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Critical boundary constants and Pohozaev identity (*)

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à mes deux filles Kénizé et Maöna

RÉSUMÉ. — La première partie de ce travail concerne le problème,

$$(P\epsilon)\begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \ dans \ B_1, \\ u > 0 \ dans \ B_1, \\ u = \epsilon \ sur \ \partial B_1, \end{cases}$$

où $B_1 = \{x \in \Re^n, \ \|x\| < 1 \}$, $n \geqslant 3$ et $\epsilon > 0$.

On démontre qu'il existe une constante critique $\epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$, telle que le problème $(P\epsilon)$ admet exactement deux solutions $u_{\epsilon 1}$ et $u_{\epsilon 2}$ ($u_{\epsilon 1} < u_{\epsilon 2}$) si $0 < \epsilon < \epsilon^*$, une solution unique si $\epsilon = \epsilon^*$ et n'admet pas de solution si $\epsilon > \epsilon^*$.Toutes ces solutions seront données explicitement. Il est démontré que quand $\epsilon \to 0$,

$$\frac{u_{\epsilon 1}(x) - \epsilon}{\epsilon} \to 0 \ \ sur \ \overline{B_1} \ et \quad \frac{u_{\epsilon 2}(x) - \epsilon}{\epsilon} \to \|x\|^{2-n} - 1, \ sur \ \overline{B_1} \backslash \{O\}.$$

Au cours de la seconde partie, on s'intéresse au problème

$$(Q\epsilon)\begin{cases} \Delta u + f(x,u) = 0 \ dans \ \Omega, \\ u > 0 \ dans \ \Omega, \\ u = \epsilon \ sur \ \partial\Omega, \end{cases}$$

où Ω est un domaine borné, régulier et étoilé par rapport à l'origine, f est continue et dépend asymptotiquement de u "comme u^{α}", $1 < \alpha$ et $\alpha \neq \frac{n+2}{n-2}$. Différents résultats d'existence de constante au bord critique ϵ^* pour le problème $(Q\epsilon)$ sont donnés.

ABSTRACT. — The first part of this work deals with the problem

$$(P\epsilon)\begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \ in \ B_1, \\ u > 0 \ in \ B_1, \\ u = \epsilon \ on \ \partial B_1. \end{cases}$$

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where $B_1 = \{x \in \Re^n,\ \|x\| < 1\ \}$, $n \geqslant 3$ and $\epsilon > 0$.
We show that there exists a critical constant $\epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$, such that the problem $(P\epsilon)$ admits two solutions $u_{\epsilon 1}$ and $u_{\epsilon 2}$ ($u_{\epsilon 1} < u_{\epsilon 2}$) if $0 < \epsilon < \epsilon^*$, only one solution if $\epsilon = \epsilon^*$ and no solution if $\epsilon > \epsilon^*$.We give all these solutions explicitly.We show that, when $\epsilon \to 0$,

$$\frac{u_{\epsilon 1}(x) - \epsilon}{\epsilon} \to 0 \ \ on\ \overline{B_1}\ and \quad \frac{u_{\epsilon 2}(x) - \epsilon}{\epsilon} \to \|x\|^{2-n} - 1,\ if\ x \in \overline{B_1}\backslash\{O\}.$$

The second part is devoted to the following problem

$$(Q\epsilon)\left\{\begin{array}{l} \Delta u + f(x,u) = 0\ in\ \Omega, \\ u > 0\ in\ \Omega, \\ u = \epsilon\ on\ \partial\Omega, \end{array}\right.$$

where Ω is a regular bounded domain which is starshaped about the origin, f is continuous and behaves like u^α in the second variable, $1 < \alpha$ and $\alpha \neq \frac{n+2}{n-2}$. We give different existence results for a boundary critical datum ϵ^* for $(Q\epsilon)$.

1. The Sobolev Exponent Growth

Let us consider the problem

$$(P\epsilon)\left\{\begin{array}{l} \Delta u + u^{\frac{n+2}{n-2}} = 0\ in\ B_1, \\ u \geqslant 0\ in\ B_1, \\ u = \epsilon\ on\ \partial B_1. \end{array}\right.$$

In [9], it is shown that there exists a critical boundary datum ϵ^* such that $\forall\ 0 < \epsilon < \epsilon^*$, $(P\epsilon)$ admits -at least- one C^2-solution. There is no solution when $\epsilon > \epsilon^*$. It is known that $(P0)$ does not admit a nontrivial solution (see [13]). According to [5], every regular solution of $(P\epsilon)$ is spherically symmetric. Let u_ϵ ($\epsilon > 0$), be a solution of $(P\epsilon)$, then

$$v_\epsilon = \frac{u_\epsilon - \epsilon}{\epsilon}$$

is a solution of

$$(A\lambda)\left\{\begin{array}{l} \Delta w + \lambda(1+w)^{\frac{n+2}{n-2}} = 0\ in\ B_1, \\ w > 0\ in\ B_1, \\ w = 0\ on\ \partial B_1, \end{array}\right.$$

where

$$\lambda = \epsilon^{\frac{4}{n-2}}.$$

The maximum principle implies

$$v_\epsilon > 0, \ \ in\ B_1.$$

It is known (see [11],[4] and [3]) that there exists a constant λ^* such that $(A\lambda)$ admits just two C^3- spherically symmetric solutions when $0 < \lambda < \lambda^*$, only one solution for $\lambda = \lambda^*$ and no solution if $\lambda > \lambda^*$. We give here the value of λ^* and the explicit solutions for every $\lambda \leqslant \lambda^*$. We show that when $\lambda \to 0$, the "small" solution tends to $v = 0$ the trivial solution of $(A0)$ and the "big" one tends to $H(x) = \|x\|^{2-n} - 1,\ x \neq O$.

PROPOSITION 1

$$\lambda^* = (\epsilon^*)^{\frac{4}{n-2}} = \frac{n(n-2)}{4}.$$

Proof. — For

$$0 < \epsilon < \epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

let u_ϵ be a regular $(P\epsilon)$ solution. As in [13], let us put

$$g(x) = \sum_{i=1}^{n} x_i D_i v_\epsilon \nabla v_\epsilon \ \ (where\ v_\epsilon = \frac{u_\epsilon - \epsilon}{\epsilon}),$$

and use the Divergence Theorem, to get

$$\epsilon^{\frac{4}{n-2}} \left\{(1-\frac{1}{2}n)\int_{B_1} [1+v_\epsilon(x)]^{\frac{n+2}{n-2}} v_\epsilon(x)dx + \frac{n-2}{2}\int_{B_1} \left[(1+v_\epsilon(x))^{\frac{2n}{n-2}} - 1\right] dx\right\}$$

$$+\frac{1}{2}\int_{\partial B_1} [x.\nu]\, \|\nabla v_\epsilon(x)\|^2 ds = \int_{\partial B_1} [x.\nabla v_\epsilon(x)]\, [\nabla v_\epsilon(x).\nu]\, ds.$$

From the identity

$$\int_{B_1} [1+v_\epsilon(x)]^{\frac{2n}{n-2}} dx = \int_{B_1} [1+v_\epsilon(x)]^{\frac{n+2}{n-2}} v_\epsilon(x)dx + \int_{B_1} [1+v_\epsilon(x)]^{\frac{n+2}{n-2}} dx,$$

we infer that

$$(*) \quad \epsilon^{\frac{4}{n-2}} \left\{\frac{2-n}{2}\int_{B_1} dx + \frac{n-2}{2}\int_{B_1} [1+v_\epsilon(x)]^{\frac{n+2}{n-2}} dx\right\}$$

$$+\frac{1}{2}\int_{\partial B_1} [x.\nu]\, \|\nabla v_\epsilon(x)\|^2 ds = \int_{\partial B_1} [x.\nabla v_\epsilon(x)]\, [\nabla v_\epsilon(x).\nu]\, ds.$$

Using again the Divergence Theorem, we get

$$\epsilon^{\frac{4}{n-2}} \int_{B_1} (1+v_\epsilon)^{\frac{n+2}{n-2}} dx = -\int_{B_1} \Delta v_\epsilon(x) dx = -\int_{\partial B_1} \frac{\partial v_\epsilon(x)}{\partial \nu} ds,$$

where ν denotes the unit outward normal to ∂B_1.

The Maximum Principle implies that

$$\frac{\partial v_\epsilon}{\partial \nu} < 0 \ on\ \partial B_1.$$

As v_ϵ is spherically symmetric and vanishs on ∂B_1, we get

$$x.\nabla v_\epsilon(x) = x.\nu \frac{\partial v_\epsilon}{\partial \nu},\ on\ \partial\Omega,\ \text{and}$$

$$\frac{\partial v_\epsilon(x)}{\partial \nu} = -\|\nabla v_\epsilon(x)\| = l,\ on\ \partial B_1,$$

where l is a constant on ∂B_1. Using the fact that $x.\nu = 1\ on\ \partial B_1$, we obtain from (*)

$$\epsilon^{\frac{4}{n-2}} \frac{2-n}{2} \int_{B_1} dx - \frac{n-2}{2} l \int_{\partial B_1} ds = \frac{1}{2} l^2 \int_{\partial B_1} ds.$$

This equation is equivalent to

$$|S_1| l^2 + (n-2)|S_1| l + \epsilon^{\frac{4}{n-2}} (n-2)|B_1| = 0.$$

$$|B_1| = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})},\ and\ |S_1| = n|B_1|,$$

where $|B_1|$ is the Lebesgue measure of the unit ball of $\Re^n$ and $|S_1| = n|B_1|$ is the surface measure of the unit sphere.

We obtain the following equation in $l = v'_\epsilon(1)$

$$(1) \qquad nl^2 + n(n-2)l + \epsilon^{\frac{4}{n-2}}(n-2) = 0.$$

When

$$0 < \epsilon < \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

this equation admits two negative solutions

$$l_1(\epsilon) = \frac{n(2-n) + \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}$$

$$l_2(\epsilon) = \frac{n(2-n) - \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}.$$

When

$$\epsilon = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

equation (1) admits a unique negative solution

$$l_1 = \frac{2-n}{2},$$

and no real solution if

$$\epsilon > \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}.$$

So it is clear that

$$\lambda^* = (\epsilon^*)^{\frac{4}{n-2}} \leqslant \frac{n(n-2)}{4}.$$

The proof will be complete , if one shows that

$$\forall\ 0 < \epsilon < \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

there exists just two regular solutions of $(P\epsilon)$. Let us recall that the problem

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \ in\ \Re^n,$$

admits the radial solutions (see [12])

$$u_\lambda(\|x\|) = \lambda^{\frac{n-2}{4}} \left(n(n-2)\right)^{\frac{n-2}{4}} \left(\lambda^2 + \|x\|^2\right)^{\frac{2-n}{2}},\ \lambda > 0.$$

Let us put

$$\phi_\lambda = u_{\lambda_{|B_1}}.$$

$$\max_{\lambda>0} \phi_\lambda(1) = \phi_1(1) = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}.$$

It is immediate to verify that

$$\phi_\lambda(1) = \phi_{\frac{1}{\lambda}}(1),\ \forall\ \lambda > 0.$$

$$\phi_\lambda \neq \phi_{\frac{1}{\lambda}},\ \forall\ \lambda \neq 1.$$

In particular,

$$\phi'_\lambda(1) \neq \phi'_{\frac{1}{\lambda}}(1),\ \forall\ \lambda \neq 1.$$

As the function

$$\lambda \to \phi_\lambda(1) = \lambda^{\frac{n-2}{2}} (n(n-2))^{\frac{n-2}{4}} \left(1+\lambda^2\right)^{\frac{2-n}{2}},$$

is continuous on $[1, \infty[$, with

$$\lim_{\lambda \to \infty} \lambda^{\frac{n-2}{2}} (n(n-2))^{\frac{n-2}{4}} \left(1+\lambda^2\right)^{\frac{2-n}{2}} = 0,$$

we obtain two distinct solutions of $(P\epsilon)$, when

$$0 < \epsilon < \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

and one solution when

$$\epsilon = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}},$$

which is ϕ_1.

So the proof of Proposition 1 is complete.

Let us study the behavior of solutions when $\epsilon \to 0$. Let $u_{\epsilon 1}$ and $u_{\epsilon 2}$ be the two solutions of $(P\epsilon)$, with

$$u'_{\epsilon 1}(1) = l_1(\epsilon) = \frac{n(2-n) + \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n},$$

and

$$u'_{\epsilon 2}(1) = l_2(\epsilon) = \frac{n(2-n) - \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}.$$

$$\forall\ 0 < \epsilon < \epsilon^*,\ \exists!\ \lambda(\epsilon) > 1,$$

such that

$$u_{\epsilon 1} = \phi_{\lambda(\epsilon)} \ \ \textit{and} \ \ u_{\epsilon 2} = \phi_{\frac{1}{\lambda(\epsilon)}}.$$

$$\lambda(\epsilon) = \frac{[n(n-2)]^{\frac{1}{2}} + \sqrt{n(n-2) - 4\epsilon^{\frac{4}{n-2}}}}{2\epsilon^{\frac{2}{n-2}}}.$$

Let us put

$$\psi_{\epsilon 1} = \frac{u_{\epsilon 1} - \epsilon}{\epsilon} \ \ \textit{and} \ \ \psi_{\epsilon 2} = \frac{u_{\epsilon 2} - \epsilon}{\epsilon}.$$

Proposition 2

$$(i) \quad \psi_{\epsilon 1} \to \psi_1 = 0, \quad in\ C^1(\overline{B_1}),\ as\ \epsilon \to 0.$$

$$(ii) \quad \psi_{\epsilon 2}(x) \to \psi_2 = \|x\|^{2-n} - 1,\ in\ C^1_{loc}\left(\overline{B_1}\backslash O\right),\ as\ \epsilon \to 0.$$

Proof. — Let us remark the following

$$l_1(\epsilon) \to 0,\ as\ \epsilon \to 0,\ and\ l_2(\epsilon) \to 2-n,\ as\ \epsilon \to 0.$$

We give here a direct proof, using the explicit knowledge of $\psi_{\epsilon i}$, $i \in \{1,2\}$. As we have seen, for every $0 < \epsilon < \epsilon^*$, there exists a unique

$$\lambda(\epsilon) = \frac{[n(n-2)]^{\frac{1}{2}} + \sqrt{n(n-2) - 4\epsilon^{\frac{4}{n-2}}}}{2\epsilon^{\frac{2}{n-2}}} > 1,$$

$$(\lambda(\epsilon) \to \infty\ as\ \epsilon \to 0)$$

such that we have

$$\psi_{\epsilon 1}(r) = \frac{[\lambda(\epsilon)]^{\frac{n-2}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ \left([\lambda(\epsilon)]^2 + r^2\right)^{\frac{2-n}{2}} - \left([\lambda(\epsilon)]^2 + 1\right)^{\frac{2-n}{2}} \right\}}{[\lambda(\epsilon)]^{\frac{n-2}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{[\lambda(\epsilon)]^2 + 1\right\}^{\frac{2-n}{2}}},$$

$$\psi_{\epsilon 2}(r) = \frac{[\lambda(\epsilon)]^{\frac{2-n}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ \left([\lambda(\epsilon)]^{-2} + r^2\right)^{\frac{2-n}{2}} - \left([\lambda(\epsilon)]^{-2} + 1\right)^{\frac{2-n}{2}} \right\}}{[\lambda(\epsilon)]^{\frac{2-n}{2}} (n(n-2))^{\frac{n-2}{4}} \left\{[\lambda(\epsilon)]^{-2} + 1\right\}^{\frac{2-n}{2}}}.$$

We finally get

$$\psi_{\epsilon 1}(r) = \left(\frac{[\lambda(\epsilon)]^2 + 1}{[\lambda(\epsilon)]^2 + r^2}\right)^{\frac{n-2}{2}} - 1\ ;\ \psi_{\epsilon 2}(r) = \left\{\frac{1 + [\lambda(\epsilon)]^2}{1 + [\lambda(\epsilon)]^2 r^2}\right\}^{\frac{n-2}{2}} - 1.$$

It is immediate to verify that

$$\psi_{\epsilon 1}(\|x\|) = \left(\frac{[\lambda(\epsilon)]^2 + 1}{[\lambda(\epsilon)]^2 + \|x\|^2}\right)^{\frac{n-2}{2}} - 1 \to 0,\ in\ C^1(\overline{B_1}),\ as\ \epsilon \to \infty$$

and

$$\psi_{\epsilon 2}(\|x\|) = \left\{\frac{1 + [\lambda(\epsilon)]^2}{1 + [\lambda(\epsilon)]^2 \|x\|^2}\right\}^{\frac{n-2}{2}} - 1 \to \|x\|^{2-n} - 1,\ on\ \overline{B_1}\backslash\{O\}.$$

Remark **1.** — *According to Theorem 1.1 in [7], it is, in general, false that every positive solution u in B_1 of $\Delta u + u^\alpha = 0$, is a restriction of a positive solution v of this problem in $\Re^n$.*

2. Nonlinearities with Noncritical Growth

2.1. The Subcritical Behavior

We deal here with the following problem

$$(Q\epsilon)\begin{cases} \Delta u + f(x,u) = 0 \ in \ \Omega, \\ u > 0 \ in \ \Omega, \\ u = \epsilon \ on \ \partial\Omega, \end{cases}$$

Let us suppose the following

(i) Ω is a bounded regular domain of $\Re^n$, which is starshaped about the origin.

(ii) $f \in C^0\left(\overline{\Omega} \times \Re_+, \Re_+\right)$,

(iii) there exist positive constants c_1, γ, α and a positive function $a \in C^0(\overline{\Omega})$ such that

$$1 < \gamma \leqslant \alpha < \frac{n+2}{n-2}; \quad c_1 t^\gamma \leqslant f(x,t), \ \forall x \in \overline{\Omega}, t > 0,$$

$$\lim_{t \to \infty} \frac{f(x,t)}{t^\alpha} = a(x) \ and \ f(x,t) = o(t) \ near \ t = 0, \ uniformly \ in \ x \in \overline{\Omega}.$$

PROPOSITION 3.— *Under the previous hypotheses on Ω and f,there exists a positive constant $\epsilon^*(\Omega, f)$, such that for every $0 \leqslant \epsilon \leqslant \epsilon^*(\Omega, f)$, the problem $(Q\epsilon)$ admits, at least, one solution $u_\epsilon \in C^{1,\delta}(\overline{\Omega})$, $0 \leqslant \delta < 1$. There is no bounded solution of $(Q\epsilon)$ if $\epsilon > \epsilon^*(\Omega, f)$.*

Proof.— The proof is nearly the same as in (Theorem 1 in [9]). The only difference is that the subsolutions and supersolutions are considered as elements of $H_0^1(\Omega) \cap L^\infty(\Omega)$, and the inequalities are in the sense of duality $H^{-1}(\Omega)$, $H_0^1(\Omega)$.

Let us recall the main steps for this proof.

1. We use the hypothesis *(iii)* and Théorème 3.1 in [2], to show that the problem $(Q\epsilon)$ admits -at least- one solution $u \in H_0^1(\Omega)$, when ϵ is "small" enough.Using the L^p-estimates (see [1]), we infer that $u \in W^{2,p}(\Omega)$, $\forall p > 1$. One can use embedding results (see [8]) to deduce that $u \in C^{1,\alpha}(\overline{\Omega})$.

2. We show that if $(Q\overline{\epsilon})$ admits a solution, so does $(Q\epsilon)$ for every $\epsilon \leqslant \overline{\epsilon}$.

3.Using the a priori estimate in [6], we show that $(Q\epsilon)$ does not admit a solution, if ϵ is great enough.

From these steps, we infer that the set I of ϵ, for which $(Q\epsilon)$ admits a solution, is a bounded interval.

4. Let $\epsilon^*(\Omega, f)$ be the upper bound of I. The blow-up argument used in [6], can be applied to show that there exists no increasing sequence (ϵ_j) in I, such that

$$\lim_{j\to\infty} \epsilon_j = \epsilon^*(\Omega, f), \ \textit{with} \ \lim_{j\to\infty} \max_{x\in\Omega} u_{\epsilon_j}(x) = \infty.$$

This last a priori L^∞–estimate of the solutions u_ϵ near $\epsilon^*(\Omega, f)$, leads to a solution of $(Q\epsilon^*(\Omega, f))$.

Remark **2.**— *When* $\Omega = B_r = \{x \in \Re^n;\ \|x\| < r\ \}$ *and* $f(x, u) = u^\alpha$, *then*

1. *every solution of* $(Q\epsilon)$ *is spherically symmetric (see [5]),*

2. $\epsilon^*(B_r, \alpha) = r^{\frac{2}{1-\alpha}} \epsilon^*(B_1, \alpha)$, *(see [10]).*

2.2. The Supercritical Growth Case

Let us consider the following problem

$$(T\epsilon)\begin{cases} \Delta u + a(x)u^\beta = 0 \ in \ \Omega, \\ u > 0 \ in \ \Omega, \\ u = \epsilon \ on \ \partial\Omega. \end{cases}$$

We suppose that

(i) Ω is a bounded regular domain, which is starshaped about the origin.

(ii) $a \in C^0(\overline{\Omega}, \Re_+^*)$ and $\beta > \frac{n+2}{n-2}$.

Under appropriate hypotheses, the following problem

$$(P)\begin{cases} \Delta u + a(x)u^\beta = 0 \ in \ \Re^n, \\ u > 0 \ in \ \Re^n, \\ u \in C^2(\Re^n), \end{cases}$$

admits solutions (see [14]).

PROPOSITION 4.— *Let us suppose that the problem* (P) *admits a solution, then under hypotheses (i) and (ii), there exists a positive constant* $\epsilon^*(\Omega, a)$ *such that* $(T\epsilon)$ *admits, at least, one solution* $u_\epsilon \in C^{1,\delta}(\overline{\Omega})$, $0 \leqslant \delta < 1$, *when* $0 < \epsilon < \epsilon^*(\Omega, a)$. *There is no* L^∞*-solution of* $(T\epsilon)$ *for* $\epsilon > \epsilon^*(\Omega, a)$.

Proof. — The proof is similar to the demonstration of Theorem 2 in [9].

Remark **3.** — *The hypothesis concerning the existence of a solution of (P) is justified by the critical growth case (see section 1).*

The $\epsilon^*(\Omega, a)$-limit case.

Before dealing with this case, let us state the following lemma.

LEMMA **1.** — *Under the hypotheses (i) and (ii), assume that (u_j) is a sequence of $C^2\left(\overline{\Omega}\right)$ – functions and (ϵ_j) is a real sequence, such that*

$$(P_j)\left\{\begin{array}{c} \Delta u_j + a(x)u_j^\beta = 0 \ in\ \Omega, \\ u_j > 0 \ in\ \Omega, \\ u = \epsilon_j > 0 \ on\ \partial\Omega. \end{array}\right.$$

Then, if the real sequence (ϵ_j) is bounded in $\Re$, so is (u_j) in $H_0^1(\Omega)$.

Proof. — Using Pohozaev Identity, we get

$$(1-\frac{n}{2})\int_\Omega \|\nabla u_j(x)\|^2dx+\frac{1}{2}\int_{\partial\Omega}(x.\nu)\|\nabla u_j(x)\|^2ds+\frac{n}{\beta+1}\int_\Omega a(x)u_j^{\beta+1}(x)dx$$

$$-\int_{\partial\Omega}(x.\nu)a(x)\epsilon_j^{\beta+1}dx = \int_{\partial\Omega}(x.\nabla u_j(x))(\nabla u_j(x).\nu)ds.$$

Using the Green's first identity, we get

$$\int_\Omega a(x)u_j^{\beta+1}dx = \int_\Omega \|\nabla u_j(x)\|^2dx - \int_{\partial\Omega}\epsilon_j\frac{\partial u_j(x)}{\partial\nu}ds.$$

So we infer that

$$(*)\quad \left(1-\frac{n}{2}+\frac{n}{\beta+1}\right)\int_\Omega \|\nabla u_j(x)\|^2dx = \int_{\partial\Omega}(x.\nabla u_j(x))(\nabla u_j(x).\nu)ds$$

$$-\frac{1}{2}\int_{\partial\Omega}(x.\nu)\|\nabla u_j(x)\|^2ds + \int_{\partial\Omega}x.\nu\ a(x)\epsilon_j^{\beta+1}ds + \frac{n}{\beta+1}\int_{\partial\Omega}\epsilon_j\frac{\partial u_j(x)}{\partial\nu}ds.$$

Using the maximum principle, and the fact that $u_j = \epsilon_j,\ on\ \partial\Omega$, we obtain

$$(1-\frac{n}{2}+\frac{n}{\beta+1})\int_\Omega \|\nabla u_j(x)\|^2dx = \frac{1}{2}\int_{\partial\Omega}\|\nabla u_j(x)\|^2x.\nu ds$$

$$+\int_{\partial\Omega}x.\nu a(x)\epsilon_j^{\beta+1}ds - \frac{n}{\beta+1}\int_{\partial\Omega}\epsilon_j\|\nabla u_j(x)\|ds.$$

As Ω is regular and starshaped, we get

$$(1-\frac{n}{2}+\frac{n}{\beta+1})\int_{\Omega}\|\nabla u_j(x)\|^2dx \geqslant c_0\int_{\partial\Omega}\|\nabla u_j(x)\|^2ds$$
$$-\frac{n}{\beta+1}\int_{\partial\Omega}\epsilon_j\|\nabla u_j(x)\|ds - c_1,$$

where,

$$c_0=\frac{1}{2}\min_{x\in\partial\Omega} x.\nu > 0 \textit{ and } \int_{\partial\Omega} x.\nu a(x)\epsilon_j^{\beta+1}ds \leqslant c_1.$$

As,

$$\beta > \frac{n+2}{n-2} \Longleftrightarrow 1-\frac{n}{2}+\frac{n}{\beta+1} < 0,$$

we get

$$\int_{\Omega}\|\nabla u_j(x)\|^2dx \leqslant c_2\int_{\partial\Omega}\|\nabla u_j(x)\|^2ds + c_3\int_{\partial\Omega}\|\nabla u_j(x)\|ds + c_4,$$

where $c_2 < 0 < c_3$ and $c_i, i = 2, ..4$ are constants not depending on j. Using Hölder's Inequality, we obtain

$$\int_{\Omega}\|\nabla u_j(x)\|^2dx \leqslant c_2\int_{\partial\Omega}\|\nabla u_j(x)\|^2ds + c_5\left(\int_{\partial\Omega}\|\nabla u_j(x)\|^2ds\right)^{\frac{1}{2}} + c_4$$
$$\leqslant \sup_{t\in\Re} c_2t^2 + c_5t + c_4 < \infty.$$

Let us put $v_j = u_j - \epsilon_j$, then $v_j \in H_0^1(\Omega)$ and

$$\|\nabla v_j\|_{L^2(\Omega)} = \|\nabla u_j\|_{L^2(\Omega)}.$$

Using Poincaré Inequality, we get

$$\exists c_0 > 0 \; ; \; \|u_j - \epsilon_j\|_{H_0^1(\Omega)} \leqslant c_0, \; \forall j.$$

As

$$u_j^2(x) = [u_j(x) - \epsilon_j + \epsilon_j]^2 \leqslant 2\left\{[u_j(x) - \epsilon_j]^2 + \epsilon_j^2\right\},$$

and ϵ_j is bounded in $\Re$, this completes the proof of Lemma 1.

Remark 4.— *The a priori estimate in Lemma 1 remains true for nonlinearities such that, there exist constants c and γ, with*

$$c + uf(x,u) \leqslant \gamma F(x,u) \;, \; \textit{where } F(x,u) = \int_0^u f(x,t)dt, \; \gamma > 2^* = \frac{2n}{n-2}.$$

PROPOSITION 5. — *Under the hypotheses of Proposition 4, if* $a \in C^{0,\delta}(\overline{\Omega}), 0 < \delta \leqslant 1,$ *then* $(T\epsilon^*(\Omega, a))$ *admits a solution.*

Proof. — Let (ϵ_j) be an increasing real sequence such that

$$0 < \epsilon_j < \epsilon_{j+1} < \lim_{i \to \infty} \epsilon_i = \epsilon^*(\Omega, a).$$

For every j, let u_j be the solution of $(T\epsilon_j)$ (see Proposition 4). As $u_j \in C^2(\overline{\Omega})$, one can use Lemma 1 to obtain

$$\exists c > 0, \ \|u_j\|_{H^1(\Omega)} \leqslant c, \ \ \forall j.$$

Then, up to a subsequence, $u_j \rightharpoonup u$ *in* $H^1(\Omega)$-weak, $u_j \rightarrow u$ *in* $L^2(\Omega)$ − *strong and* $u_j \rightarrow u$, a.e. in Ω. One can multiply (P_j) by u_j to verify that

$$a(x)u_{\epsilon_j}^{\beta} \in L^{\frac{\beta+1}{\beta}}(\Omega).$$

By using the L^p-estimates and a bootstrap argument, one can show that u is a solution of $(T\epsilon^*(\Omega, a))$.

PROPOSITION 6. — *Let* u *be a spherically symmetric* $L^{\infty}_{loc}(\Re^n)$− *solution of*

$$\begin{cases} \Delta u + u^{\beta} = 0 \ in \ \Re^n \\ u > 0 \ in \ \Re^n, \end{cases}$$

then $u \in C^2(\Re^n)$ *and* $u \in L^p(\Re^n), \ \forall p > \frac{n(\beta-1)}{2}$.

Proof. — Let us choose $r > 0$, such that $u(r) < \infty$. As $u \in L^{\infty}(B_r)$, one can use the L^p-estimates (see [1]) to infer that $u \in W^{2,p}(B_r), \ \forall \ p > 1$. We infer that (see [8])

$$u \in C^{1,\delta}(\overline{B_r}), \ \forall \ 0 < \delta < 1.$$

From the previous line, we see that $u^{\beta} \in C^{0,\delta}(\overline{B_r}), \ \forall \ 0 < \delta < 1$, So we can use the Schauder Estimates to deduce that $u \in C^{2,\delta}(\overline{B_r})$. We can use Proposition 4, to infer that

$$\exists \ \epsilon^*(B_r, \beta) \ such \ that \ u(r) \leqslant \epsilon^*(B_r, \beta).$$

It is easy to verify(see [10]) that

$$\epsilon^*(B_r, \beta) \leqslant \epsilon^*(B_1, \beta) r^{\frac{2}{1-\beta}},$$

so we deduce that, if $p > \frac{n(\beta-1)}{2}$, then $u \in L^p(\Re^n)$.

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