $\Gamma(d)$, and since $W^{-1}VW = W'^{-1}VW'$, the groups $G(d)$ and $\Gamma(d)$ have F as a common Fuchsian subgroup.

Denote by $\mathbf{D}$ the hyperbolic plane in $\mathbf{H}^3$ containing $\mathrm{Ax}\, V$ and $\mathrm{Ax}\, W^{-1}VW$. Since G_d is Fuchsian, its action leaves $\mathbf{D}$ invariant. We are going to define a pleated surface map from $(\mathbf{D}, G(d))$ to a bent surface $(\hat{\mathbf{D}}, \Gamma(d))$ in $\mathbf{H}^3$. Let N be the Nielsen region of F acting in $\mathbf{D}$, and let $\hat{\mathbf{D}} = \Gamma(d) \cdot N \subset \mathbf{H}^3$. Define $\phi : \mathbf{D} \to \hat{\mathbf{D}}$ by $\phi(g(z)) = \chi(g)(\phi(z)), g \in G_d, z \in N$. It is not hard to see that ϕ is a pleated surface map whose image surface $\hat{\mathbf{D}}$ is bent along the $\Gamma(d)$-orbit of $\mathrm{Ax}\, V$ and whose flat pieces are the $\Gamma(d)$-orbit of N. The bending angle ϑ is the angle between the planes containing N and $W(N)$, which can be measured as the imaginary part of the complex distance between δ and $W(\delta)$. We want to conclude that for d near $\mathcal{F}_\theta$, $\hat{\mathbf{D}}$ is a component of $\partial\mathcal{C}(d)$.

As in [13] proposition 7.2, if $|\vartheta|$ is sufficiently small, then $\phi : \mathbf{D} \to \hat{\mathbf{D}}$ is an embedding and $\hat{\mathbf{D}}$ is one of the two components of $\partial\mathcal{C}(d)$. In our case this condition is satisfied since $\vartheta = \vartheta(d) \to 0$ whenever $d \to \mathcal{F}_\theta$ along $\mathcal{H}_{p/q}$. (In [13] we studied only deformations in which the length of $\gamma_{p/q}$ was held constant. However it is clear that for sufficiently small deformations all the necessary estimates work even if $l(\gamma_{p/q})$ varies, see also the discussion in [13] proposition 7.6.) From its construction, $\hat{\mathbf{D}}$ has $\gamma_{p/q}$ as its pleating locus, hence $d \in \mathcal{P}_{p/q} \cup \mathcal{P}_{q/p}$. $\square$

5. Rational rays: the main theorem

We are now able to deduce our main result about the rational pleating varieties $\mathcal{P}_{p/q}$.

THEOREM 5.1.— *For any $p/q \in \hat{\mathbf{Q}}$ with $p/q \neq \pm 1$, $\mathcal{P}_{p/q}$ is an embedded arc in $\mathcal{E}_\theta$. The set of limit points of $\mathcal{P}_{p/q}$ in $\mathbf{C} \setminus \mathcal{P}_{p/q}$ consists of the two endpoints of the arc: a point $c_{p/q}$ on $\partial\mathcal{E}_\theta$ at which $|T_{p/q}(c_{p/q})| = 2$, and a point $b_{p/q} \in \mathcal{F}_\theta$ which is the unique critical point of $T_{p/q}(d)$ on $\mathcal{F}_\theta$.*

This theorem justifies the terminology *pleating rays*. The rays are the arcs shown in figure 1 on page 73. Figure 2 shows the real loci of T_2 (left) and T_3 (right) in the first quadrant of the d-plane $\mathbf{C}$.

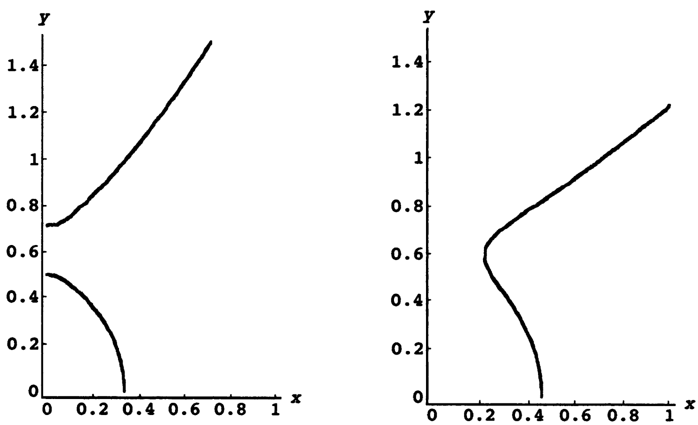

Fig. 2. Real loci of the trace functions T_2 and T_3

We need several lemmas.

LEMMA 5.2.— *For any $p/q \in \hat{\mathbf{Q}}$ with $p/q \neq \pm 1$, the trace function $T_{p/q}$ restricted to the Earle line $\mathcal{F}_\theta$ is proper and has at least one maximum or minimum on $\mathcal{F}_\theta$.*

Proof. — Let $\gamma_{p/q}$ be the simple closed geodesic corresponding to p/q on $\Omega(d)^+/\Gamma(d)$. For $d \in \mathcal{F}_\theta$, the group $\Gamma(d)$ is Fuchsian and the hyperbolic lengths of curves on $\Omega(d)^\pm/\Gamma(d)$ (with respect to the Fuchsian uniformisation of the complex structure) are given by the corresponding traces of elements in $\Gamma(d)$ as $2\cosh^{-1}\frac{|T_{p/q}(d)|}{2}$. Because $T_{1/1}(d) = \operatorname{Tr} A(d)^{-1}B(d) =$

$4d^2+2$ and $T_{-1/1}(d) = \operatorname{Tr} A(d)B(d) = 2+\frac{1}{d^2}$, the hyperbolic length of $\gamma_{1/1}$ goes to 0 when $d \to 0$ along $\mathcal{F}_\theta$, and similarly the hyperbolic length of $\gamma_{-1/1}$ tends to 0 as $d \to \infty$. The axis of any simple closed geodesic $\gamma_{p/q}, p/q \neq \pm 1$ intersects both of $\gamma_{\pm 1/1}$ on $\partial\mathcal{C}_d^+/\Gamma(d)$ and so whenever the length of $\gamma_{\pm 1/1}$ tends to zero, the length of $\gamma_{p/q}$ tends to infinity by the collar lemma. The conclusion about the critical point follows since the sign of $T_{p/q}$ is constant on $\mathcal{F}_\theta$. □

To make precise the connection between the classical upper half plane picture of Teich($\mathcal{T}_1$) and the Earle slice, we have to use conformal invariants, so we need to consider extremal length, see for example [1]. We can then make a standard comparison between extremal and hyperbolic lengths of short curves.

LEMMA 5.3.— *Consider* $\mathbf{H}$ *as the Teichmüller space* Teich($\mathcal{T}_1$) *as explained in section 3.3. Let* $C_{p/q}(k)$ *be the set of* $\tau \in \mathbf{H}$ *such that the extremal length of the homotopy class* $[L_{p/q}]$ *in* $\mathbf{C}_\tau/G_\tau$ *is less than* k*, and let* $\overline{C_{p/q}(k)}$ *be the closure of* $C_{p/q}(k)$ *in* $\hat{\mathbf{C}}$*. Then:* $\overline{C_{p/q}(k)} \cap \partial\mathbf{H} = \{p/q\}$*; if* $k_1 > k_2$ *then* $C_{p/q}(k_1) \supset C_{p/q}(k_2)$*; and* $\cap_{k>0}\overline{C_{p/q}(k)} = \{p/q\}$*.*

Proof. — By definition, $[L_\infty]$ corresponds to the element $\hat{A} \in G(\tau)$, $\hat{A}(z) = z+1$. The boundary $\partial C_\infty(k)$ is the horizontal line $\Im\tau = 1/k$ (see e.g. example 1 in P. 12 of [1]), so that in this case all the claims are immediate. Now the modular group $PSL_2(\mathbf{Z})$ acts on $\mathbf{H}$ as the mapping class group of $\mathcal{T}_1$, inducing the obvious action on $G(\tau) \equiv \mathbf{Z}^2$. Thus $\partial C_{p/q}(k)$ is the image of $\Im\tau = 1/k$ under any element $g \in PSL_2(\mathbf{Z})$ which takes $[L_\infty]$ to $[L_{p/q}]$. Now as noted in remark 4.2, the extremal length of the class $[L_{p/q}]$ goes to zero at the point $p/q \in \mathbf{R} \cup \{\infty\}$. Hence $g(\infty) = p/q$ and $C_\infty(k)$ is conjugated into $C_{p/q}(k)$ which is therefore a horocycle at p/q. The results follow. □

We now want a similar result for hyperbolic length in the Fuchsian uniformisation of the complex structure of $\Omega(d)^+/\Gamma(d)$.

LEMMA 5.4.— *Let* $D_{p/q}(k) \subset \mathbf{H}$ *be the inverse image under the Earle embedding* ψ *of the set of* $d \in \mathcal{E}_\theta$ *such that the hyperbolic length of the curve* $\gamma_{p/q}$ *in* $\Omega(d)^+/\Gamma(d)$ *is less than* k*. Then there exist constants* $c, c_1, c_2 > 0$ *such that for* $k < c$*,* $C_{p/q}(c_1 k) \supset D_{p/q}(k) \supset C_{p/q}(c_2 k)$*.*

Proof. — By construction, for $d = \psi(\tau)$, the surfaces $\Omega^+(d)/\Gamma(d)$ and $\mathbf{C}(\tau)/G(\tau)$ are conformally equivalent. The result follows from a well known comparison of extremal and hyperbolic lengths due to Maskit [22]. □

Finally we can compare to hyperbolic length in $\mathbf{H}^3/\Gamma(d)$.

LEMMA 5.5.— *Suppose that* $d \in \mathcal{P}_{p/q}$ *and* $|T_{p/q}(d)| \to 2$. *Then* $d \in \psi(D_{p/q}(k))$ *as* $k \to 0$.

Proof. — For $d \in \mathcal{P}_{p/q}$, the hyperbolic length of $\gamma_{p/q}$ in $\mathbf{H}^3/\Gamma(d)$ is the same as the hyperbolic length in the hyperbolic structure of the pleated surface $\partial\mathcal{C}(d)^+/\Gamma(d)$. Thus it suffices to compare the hyperbolic lengths l_1, l_2 of $\gamma_{p/q}$ on $\partial\mathcal{C}(d)^+/\Gamma(d)$ and $\Omega(d)^+/\Gamma(d)$ respectively. The result follows either by a theorem of Sullivan [6] which states that there is a Lipschitz map with universally bounded constant between these two structures; or by the inequality $l_2 < l_1$ which is a strengthened form of Bers inequality due to McMullen, [24] corollary 3.2. □

We now consider the image of the sets $C_{p/q}(k), k > 0, p/q \in \hat{\mathbf{Q}}$ under the Riemann map $\psi : \mathbf{H} \to \mathcal{E}_\theta$. Recall that a prime end of such a map is an equivalence class of cross cuts, and that an accessible boundary point defines such an end. The following is part of Carathéodory's theory of prime ends.

PROPOSITION 5.6.— *Let* U *be a simply connected domain in* $\mathbf{C}$ *and let* D *be the unit disc. Then any arc in* U *which lands at one point of* ∂U *corresponds, under the Riemann map, to an arc in* D *which lands at one point of* ∂D. *Arcs which define distinct prime ends of* ∂U *necessarily correspond to arcs which land at distinct points of* ∂D.

Proof. — See proposition 2.14 and theorem 2.15 in [29]. □

Proof of theorem 5.1.— Suppose $p/q \in \hat{\mathbf{Q}} \setminus \pm 1$. By lemma 5.2, the trace function $T_{p/q}$ has a critical point $b_{p/q}$ on $\mathcal{F}_\theta$, so that in a neighbourhood we may choose a branch of $\mathcal{H}_{p/q}$ in $\mathbf{C}^* \setminus \mathcal{F}_\theta$. Since $T_{p/q}$ is a rational function, it is a branched covering of $\hat{\mathbf{C}}$. Therefore we can follow this branch, possibly through a critical point but always decreasing $|T_{p/q}|$, until we reach a point $c_{p/q}$ at which $|T_{p/q}(c_{p/q})| = 2$. This gives a simple arc $\lambda_{p/q}$ from $b_{p/q}$ to $c_{p/q}$. Corollary 4.9 and theorem 4.10 show that the interior of $\lambda_{p/q}$ is contained in $\mathcal{P}_{p/q} \cup \mathcal{P}_{q/p}$. Hence $\lambda_{p/q}$ is contained in $\mathcal{E}_\theta \setminus \mathcal{F}_\theta$ and $c_{p/q} \in \partial\mathcal{E}_\theta$. Thus $\lambda_{p/q}$ defines a prime end of $\mathcal{E}_\theta$. By taking its image under complex conjugation if necessary, we may assume by corollary 4.4 that $\lambda_{p/q} \subset \mathcal{P}_{p/q}$.

Now suppose that $a_{p/q} \in \mathcal{P}_{p/q} \setminus \lambda_{p/q}$. By proposition 4.7, $a_{p/q} \in \mathcal{H}_{p/q}$. As before, using analytic continuation following the branch of $\mathcal{H}_{p/q}$ starting from $a_{p/q}$ in the direction of decreasing $|T_{p/q}|$, we find using corollary 4.9 another arc $\lambda \subset \mathcal{P}_{p/q}$ which ends in a boundary point $c \in \partial\mathcal{E}_\theta$ such that $|T_{p/q}(c)| = 2$, which again defines a prime end of $\mathcal{E}_\theta$. If the arc from $a_{p/q}$

meets $\lambda_{p/q}$ it must do so in a critical point of $T_{p/q}(b)$, so that by following different branches of $\mathcal{H}_{p/q}$ through the critical point we can always arrange that any intersections of the arcs λ and $\lambda_{p/q}$ are transversal. We claim that the prime ends defined by $\lambda_{p/q}$ and λ are distinct. If $c \neq c_{p/q}$, this is obvious, so suppose that $c = c_{p/q}$. Since $\lambda_{p/q}$ and λ are distinct, $c_{p/q}$ must be a critical point of $T_{p/q}$. Thus $\lambda_{p/q}$ and λ are separated on both sides in a neighbourhood of $c_{p/q}$ by arcs on which $T_{p/q} \in \mathbf{R}$ and $2 > |T_{p/q}| > 0$, which are certainly outside $\mathcal{E}_\theta$.

Now, by Carathéodory's theorem, $\psi^{-1}(\lambda_{p/q})$ and $\psi^{-1}(\lambda)$ must have distinct endpoints in $\partial\mathbf{H}$. On the other hand, both branches satisfy the conditions of lemma so that by lemmas 5.3, 5.4, 5.5, $\psi^{-1}(\lambda_{p/q})$ and $\psi^{-1}(\lambda)$ both limit on $p/q \in \partial\mathbf{H}$. This is impossible. This also shows the uniqueness of the critical point $b_{p/q}$ of the trace function $T_{p/q}$ on $\mathcal{F}_\theta$. □

Remark. — The uniqueness of the critical point $b_{p/q}$ of $T_{p/q}$ on $\mathcal{F}_\theta$ also follows from proposition 6.12 in section 6, where we also prove that the map $p/q \mapsto b_{p/q}$ extends to a strictly monotone map from $(-1,1) \to \mathcal{F}_\theta$, so that if $p/q \neq r/s$, then $b_{p/q} \neq b_{r/s}$.

The following propositions complete our picture of the rational rays.

PROPOSITION 5.7.— *The endpoints of distinct pleating rays on $\partial\mathcal{E}_\theta$ are distinct: if $p/q, r/s \in \hat{\mathbf{Q}} \setminus \pm 1, p/q \neq r/s$, then $c_{p/q} \neq c_{r/s}$.*

Proof. — The group $\Gamma(c_{p/q})$ is a maximal parabolic group in the sense of [9], with parabolic elements $W_{p/q}$ and $W_{q/p}$ in addition to the parabolic commutator $[A,B]$. Thus if $r/s \notin \{p/q, q/p\}$, we cannot have in addition $W_{r/s}$ parabolic, so $c_{p/q} \neq c_{r/s}$. From corollary 4.4 we have $\mathcal{P}_{q/p} = \iota(\mathcal{P}_{p/q})$ and hence that $c_{q/p} = \iota(c_{p/q})$. Since $\Gamma(c_{p/q})$ is certainly not Fuchsian, $c_{q/p} \neq \iota(c_{p/q})$ and the result follows. □

Remark. — By theorem 3 of [9], a maximally parabolic group is uniquely determined up to conjugacy in $PSL_2(\mathbf{C})$ by its abstract isomorphism class and its parabolic elements. The groups $\Gamma(c_{p/q})$ and $\Gamma(c_{q/p})$ have the same parabolic elements but the conjugacy between them reverses orientation, so that they represent distinct points on $\partial\mathcal{E}_\theta$, see also [17].

Define $\mathcal{E}_\theta^+ = \{d \in \mathcal{E}_\theta : \Im d > 0\}$. Now we determine for which p/q, $\mathcal{P}_{p/q} \subset \mathcal{E}_\theta^+$.

LEMMA 5.8.— $\mathcal{P}_0 \subset \mathcal{E}_\theta^+$.

Proof. — Since $\mathcal{P}_0$ is contained in the real locus of $\operatorname{Tr} B$, direct calculation and theorem 4.8 shows that either $\mathcal{P}_0 = \{d \in \mathbf{C} : d = \frac{1}{\sqrt{2}}e^{i\theta}, \ -\pi/4 < \theta < 0\}$, or $\mathcal{P}_0 = \{d \in \mathbf{C} : d = \frac{1}{\sqrt{2}}e^{i\theta}, \ 0 < \theta < \pi/4\}$. If a group is in $\mathcal{P}_0$, then by definition the pleating locus of $\partial\mathcal{C}^+$, consisting of the axis of B and its conjugates, faces the component Ω^+. Applying proposition 6.2 in [27] to our situation (see also figure 5 of p.193 in [27]), the marked group $\Gamma(d) = \langle A(d), B(d)\rangle$, $d \in \mathcal{P}_0$, is conjugate to $\Gamma = \langle A, B\rangle$ where

$$A = \begin{pmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{pmatrix}, \quad B = \begin{pmatrix} \cosh a & ie^b \sinh a \\ -ie^{-b}\sinh a & \cosh a \end{pmatrix},$$

and in addition $a > 0, b < 0$ and $(\sinh a)^2 \cosh b = 1$. By considering $\operatorname{Tr} AB$, we can conclude that $\mathcal{P}_0 = \{d \in \mathbf{C} : d = \frac{1}{\sqrt{2}}e^{i\theta}, \ 0 < \theta < \pi/4\}$. $\square$

Remark 5.9. — Another way to see the above result is as follows. By lemmas 5.3, 5.4, 5.5, for $p/q \neq \pm 1$, the Earle embedding $\psi : \mathbf{H} = \operatorname{Teich}\mathcal{T}_1 \to \mathcal{E}_\theta$ extends to map the boundary point $p/q \in \partial D$ to the cusp $c_{p/q}$. As in lemma 5.2, it also maps $1 \in \partial\mathbf{H}$ to $0 \in \partial\mathcal{F}_\theta$ and $-1 \in \partial\mathbf{H}$ to $\infty \in \partial\mathcal{F}_\theta$. This map preserves anticlockwise order round $\partial\mathbf{H}$, which forces the result.

There are two further results about how the rational rays sit in the Earle slice. The proofs depend on facts about irrational rays and are given in the next section, but to complete our picture we state the results here.

PROPOSITION 5.10. — *If $-1 < p/q < 1$, then $\mathcal{P}_{p/q} \subset \mathcal{E}_\theta^+$. Moreover $b_{p/q} > b_{r/s}$ whenever $-1 < p/q < r/s < 1$.*

This result is proved at the end of section 6.4.

THEOREM 5.11. — *The union of the rational pleating rays is dense in $\mathcal{E}_\theta$.*

This result is proved at the end of section 6.3.

6. Irrational rays

In order to make the extension of theorem 5.1 from simple closed geodesics to "irrational" laminations, we have to use a number of results from [14]. We begin by discussing in more detail exactly what the irrational laminations are.

6.1. Geodesic laminations

A *geodesic lamination* on a hyperbolic surface is a closed set that is a union of pairwise disjoint simple geodesics called its *leaves*, see [31, 4]. A geodesic lamination is *measured* if it carries a transverse measure μ, that is, an assignment of a finite Borel measure to each transversal which is invariant under isotopies preserving leaves. We use μ to denote both the measured lamination and the transverse measure and denote the underlying lamination by $|\mu|$. In particular, if γ is a simple closed non-boundary parallel curve, then the unique geodesic in the homotopy class of γ is a geodesic lamination which carries a transverse measure of the form $c\delta_\gamma, c > 0$, which assigns mass c to each intersection with γ. Such laminations we call *rational* and denote the set of rational measured laminations by ML_Q. Two measured laminations $\mu, \mu' \in ML$ are *projectively equivalent* if $|\mu| = |\mu'|$ and if there exists $c > 0$ such that for any arc σ transverse to the leaves of $|\mu|$, $\mu'(\sigma) = c\mu(\sigma)$. Thus the family of measured laminations $c\delta_\gamma, c > 0$ are projectively equivalent.

We denote the set of all projective measured laminations on $\mathcal{T}_1$ with no leaves that end in the cusp by PML; this set is independent of the hyperbolic structure on $\mathcal{T}_1$ and is well known to be homeomorphic to $S^1 \simeq \mathbf{R} \cup \{\infty\}$ (see for example [31] or [3]). Proposition 4.1 gives an embedding $\hat{\mathbf{Q}} \to PML$ which identifies a curve in $\mathcal{S}$ with its corresponding projective class. The remaining projective laminations, corresponding to $\mathbf{R} \setminus \mathbf{Q}$, may be obtained by projecting families of parallel lines of irrational slope in $\mathbf{C}$ to $\mathcal{T}_1$ and "pulling tight" to obtain a family of non-closed and pairwise disjoint geodesics which intersect local transversals in Cantor like sets (whose Hausdorff dimension is however zero). This picture is explained in more detail in [30]. The identification is continuous with respect to the natural weak (measure) topology on PML and we always identify PML with $\hat{\mathbf{R}}$ in this explicit fixed way.

6.2. Normalised complex length

In order to deal with irrational laminations, we need a substitute for the trace function $T_{p/q}$. Suppose first that Σ is a hyperbolic surface $\mathbf{H}^2/G, G \subset PSL_2(\mathbf{R})$ and let $g \in G$ represent a geodesic γ on Σ. The hyperbolic length $l(\gamma)$ is related to the trace by $|\operatorname{Tr} g| = 2\cosh(l(\gamma)/2)$. Now let $\mu \in ML(\Sigma)$ be a measured geodesic lamination on Σ. One can define the hyperbolic length $l(\mu)$ of μ by integrating the length of leaves against the measure μ on transversals, for details see [15]. With this definition, the length function is continuous on ML, and, for a geodesic $\gamma \in \mathcal{S}$, $l(c \cdot \delta_\gamma) = c \cdot l(\gamma), c \geqslant 0$.

Now for $G \subset PSL_2(\mathbf{C})$ one has the similar formula $\operatorname{Tr} g = 2\cosh(\lambda(\gamma)/2)$, where now $\lambda(\gamma)$ is the complex length of γ. (For a more geometrical interpretation, see for example [27].) We can pick a well defined branch of $\lambda(\gamma)$ by choosing it to be positive on $\mathcal{F}$. In [14] section 5, a normal family argument is used to extend the *complex* length function to a holomorphic function on $\mathcal{QF}$, whose restriction to $\mathcal{F}$ is exactly the lamination length above. For $\mu \in ML_Q$ and $q \in \mathcal{QF}$, let $\lambda_\mu(q)$ be the complex length of μ at q. The following is [14] theorem 5.3.

PROPOSITION 6.1.— *The function $ML_Q \times \mathcal{QF} \to \mathbf{C}$ defined by $(\mu, q) \mapsto \lambda_\mu(q)$ extends to a continuous function $\lambda_\mu : ML \times \mathcal{QF} \to \mathbf{C}$ defined by $(\mu, q) \mapsto \lambda_\mu(q)$. The length function $q \mapsto \lambda_\mu(q)$ is non-constant and holomorphic for all μ. For $K \subset ML$ compact, the family $\{\lambda_\mu\}_{\mu \in K}$ is uniformly bounded and equicontinuous on compact subsets of $\mathcal{QF}$.*

We want to restrict to $\mathcal{E}_\theta$. Rather than consider all lengths of laminations in the same projective class, it will be sufficient to normalise the length functions by choosing a continuous section $PML \to ML$. This we do by choosing μ_x to be the unique lamination in the class x for which $l_{\mu_x}(d_0) = 1$, where $d_0 = 1/\sqrt{2}$ represents the square torus. (Note that $l_{\mu_x}(d_0) = \lambda_{\mu_x}(d_0)$, and that this normalisation is equivalent to taking the length to be $l_\mu(d)/l_\mu(d_0)$ for all $\mu \in ML$.) Thus for each $x \in \hat{\mathbf{R}}$, we define the normalised length function $L_x : \mathcal{E}_\theta \to \mathbf{C}$ by $L_x(d) = \lambda_{\mu_x}(d)$. Applied to our situation, we immediately deduce:

THEOREM 6.2.— *The function $\hat{\mathbf{R}} \times \mathcal{E}_\theta \to \mathbf{C}$ defined by $(x, d) \mapsto L_x(d)$ has the following properties:*

1. *It is holomorphic in d for fixed x.*
2. *It is continuous in x for fixed d.*
3. *If $x_n \to x$, then $L_{x_n} \to L_x$, uniformly on compact subsets of $\mathcal{E}_\theta$.*

We can also prove the analogue of lemma 5.2 for irrational rays.

LEMMA 6.3.— *For $x \in \hat{\mathbf{R}} \setminus \pm 1$, the length function L_x has a critical point on the Earle line $\mathcal{F}_\theta$.*

Proof. — The proof is similar to that of lemma 5.2. As in that proof, the hyperbolic length of $l(\gamma_{1/1})$ of $\gamma_{1/1}$ tends to 0 as $d \to 0$ along $\mathcal{F}_\theta$, and $l(\gamma_{-1/1}) \to 0$ as $d \to \infty$. Then $|\mu_x|$ intersects both $\gamma_{\pm 1/1}$. The length of $|\mu_x|$ can be computed as an integral of first return lengths along the transversal $\gamma_{\pm 1/1}$, against a fixed and non-zero measure on the transversal. Thus if $l(\gamma_{\pm 1/1}) < \epsilon$, say, then by the collar lemma, the return length of any leaf of $|\mu_x|$ to the transversal $\gamma_{\pm 1/1}$ is greater than $-\log \epsilon$ so that $L_x(d) \to \infty$ as $d \to 0, \infty$ as required. $\square$

We immediately deduce

COROLLARY 6.4.— *For any $x \in \hat{\mathbf{R}}$, L_x is non-constant on $\mathcal{E}_\theta$.*

6.3. Pleating varieties

As explained in 4.2, the pleating locus of $\partial\mathcal{C}(d)^{\pm}/\Gamma(d)$ is a geodesic lamination carrying a natural transverse measure, the bending measure. We denote the projective classes of these measures $pl^{\pm}(d)$ respectively. The following is a special case of one of the main results of [12].

THEOREM 6.5 ([12] Theorem 4.6). — *The map $pl^{\pm} : \mathcal{E}_\theta \setminus \mathcal{F}_\theta \to PML$, $d \mapsto pl^{\pm}(d)$, is continuous.*

Extending the definitions of section 4.2, for $\xi, \eta \in PML$, we define

$$\mathcal{P}_{\xi,\eta} = \{d \in \mathcal{E}_\theta \setminus \mathcal{F}_\theta : pl^{+}(d) = \xi, pl^{-}(d) = \eta\}$$

and also $\mathcal{P}^{\pm}(\xi) = \{d \in \mathcal{E}_\theta | \, pl^{\pm}(d) = \xi\}$.

We note that $\mathcal{P}_{\xi,\eta} = \mathcal{P}^{+}_{\xi} \cap \mathcal{P}^{-}_{\eta}$, and that, identifying the geodesic $\gamma \in \mathcal{S}$ with the projective measure class $[c\delta_\gamma]$, this definition is consistent with the previous definition of $\mathcal{P}_{p/q,r/s}$. As before, we call $\mathcal{P}_{\xi,\eta}$ the (ξ,η)-*pleating variety* or *pleating ray*. It is clear from the definitions that $\mathcal{P}_{x,y} \cap \mathcal{P}_{x',y'} = \emptyset$ unless $x = x', y = y'$, and that $\mathcal{E}_\theta \setminus \mathcal{F}_\theta = \cup_{x,y\in\hat{\mathbf{R}}}\mathcal{P}_{x,y}$. Applying 6.5 and 5.11, we obtain the following extension of proposition 4.5.

PROPOSITION 6.6. — *The sets $\mathcal{P}_{x,y}, x, y \in \hat{\mathbf{R}}$ are pairwise disjoint and:*

1. $\mathcal{P}_{x,x} = \emptyset$.
2. $\mathcal{P}_{x,y} = \emptyset$ *unless* $y = 1/x$.
3. $\iota(\mathcal{P}_{x,1/x}) = \mathcal{P}_{1/x,x}$ *and* $\sigma(\mathcal{P}_{x,1/x}) = \mathcal{P}_{-1/x,-x}$
4. $\mathcal{E}_\theta \setminus \mathcal{F}_\theta = \cup_{x\in\hat{\mathbf{R}},x\neq\pm1}\mathcal{P}_{x,1/x}$.

From now on, we shall write $\mathcal{P}_x$ for $\mathcal{P}_{x,\frac{1}{x}}$.

Let $\mathcal{R}_x = \{d \in \mathcal{E}_\theta : L_x(d) \in \mathbf{R}\}$. We call $\mathcal{R}_x$ the *real locus* of L_x. We note that for $x \in \hat{\mathbf{Q}}$, $\mathcal{R}_x$ is almost the same as the hyperbolic locus $\mathcal{H}_x$. The difference is that the trace is defined on the whole d-plane minus zero, and $\mathcal{H}_x$ is the subset of the whole d-plane minus zero on which the complex length of the corresponding group element is real, while L_x is only defined on $\mathcal{E}_\theta$ and $\mathcal{R}_x \subset \mathcal{E}_\theta$. The following extension of proposition 4.7 (3) is an important result in [14].

THEOREM 6.7. — *([14] theorem 6.9) For $x \in \hat{\mathbf{R}}, \mathcal{P}_x \subset \mathcal{R}_x$.*

Proof of theorem 5.11 (See [14] theorem 4). — Let $d \in \mathcal{P}_x$ for some $x \in \hat{\mathbf{R}} \setminus \hat{\mathbf{Q}}$, so that $L_x(d) \in \mathbf{R}$. Since L_x is holomorphic and non-constant, there is a point d' near d at which $L_x(d') \notin \mathbf{R}$, hence, by theorem 6.7, $d' \notin \mathcal{P}_x$. Join d to d' by an arc in $\mathcal{E}_\theta$. Along this arc, the function $pl^\pm$ is continuous and non-constant, so that there must be points at which $pl^\pm$ takes rational values arbitrarily close to d. $\square$

6.4. Critical points

In order to complete our picture of the irrational pleating varieties, we need to study the critical points of the length function L_x on the Earle line $\mathcal{F}_\theta$. This we do by considering the effect of the Earle symmetry on the space $\mathcal{F}$ of Fuchsian groups. By combining with results of Kerckhoff about geodesic length functions for hyperbolic surfaces, this will allow us to extend the results of section 5 about the uniqueness of the critical point to irrational laminations.

The classical uniformisation theorem gives a bijection between the Teichmüller space of the flat torus and the space $\mathcal{F}$ of marked Fuchsian once punctured torus groups $\rho : \pi_1(\mathcal{T}_1) \to PSL(2,\mathbf{R})$, modulo conjugation in $PSL(2,\mathbf{R})$. We denote by $F(\tau)$ the marked Fuchsian group corresponding to the marked torus $\mathbf{C}(\tau)/G(\tau)$ under this map. (Strictly, $F(\tau)$ is only defined up to conjugation in $PSL(2,\mathbf{R})$; the exact normalisation will not matter in what follows.) We denote by $l_\mu(\tau)$ the length of the measured lamination μ on the hyperbolic surface $\mathbf{H}/F(\tau)$, and remark that for $x \in PML$, μ_x is the lamination such that $l_{\mu_x}(i) = 1$, since $i \in \mathbf{H}$ represents the square torus and hence corresponds to the point $d_0 = 1/\sqrt{2} \in \mathcal{E}_\theta$. We write l_x for l_{μ_x}, and emphasize that l_x is not the same as the complex length L_x; l_x is defined on $\mathcal{F}$ and takes values in $\mathbf{R}^+$ while L_x is a holomorphic function on $\mathcal{E}_\theta$. There is however a natural identification of the line of groups $F(\tau), |\tau| = 1$ with groups $\Gamma(d)$ for d on the Earle line $\mathcal{F}_\theta$, and on this line the functions l_x and L_x coincide.

LEMMA 6.8. — *1. The map $\tau \mapsto 1/\bar{\tau}$ induces an anticonformal bijection between the marked torus $(G(\tau); A, B(\tau))$ and the marked torus $(G(\tau); B(1/\bar{\tau}), A)$. The fixed point set of the induced involution $J : \mathcal{F} \to \mathcal{F}$ consists exactly of groups $F(\tau)$ on the rhombus line $|\tau| = 1$.*

2. $l_x(1/\bar{\tau}) = l_{1/x}(\tau)$.

Proof. — For (1) we note that complex conjugation gives a natural anticonformal bijection between the marked tori $(G(\tau); A, B(\tau))$ and $(G(\bar{\tau}); A, B(\bar{\tau}))$. The second of these tori is conformally the same as the marked torus $(G(1/\bar{\tau}); B(1/\bar{\tau}), A)$.

To prove (2) we proceed as follows. By definition, the word $W_{p/q}(A,B)$ represents the homotopy class $\hat{A}^{-q}\hat{B}^{p}$ on the flat torus $\mathbf{C}(\tau)/G(\tau)$. Thus $W_{q/p}(B,A)$ represents the homotopy class $\hat{B}^{-p}\hat{A}^{q}$. Since by proposition 4.1 there is a unique homotopy class of curves on $\mathcal{T}_1$ with this property, it follows that $W_{p/q}(A,B)$ is conjugate in the free group $F_2=\langle A,B\rangle$ to $W_{q/p}(B,A)^{-1}$.

The (orientation reversing) map of marked groups

$$(F(\tau);A,B)\to(F(1/\bar{\tau});A,B)$$

induced by $\tau\mapsto 1/\bar{\tau}$ is clearly an isometry. Thus if $V(A,B)$ is any word in F_2 and if $l(V)(\tau)$ denotes hyperbolic length of the axis corresponding to V on the surface $\mathbf{H}/F(\tau)$, then $l(V(A,B))(\tau)=l(V(A,B))(1/\bar{\tau})$. In particular, $l(W_{q/p}(A,B))(\tau)=l(W_{q/p}(A,B))(1/\bar{\tau})$.

On the other hand, by the remarks above,

$$l(W_{q/p}(A,B))(1/\bar{\tau})=l(W_{q/p}(A,B)^{-1})(1/\bar{\tau})=l(W_{p/q}(B,A))(1/\bar{\tau}).$$

Thus $l(W_{q/p}(A,B))(\tau)=l(W_{p/q}(B,A))(1/\bar{\tau})$. This shows that the length of $\gamma_{q/p}$ in the marked group $(F(\tau);A,B)$ is equal to length of $\gamma_{p/q}$ in the marked group $(F(1/\bar{\tau});B,A)$, which is exactly the result (2) in the case $x=p/q$. The result for general x follows by taking limits. $\square$

Recall that for any hyperbolic surface Σ, and for any measured lamination $\mu\in ML(\Sigma)$, the time t earthquake $\mathcal{E}_\mu(t)$ along μ is a generalisation of distance t Fenchel-Nielsen twist about a simple closed geodesic on Σ. We call $\{\mathcal{E}_\mu(t)(p):t\in\mathbf{R}\}\subset\mathcal{F}$ the *earthquake path* along μ through p. Recall Kerckhoff's theorem [15] about lengths of geodesics along earthquake paths: if $\nu\in ML(\Sigma)$ has non-zero intersection with μ, then the lamination length l_ν on a earthquake path $\mathcal{E}_\mu(t)(p)$ is a convex function of t with a unique minimum at a point $p_0\in\mathcal{E}_\mu(t)(p)$ which is also the unique minimum for the length l_μ along $\mathcal{E}_\nu(t)(p_0)$. Wolpert has shown further, that this minimum is a simple critical point for the length function. It follows immediately that any two earthquake paths intersect in at most two points.

Using these results, we showed in [14] section 3 that for each $\mu\in ML(\mathcal{T}_1)$ and each $c>0$, all points $p\in\mathcal{F}$ for which $l_\mu(p)=c$ lie on a unique earthquake path $\mathcal{E}_{\mu,c}$ obtained by earthquaking along μ from the same base point $p_0=p_0(\mu,c)$. For $k>0$, define $E_x(k)=\{\tau\in\mathbf{H}:l_x(\tau)<k\}$, where as above, $l_x(\tau)$ means the hyperbolic length of the lamination μ_x on the surface $\mathbf{H}/F(\tau)$. Clearly, $\partial E_x(k)$ is exactly the earthquake path along μ_x through any point for which $l_x(\tau)=k$. It follows that any two sets $\partial E_x(k)$ and $\partial E_{x'}(k')$ intersect in at most two points.

LEMMA 6.9.— *An earthquake path $\partial E_x(k)$ intersects the rhombus line $\mathcal{F}_\theta$ in at most two points.*

Proof. — If $\tau \in \mathcal{F}_\theta \cap \partial E_x(k)$, then $J(\tau) = \tau$ and so by the symmetry lemma 6.8, $\tau \in \partial E_{1/x}(k)$. The result follows since $\partial E_x(k)$ and $\partial E_{1/x}(k)$ intersect at most twice. □

LEMMA 6.10.— *For $x \in \hat{\mathbf{R}} \setminus \pm 1$, there exists a unique $k = k_x > 0$ such that the earthquake paths $\partial E_x(k)$ and $\partial E_{1/x}(k)$ are tangent in $\mathcal{F}$. The tangency point u_x is on the rhombus line $|\tau| = 1$.*

Proof. — The projective laminations x and $1/x$ are distinct and thus, as in the proof of corollary 6.3, cannot be simultaneously short. Thus for small k, the sets $E_x(k)$ and $E_{1/x}(k)$ are certainly disjoint. On the other hand, certainly $E_x(k) \cap E_{1/x}(k') \neq \emptyset$ for some k' and since the sets $E_x(k)$ are nested, it follows that $E_x(k') \cap E_{1/x}(k') \neq \emptyset$ for sufficiently large k'. Since $\partial E_x(k)$ and $\partial E_{1/x}(k)$ intersect in at most two points, the first part of the statement follows.

We denote the tangency point by u_x. At this point, $k_x = L_x(u_x) = L_{1/x}(u_x)$. From the symmetry lemma 6.8 we see that $J(E_x(k_x)) = E_{1/x}(k_x)$, and it follows that u_x is a fixed point of J, and hence $u_x \in \mathcal{F}_\theta$ as required. □

COROLLARY 6.11.— *The length function L_x restricted to the Earle line $\mathcal{F}_\theta$ has a unique critical point on $\mathcal{F}_\theta$. This point is a simple critical point and is exactly the tangency point u_x of the earthquake paths $\partial E_x(k_x)$ and $\partial E_{1/x}(k_x)$.*

Proof. — Since $\partial E_x(k_x)$ is tangent to $\mathcal{F}_\theta$ at u_x, points on $\mathcal{F}_\theta$ near u_x are outside $E_x(k_x)$ and hence u_x is a local minimum for $L_x|_{\mathcal{F}_\theta}$. Suppose that v_x were another critical point on $\mathcal{F}_\theta$. Then $\frac{\partial L_x}{\partial t}|_{v_x} = 0$, where $\frac{\partial}{\partial t}$ denotes differentiation along $\mathcal{F}_\theta$. Since $l_x|_{\mathcal{F}_\theta} = L_x|_{\mathcal{F}_\theta}$, the same is true of the derivative of l_x along $\mathcal{F}_\theta$. Now $\partial E_x(L_x(v_x))$ is a level set of l_x and hence must be tangent to $\mathcal{F}_\theta$ at v_x. However by lemma 6.9, $\partial E_x(L_x(v_x))$ cuts $\mathcal{F}_\theta$ at most twice, so that $\partial E_x(L_x(v_x)) \cap \mathcal{F}_\theta = \{v_x\}$. Finally, since the sets $E_x(c)$ as c varies are nested, we conclude that $L_x(v_x) = k_x$ and hence that $v_x = u_x$ as required. □

In the case $x = p/q \in \hat{\mathbf{Q}}$, this result allows us to identify the critical point u_x with the critical point $b_{p/q}$ of theorem 5.1. Thus from now on, we write b_x for u_x, and note from the construction that $b_{1/x} = b_x$.

PROPOSITION 6.12.— *The map $x \mapsto b_x$ from $(-1, 1)$ to $\mathcal{F}_\theta$ is a mono-*

tonically decreasing homeomorphism.

Proof. — Kerckhoff shows in [16] that for any $p \in \mathcal{F}$, the map which associates $\mu \in ML$ to the tangent to the earthquake path along μ at p is a homeomorphism between the space of measured laminations ML and the tangent space $T_p(\mathcal{F})$ to $\mathcal{F}$ at p. The inverse of this map induces a map f on rays which takes the ray $[\mathbf{v}]$ of tangent vectors $\lambda\mathbf{v}, \lambda \in \mathbf{R}^+$ at the point $p \in \mathcal{F}$ to the projective measured lamination x for which the forward tangent to the earthquake path along x through p is in $[\mathbf{v}]$. (This makes sense since for any $\mu \in ML$, $\mathcal{E}_{c\mu}(t) = \mathcal{E}_\mu(t/c)$.) Since $\partial E_x(k_x)$ is tangent to $\mathcal{F}_\theta$ at b_x, we see that the restriction of f to the space of forward pointing tangent rays to $\mathcal{F}_\theta$ is exactly the inverse of the map $x \mapsto b_x$. The fact that earthquake paths intersect at most twice easily implies that this restriction of f is monotone, and the result follows. □

Proof of proposition 5.10. — We find by direct calculation that $b(-1/2) > b(0)$. (In fact $b(-1/2) = \frac{\sqrt{3+\sqrt{33}}}{2}$ and $b(0) = 1/\sqrt{2}$.) From proposition 6.12 we conclude that $b_{p/q} > b_{r/s}$ whenever $-1 < p/q < r/s < 1$. The result follows from theorem 6.5 which asserts the continuity of the map $pl^+ : \mathcal{E}_\theta^+ \to (-1,1)$ and lemma 5.8 above. □

6.5. Irrational pleating rays

Finally, we are able to complete our picture of the structure of irrational rays. We need the extension of theorem 4.8. This is given by the fundamental local and limit pleating theorems of [14].

THEOREM 6.13. — *For $x \in \hat{\mathbf{R}}$,*

1. *$\mathcal{P}_x \cup \mathcal{P}_{1/x} \cup \mathcal{F}_\theta$ is open and closed in $\mathcal{R}_x$.*

2. *If $d_n \in \mathcal{P}_x$ and if $L_x(d_n) \to c$, then a subsequence of the groups $\Gamma(d_n)$ have an algebraic limit Γ_∞. The limit is in $\mathcal{E}_\theta$ unless $c = 0$.*

Proof. — (1) is just [14] theorem 8.1. (2) follows from [14] theorem 5.1 with the easy observation that if Γ_∞ is an algebraic limit of groups in $\mathcal{E}_\theta$, and if $\Gamma_\infty \in \mathcal{QF}$, then $\Gamma_\infty \in \mathcal{E}_\theta$. □

COROLLARY 6.14. — *1. If $x \in \hat{\mathbf{R}}$ and $d \in \mathcal{R}_x \setminus \mathcal{F}_\theta$ is sufficiently near $\mathcal{F}_\theta$, then $d \in \mathcal{P}_x \cup \mathcal{P}_{1/x}$. In particular, $\mathcal{P}_x \neq \emptyset$ for $x \in \hat{\mathbf{R}} \setminus \pm 1$.*

2. $\mathcal{P}_x$ is a union of connected components of $\mathcal{R}_x$ in $\mathcal{E}_\theta \setminus \mathcal{F}_\theta$.

3. Let $K \subset \mathcal{P}_x$ be a connected component of $\mathcal{P}_x$. Then $L_x(K) = (0, \infty)$, $(0, k_x)$ or (k_x, ∞), where $k_x = L_x(b_x)$ is the minimum of L_x on $\mathcal{F}_\theta$. In the last two cases, b_x is in the closure of K in $\mathcal{E}_\theta$.

Proof. — Part 1 is a direct application of theorem 6.13 (1), see also [14] theorem 8.9. Part 2 is immediate from theorem 6.13. Part 3 is like [14] lemma 9.3. The local pleating theorem shows that $L_x|_K$ is an open map. If $r \in \overline{L_x(K)}, r > 0$, then we have a sequence $d_n \in K$ with $L_x(d_n) \to r$, so by the limit pleating theorem, a subsequence $\Gamma(d_{n_r})$ has algebraic limit $\Gamma(d_\infty) \in \mathcal{E}_\theta$. By theorem 6.5, either $\Gamma(d_\infty) \in \mathcal{P}_x$ or $\Gamma(d_\infty) \in \mathcal{F}_\theta$. In the second case, we must have $d_\infty = b_x$. The result follows from theorem 6.13 (1). □

PROPOSITION 6.15.— *For $x \in \hat{\mathbf{R}} \setminus \pm 1$, the map $L_x|_{\mathcal{P}_x}$ has bounded range.*

Proof. — Without loss of generality, we may assume $-1 < x < 1$. Choose $-1 < a < x < b < 1$. By proposition 6.12, the map $y \mapsto b_y$ is continuous on $(-1, 1)$, and hence there exists $M > 0$ such that $L_y(b_y) \leqslant M$ for $a \leqslant y \leqslant b$. Now it follows from theorem 5.1 that $L_{p/q}(d) \leqslant L_{p/q}(b_{p/q})$ for all $p/q \in \hat{\mathbf{Q}} \setminus \pm 1$ and $d \in \mathcal{P}_{p/q}$. Now suppose $d \in \mathcal{P}_x$. Pick $p_n/q_n \in \hat{\mathbf{Q}} \cap [a, b], p_n/q_n \to x$ and use corollary 5.11 to find $d_n \in \mathcal{P}_{p_n/q_n}, d_n \to d$. Using the equicontinuity of the family $y \to L_y$ we conclude that $L_x(d) \leqslant M$. This gives the result. □

THEOREM 6.16.— *For $x \neq \pm 1$, L_x has a unique critical point b_x on $\mathcal{F}_\theta$. The pleating variety $\mathcal{P}_x$ is an embedded arc of $\mathcal{E}_\theta$ whose boundary in $\mathcal{F}_\theta$ is exactly b_x.*

Proof. — The only points remaining to prove are that $\mathcal{P}_x$ is connected and contains no critical points of L_x. By proposition 6.15 the range of L_x on any component of $\mathcal{P}_x$ is bounded, so that analytically continuing along any branch of $\mathcal{P}_x \subset \mathcal{R}_x$ in the direction of increasing length, we must eventually reach the critical point $b_x \in \mathcal{F}_\theta$. By corollary 6.11, the critical point b_x is simple so at most two arcs of $\mathcal{P}_x$ meet $\mathcal{F}_\theta$ at b_x. However, one of these arcs must belong to $\mathcal{P}_x$ and one to $\mathcal{P}_{1/x}$ which are certainly distinct, so we conclude that exactly one arc of $\mathcal{P}_x$ meets $\mathcal{F}_\theta$ at b_x. Finally, if $\mathcal{P}_x$ contained a critical point, we could continue along at least two distinct arcs in the direction of increasing length, which by the above is impossible. □

The above results combine to give our main theorem about pleating coordinates for the Earle slice.

THEOREM 6.17.— *The maps*

$$\Pi^{\pm} : \mathcal{E}_\theta^{\pm} \to \mathbf{R} \times \mathbf{R}^{+}$$

defined by $\Pi^{\pm}(d) = (pl^{\pm}(d), L_{pl^{\pm}(d)}(d))$*, are homeomorphisms onto their images.*

7. Appendix 1: Computing Pleating coordinates for the Earle embedding, by P. Liepa

The calculations for the drawing in Figure 1 were based on Wright's recursive scheme. For a given rational p/q, let $T_{p/q}(d), c_{p/q}$ and $b_{p/q}$ be defined as in sections 4 and 5. Using the Farey recursion, if we have the above data for $p/q < r/s$ and $ps - rq = -1$, then we can calculate the corresponding data for $(p+r)/(q+s)$ from $T_{(p+r)/(q+s)} = T_{r/s}T_{p/q} - T_t$, where $t = \frac{p-r}{q-s}$. (See [10] proposition 3.1.)

The value $c_{(p+r)/(q+s)}$ is calculated numerically, using an initial guess based on a weighted average of $c_{p/q}$ and $c_{r/s}$ derived from the position of $(p+r)/(q+s)$ in the interval $[p/q, r/s]$. Likewise $b_{(p+r)/(q+s)}$ is calculated numerically, using an initial guess derived from $b_{p/q}$ and $b_{r/s}$.

For selected values of p/q the rays are computed as follows. Let $\lambda = 2\cosh^{-1}(T_{p/q}(b_{p/q})/2)$. To compute an n-segment approximation of the ray, $n+1$ approximately equally spaced points on the ray must be computed. The ith position can be computed by finding the value d for which $T_{p/q}(d) = 2\cosh((1-(n-i)^2/n^2)\lambda/2)$.

The value d is found numerically, using the $(i-1)$th sample as an initial guess. In Figure 1, $n = 50$.[1]

8. Appendix 2: Geodesics and pleating rays in $\mathcal{E}_\theta$

As we saw in section 3.3, the Teichmüller space $\mathrm{Teich}(\mathcal{T}_1)$ of once punctured tori can be naturally identified with the upper half plane $\mathbf{H}$ and the Earle slice $\mathcal{E}_\theta$, hence we have a conformal homeomorphism $\psi : \mathbf{H} \to \mathcal{E}_\theta$, called the *Earle embedding.* Proposition 3.8 shows that under this Riemann map ψ, the semicircle $C = \{\tau \in \mathbf{H} : |\tau| = 1\}$ corresponds to the Earle line $\mathcal{F}_\theta$, and the imaginary axis corresponds to $\mathcal{P}_{0,\infty} \cap \{b_0\} \cup \mathcal{P}_{\infty,0}$, where $b_0 = 1/\sqrt{2}$. For any $x \in (-1,1)$, let $\mathcal{G}_x$ be the semicircle from x to $1/x$ which is a hyperbolic geodesic in $\mathbf{H}$. Then it is easy to check that $\{\mathcal{G}_x\}_{x\in(-1,1)}$ satisfies the following conditions:

---

(1) Extensive details about similar computations of pleating rays and boundaries, and about making limit set plots, are to be found in [26].

1. $\mathcal{G}_x$ is a real analytic curve and $\mathcal{G}_{x_1} \cap \mathcal{G}_{x_2} = \emptyset$ if $x_1 \neq x_2$.
2. $\mathcal{G}_x$ intersects C orthogonally.
3. $\mathbf{H} = \cup_{x \in (-1,1)} \mathcal{G}_x$.

For any $x \in (-1,1)$, put $\hat{\mathcal{P}}_x = \mathcal{P}_x \cup \{b_x\} \cup \mathcal{P}_{1/x}$. From the results in sections 5 and 6, $\{\hat{\mathcal{P}}_x\}$ satisfies similar conditions:

1. $\hat{\mathcal{P}}_x$ is a real analytic curve and $\hat{\mathcal{P}}_{x_1} \cap \hat{\mathcal{P}}_{x_2} = \emptyset$ if $x_1 \neq x_2$.
2. $\hat{\mathcal{P}}_x$ intersects $\mathcal{F}_\theta$ orthogonally.
3. $\mathcal{E}_\theta = \cup_{x \in (-1,1)} \hat{\mathcal{P}}_x$.

From these facts it seems natural to ask whether $\psi(\mathcal{G}_x) = \hat{\mathcal{P}}_x$ for any $x \in (-1,1)$. The answer is negative; in fact

PROPOSITION 8.1.— $\psi(\mathcal{G}_{2-\sqrt{3}}) \neq \hat{\mathcal{P}}_{2-\sqrt{3}}$

Proof. — First we remark that $e^{\pi i/3}$ and $e^{2\pi i/3}$ are both on the semicircle C. It is well-known that the marked Fuchsian group $F = \langle A, B \rangle$ corresponding to $e^{2\pi i/3} \in \mathbf{H}$ is characterized by the trace values $(\operatorname{Tr} A, \operatorname{Tr} B, \operatorname{Tr} AB)$ $= (3,3,3)$. Hence it follows that $\psi(e^{2\pi i/3}) = 1$ and by proposition 3.8 (1), $\psi(e^{\pi i/3}) = 1/2$. It also easy to check that $e^{\pi i/3} \in \mathcal{G}_{2-\sqrt{3}}$. Therefore if we assume that $\psi(\mathcal{G}_{2-\sqrt{3}}) = \hat{\mathcal{P}}_{2-\sqrt{3}}$, then $b_{2-\sqrt{3}}$ should be equal to $1/2$. Because $2-\sqrt{3}$ is a real quadratic number, its continued fractional expansion is eventually periodic, and we can easily find its rational approximations: $1/3, 1/4, 2/7, 3/11, \cdots$. If x is rational, b_x is the unique critical point of T_x on $\mathcal{F}_\theta$, we can calculate it by using recursive formula for T_x in Appendix 1. The result is the following table. From proposition 6.12, $2-\sqrt{3} < 3/11$ should imply $b_{3/11} < b_{2-\sqrt{3}}$ which contradicts the values in the table. □

x	b_x
$1/4$	0.521228
$2-\sqrt{3}$	0.500000
$3/11$	0.503878
$2/7$	0.493983
$1/3$	0.458063

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