ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

FRIEDRICH HASLINGER

# The canonical solution operator to $\bar\partial$ restricted to spaces of entire functions

*Annales de la faculté des sciences de Toulouse 6e série*, tome 11, nº 1 (2002), p. 57-70

<http://www.numdam.org/item?id=AFST_2002_6_11_1_57_0>

Annales de la Faculté des Sciences de Toulouse Vol. XI, n° 1, 2002
pp. 57–70

# The canonical solution operator to $\overline{\partial}$ restricted to spaces of entire functions (*)

FRIEDRICH HASLINGER (1)

RÉSUMÉ. — Dans cet article, nous étudions l'opérateur canonique de solution du $\overline{\partial}$ dans les espaces $L^2(\mathbb{C}^n, e^{-p})$ a poids et discutons de ses propriétés de compacité et d'être de Hilbert-Schmidt. Dans le cas d'une seule variable complexe, nous montrons que cet opérateur solution n'est pas compact dans $L^2(\mathbb{C}, e^{-|z|^2})$ même si on se restreint au sous-espace correspondant de fonctions entières. L'opérateur solution est compact quand on le restreint au sous-espace des fonctions entières pour les poids $e^{-|z|^m}, m > 2$, mais n'est pas Hilbert-Schmidt. Dans la seconde partie, nous montrons que, dans un contexte légerement different, nous obtenons la propriété d'être de type Hilbert-Schmidt pour une classe très large d'espaces a poids de fonctions entières de plusieurs variables complexes.

ABSTRACT. — In this paper we discuss compactness and the Hilbert-Schmidt property of the canonical solution operator to $\overline{\partial}$ in weighted $L^2(\mathbb{C}^n, e^{-p})$ spaces. In the case of one complex variable we show that the solution operator is not compact on $L^2(\mathbb{C}, e^{-|z|^2})$ even when restricted to the corresponding subspace of entire functions; the solution operator is compact when restricted to the subspace of entire functions for the weights $e^{-|z|^m}$, $m > 2$, but fails to be Hilbert-Schmidt. In the second part we show that we get the Hilbert-Schmidt property in a slightly different setting for a large class of weighted spaces of entire functions in several variables.

(*) Reçu le 11 février 2002, accepté le 4 septembre 2002
(1) Supported by a FWF grant P15279.
Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria
e-mail: friedrich.haslinger@univie.ac.at

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. In [7] it is shown that the canonical solution operator $S$ to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can be expressed by an integral operator using the Bergman kernel:

$$S(g)(z) = \int_\Omega K(z,w)\,\langle g(w), z-w\rangle\, d\lambda(w),$$

where $g = \sum_{j=1}^n g_j\, d\overline{z}_j \in A^2_{(0,1)}(\Omega)$ is a $(0,1)$-form with holomorphic coefficients, $\langle g(w), z-w\rangle = \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j)$ and $K(z,w)$ is the Bergman kernel of $\Omega$. The canonical solution operator to $\overline{\partial}$ has the properties : $\overline{\partial} S(g) = g$ and $S(g) \perp A^2(\Omega)$.

The canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can also be interpreted as the Hankel operator

$$H_{\overline{z}}(g) = (I - P)(\overline{z} g),$$

where $P : L^2(\Omega) \longrightarrow A^2(\Omega)$ denotes the Bergman projection. See [1], [2], [9], [14] and [16] for details.

For the unit disk $\mathbb{D}$ in $\mathbb{C}$ the canonical solution operator restricted to $A^2(\mathbb{D})$ is a Hilbert-Schmidt operator, whereas for the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$ the canonical solution operator fails to be Hilbert-Schmidt (see [7]).

In many cases non-compactness of the canonical solution operator already happens when the solution operator is restricted to the corresponding subspace of holomorphic functions (or $(0,1)$-forms with holomorphic coefficients, in the case of several variables.)(see [4] , [13] , [11]). In this paper we will show that this phenomenon also occurs in the Fock space in one variable.

It is pointed out in [4] that in the proof that compactness of the solution operator for $\overline{\partial}$ on $(0,1)$-forms implies that the boundary of $\Omega$ does not contain any analytic variety of dimension greater than or equal to 1, it is only used that there is a compact solution operator to $\overline{\partial}$ on the $(0,1)$-forms with holomorphic coefficients. In this case compactness of the solution operator restricted to $(0,1)$-forms with holomorphic coefficients implies already compactness of the solution operator on general $(0,1)$-forms.

A similar situation appears in [13] where the Toeplitz $C^*$ -algebra $\mathcal{T}(\Omega)$ is considered and the relation between the structure of $\mathcal{T}(\Omega)$ and the $\overline{\partial}$-Neumann problem is discussed (see [13] , Corollary 4.6). The question of compactness of the $\overline{\partial}$-Neumann operator is of interest for various reasons (see the survey article [5] ).

In this paper we show that the canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is not compact. This follows from the result that the canonical solution operator for $\overline{\partial}$ restricted to the weighted space of entire functions $A^2(\mathbb{C}, e^{-|z|^2})$ (Fock space) into $L^2(\mathbb{C}, e^{-|z|^2})$ already fails to be compact. Further it is shown that the restriction to $A^2(\mathbb{C}, e^{-|z|^m})$, $m > 2$, is compact but not Hilbert-Schmidt. When using the methods of [7] in this case the main difficulty is it that there are functions $g \in A^2(\mathbb{C}, e^{-|z|^m})$ such that $\overline{z}g \notin L^2(\mathbb{C}, e^{-|z|^m})$. Hence the formula for the canonical solution operator using the Bergamn kernel can't be used directly, but it will turn out that the expression $\overline{z}g(z) - P(\overline{z}g)(z)$ makes sense in $L^2(\mathbb{C}, e^{-|z|^m})$.

In the sequel we also consider the case of several complex variables in a slightly different situation and show that the canonical solution operator to $\overline{\partial}$ is a Hilbert-Schmidt operator for a wide class of weighted spaces of entire functions using various methods from abstract functional analysis (see [12]).

## 2. Spaces of entire functions in one variable

We consider weighted spaces on entire functions

$$A^2(\mathbb{C}, e^{-|z|^m}) = \{f : \mathbb{C} \longrightarrow \mathbb{C} \ : \ \|f\|_m^2 := \int_{\mathbb{C}} |f(z)|^2 \, e^{-|z|^m} \, d\lambda(z) < \infty\},$$

where $m > 0$. Let

$$c_k^2 = \int_{\mathbb{C}} |z|^{2k} \, e^{-|z|^m} \, d\lambda(z).$$

Then

$$K_m(z, w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}$$

is the reproducing kernel for $A^2(\mathbb{C}, e^{-|z|^m})$.

In the sequel the expression

$$\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2}$$

will become important. Using the integral representation of the $\Gamma$-function one easily sees that the above expression is equal to

$$\frac{\Gamma\left(\frac{2k+4}{m}\right)}{\Gamma\left(\frac{2k+2}{m}\right)} - \frac{\Gamma\left(\frac{2k+2}{m}\right)}{\Gamma\left(\frac{2k}{m}\right)}.$$

For $m = 2$ this expression equals to 1 for each $k = 1, 2, \ldots$. We will be interested in the limit behavior for $k \to \infty$. By Stirlings formula the limit behavior is equivalent to the limit behavior of the expression

$$\left(\frac{2k+2}{m}\right)^{2/m} - \left(\frac{2k}{m}\right)^{2/m},$$

as $k \to \infty$. Hence we have shown the following

LEMMA 1.— *The expression*

$$\frac{\Gamma\left(\frac{2k+4}{m}\right)}{\Gamma\left(\frac{2k+2}{m}\right)} - \frac{\Gamma\left(\frac{2k+2}{m}\right)}{\Gamma\left(\frac{2k}{m}\right)}$$

*tends to* $\infty$ *for* $0 < m < 2$, *is equal to* 1 *for* $m = 2$ *and tends to zero for* $m > 2$ *as* $k$ *tends to* $\infty$.

Let $0 < \rho < 1$, define $f_\rho(z) := f(\rho z)$ and $\tilde{f}_\rho(z) = \overline{z} f_\rho(z)$, for $f \in A^2(\mathbb{C}, e^{-|z|^m})$. Then it is easily seen that $\tilde{f}_\rho \in L^2(\mathbb{C}, e^{-|z|^m})$, but there are functions $g \in A^2(\mathbb{C}, e^{-|z|^m})$ such that $\overline{z} g \notin L^2(\mathbb{C}, e^{-|z|^m})$.

Let $P_m : L^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow A^2(\mathbb{C}, e^{-|z|^m})$ denote the orthogonal projection. Then $P_m$ can be written in the form

$$P_m(f)(z) = \int_{\mathbb{C}} K_m(z,w) f(w) e^{-|w|^m} \, d\lambda(w) \ , \ f \in L^2(\mathbb{C}, e^{-|z|^m}).$$

PROPOSITION 1.— *Let* $m \geq 2$. *Then there is a constant* $C_m > 0$ *depending only on* $m$ *such that*

$$\int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z),$$

*for each* $0 < \rho < 1$ *and for each* $f \in A^2(\mathbb{C}, e^{-|z|^m})$.

*Proof.* — First we observe that for the Taylor expansion of $f(z) = \sum_{k=0}^\infty a_k z^k$ we have

$$\begin{aligned} P_m(\tilde{f}_\rho)(z) &= \int_{\mathbb{C}} \sum_{k=0}^\infty \frac{z^k \overline{w}^k}{c_k^2} \left( \overline{w} \sum_{j=0}^\infty a_j \rho^j w^j \right) e^{-|w|^m} \, d\lambda(w) \\ &= \sum_{k=1}^\infty a_k \frac{c_k^2}{c_{k-1}^2} \rho^k z^{k-1}. \end{aligned}$$

Now we obtain

$$\begin{aligned}
&\int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m}\, d\lambda(z) \\
&= \int_{\mathbb{C}} \left( \overline{z} \sum_{k=0}^{\infty} a_k \rho^k z^k - \sum_{k=1}^{\infty} a_k \frac{c_k^2}{c_{k-1}^2} \rho^k z^{k-1} \right) \\
&\times \left( z \sum_{k=0}^{\infty} a_k \rho^k \overline{z}^k - \sum_{k=1}^{\infty} \overline{a_k} \frac{c_k^2}{c_{k-1}^2} \rho^k \overline{z}^{k-1} \right) e^{-|z|^m}\, d\lambda(z) \\
&= \int_{\mathbb{C}} (\sum_{k=0}^{\infty} |a_k|^2 \rho^{2k} |z|^{2k+2} - 2 \sum_{k=1}^{\infty} |a_k|^2 \frac{c_k^2}{c_{k-1}^2} \rho^{2k} |z|^{2k} \\
&+ \sum_{k=1}^{\infty} |a_k|^2 \frac{c_k^4}{c_{k-1}^4} \rho^{2k} |z|^{2k-2})\, e^{-|z|^m}\, d\lambda(z) \\
&= |a_0|^2 c_1^2 + \sum_{k=1}^{\infty} |a_k|^2 c_k^2 \rho^{2k} \left( \frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right).
\end{aligned}$$

Now the result follows from the fact that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z) = \sum_{k=0}^{\infty} |a_k|^2 c_k^2,$$

and that the sequence $\left( \frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right)_k$ is bounded. □

*Remark 1.— Already in the last proposition the sequence* $\left( \frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right)_k$ *plays an important role and it will turn out that this sequence is the sequence of eigenvalues of the operator* $S_m^* S_m$ *(see below).*

PROPOSITION 2.— *Let* $m \geq 2$ *and consider an entire function* $f \in A^2(\mathbb{C}, e^{-|z|^m})$ *with Taylor series expansion* $f(z) = \sum_{k=0}^{\infty} a_k z^k$. *Let*

$$F(z) := \overline{z} \sum_{k=0}^{\infty} a_k z^k - \sum_{k=1}^{\infty} a_k \frac{c_k^2}{c_{k-1}^2} z^{k-1}$$

*and define* $S_m(f) := F$. *Then* $S_m : A^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^m})$ *is a continuous linear operator, representing the canonical solution operator to* $\overline{\partial}$ *restricted to* $A^2(\mathbb{C}, e^{-|z|^m})$, *i.e.* $\overline{\partial} S_m(f) = f$ *and* $S_m(f) \perp A^2(\mathbb{C}, e^{-|z|^m})$.

*Proof.* — By Fatou's theorem

$$\begin{aligned}
&\int_{\mathbb{C}} \lim_{\rho\to 1} \left|\tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z)\right|^2 e^{-|z|^m}\, d\lambda(z) \\
&\leq \sup_{0<\rho<1} \int_{\mathbb{C}} \left|\tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z)\right|^2 e^{-|z|^m}\, d\lambda(z) \\
&\leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z)
\end{aligned}$$

and hence the function

$$F(z) := \overline{z} \sum_{k=0}^{\infty} a_k z^k - \sum_{k=1}^{\infty} a_k \frac{c_k^2}{c_{k-1}^2} z^{k-1}$$

belongs to $L^2(\mathbb{C}, e^{-|z|^m})$ and satisfies

$$\int_{\mathbb{C}} |F(z)|^2 e^{-|z|^m}\, d\lambda(z) \leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m}\, d\lambda(z).$$

The above computation also shows that $\lim_{\rho\to 1} \|\tilde{f}_\rho - P_m(\tilde{f}_\rho)\|_m = \|F\|_m$ and by a standard argument for $L^p$-spaces (see for instance [3])

$$\lim_{\rho\to 1} \|\tilde{f}_\rho - P_m(\tilde{f}_\rho) - F\|_m = 0.$$

A similar computation as in the proof of Proposition 1 in [7] shows that the function $F$ defined above satisfies $\overline{\partial} F = f$. Let $S_m(f) := F$. Then, by the last remarks, $S_m : A^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^m})$ is a continuous linear solution operator for $\overline{\partial}$. For arbitrary $h \in A^2(\mathbb{C}, e^{-|z|^m})$ we have

$$(h, S_m(f))_m = (h, F)_m = \lim_{\rho\to 1} (h, \tilde{f}_\rho - P_m(\tilde{f}_\rho))_m = \lim_{\rho\to 1} (h - P_m(h), \tilde{f}_\rho)_m = 0,$$

where $(.,.)_m$ denotes the inner product in $L^2(\mathbb{C}, e^{-|z|^m})$. Hence $S_m$ is the canonical solution operator for $\overline{\partial}$ restricted to $A^2(\mathbb{C}, e^{-|z|^m})$. □

*Remark 2.* — *The expression for the function $F$ in the last theorem corresponds formally to the expression $\overline{z}f - P_m(\overline{z}f)$; in general $\overline{z}f \notin L^2(\mathbb{C}, e^{-|z|^m})$, for $f \in A^2(\mathbb{C}, e^{-|z|^m})$, but $f \mapsto F$ defines a bounded linear operator from $A^2(\mathbb{C}, e^{-|z|^m})$ to $L^2(\mathbb{C}, e^{-|z|^m})$.*

THEOREM 1. — *The canonical solution operator to $\overline{\partial}$ restricted to the space $A^2(\mathbb{C}, e^{-|z|^m})$ is compact if and only if*

$$\lim_{k\to\infty} \left( \frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right) = 0.$$

*Proof.* — For a complex polynomial $p$ the canonical solution operator $S_m$ can be written in the form

$$S_m(p)(z) = \int_{\mathbb{C}} K_m(z,w)p(w)(\overline{z}-\overline{w})e^{-|w|^m}\, d\lambda(w),$$

therefore we can express the conjugate $S_m^*$ in the form

$$S_m^*(q)(w) = \int_{\mathbb{C}} K_m(w,z)q(z)(z-w)e^{-|z|^m}\, d\lambda(z),$$

if $q$ is a finite linear combination of the terms $\overline{z}^k\, z^l$. This follows by considering the inner product $(S_m(p),q)_m = (p, S_m^*(q))_m$.

Now we claim that

$$S_m^* S_m(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) u_n(w) \quad , n = 1,2,\ldots$$

and

$$S_m^* S_m(u_0)(w) = \frac{c_1^2}{c_0^2}\, u_0(w),$$

where $\{u_n(z) = z^n/c_n, k = 0,1,\ldots\}$ is the standard orthonormal basis of $A^2(\mathbb{C}, e^{-|z|^m})$.

From [7] we know that

$$S_m(u_n)(z) = \overline{z}u_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2},\ n = 1,2,\ldots.$$

Hence

$$\begin{aligned} S_m^* S_m(u_n)(w) &= \int_{\mathbb{C}} K_m(w,z)(z-w)\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z) \\ &= \int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}(z-w)\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z). \end{aligned}$$

This integral is computed in two steps: first the multiplication by $z$

$$\int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\left(\frac{\overline{z}z^{n+1}}{c_n} - \frac{c_n z^n}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z)$$

$$= \int_{\mathbb{C}} \frac{z^{n+1}}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} e^{-|z|^m}\, d\lambda(z) - \frac{c_n}{c_{n-1}^2} \int_{\mathbb{C}} z^n \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} e^{-|z|^m}\, d\lambda(z)$$

$$= \frac{w^n}{c_n^3}\int_{\mathbb{C}} |z|^{2n+2}e^{-|z|^m}\, d\lambda(z) - \frac{w^n}{c_{n-1}^2 c_n^2}\int_{\mathbb{C}} |z|^{2n}e^{-|z|^m}\, d\lambda(z)$$
$$= \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2}\right) w^n.$$

And now the multiplication by $w$

$$w\int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) e^{-|z|^m}\, d\lambda(z)$$

$$= w\int_{\mathbb{C}} \frac{z^n}{c_n}\sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} e^{-|z|^m}\, d\lambda(z) - w\int_{\mathbb{C}} \frac{c_n z^{n-1}}{c_{n-1}^2}\sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} e^{-|z|^m}\, d\lambda(z)$$
$$= w\left(\frac{c_n w^{n-1}}{c_{n-1}^2} - \frac{c_n w^{n-1}}{c_{n-1}^2}\right)$$
$$= 0,$$

which implies that

$$S_m^* S_m(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) u_n(w) \quad , n = 1, 2, \dots,$$

the case $n = 0$ follows from an analogous computation.

The last statement says that $S_m^* S_m$ is a diagonal operator with respect to the orthonormal basis $\{u_n(z) = z^n/c_n\}$ of $A^2(\mathbb{C}, e^{-|z|^m})$. Therefore it is easily seen that $S_m^* S_m$ is compact if and only if

$$\lim_{n\to\infty}\left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) = 0.$$

Now the conclusion follows, since $S_m^* S_m$ is compact if and only if $S_m$ is compact (see for instance [15]). □

THEOREM 2. — *The canonical solution operator for $\overline{\partial}$ restricted to the space $A^2(\mathbb{C}, e^{-|z|^m})$ is compact, if $m > 2$. The canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is not compact.*

*Proof.* — The first statement follows immediately from Theorem 1 and Lemma 1 For the second statement we use Hörmander's $L^2$-estimate for the solution of the $\overline{\partial}$ equation [8] : for each $f \in L^2(\mathbb{C}, e^{-|z|^2})$ there is a function $u \in L^2(\mathbb{C}, e^{-|z|^2})$ such that $\overline{\partial} u = f$ and

$$\int_{\mathbb{C}} |u(z)|^2\, e^{-|z|^2}\, d\lambda(z) \le 4\int_{\mathbb{C}} |f(z)|^2\, e^{-|z|^2}\, d\lambda(z).$$

Hence the canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is continuous and its restriction to the closed subspace $A^2(\mathbb{C}, e^{-|z|^2})$ fails to be compact by Propositon 1 and Lemma 1. By the definition of compactness this implies that the canonical solution operator is not compact as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself. □

*Remark 3.— In the case of the Fock space* $A^2(\mathbb{C}, e^{-|z|^2})$ *the composition* $S_2^* S_2$ *equals to the identity on* $A^2(\mathbb{C}, e^{-|z|^2})$, *which follows from the proof of Theorem 1.*

THEOREM 3.— *Let* $m \geq 2$. *The canonical solution operator for* $\overline{\partial}$ *restricted to* $A^2(\mathbb{C}, e^{-|z|^m})$ *fails to be Hilbert Schmidt.*

*Proof.* — By Proposition 2 we know that the canonical solution operator is continuous and we can use the techniques from [7]

$$\begin{aligned} \|S_m(u_n)\|^2 &= \frac{1}{c_n^2} \int_{\mathbb{C}} \left| \overline{z}\, z^n - \frac{c_n^2}{c_{n-1}^2}\, z^{n-1} \right|^2 e^{-|z|^m}\, d\lambda(z) \\ &= \frac{1}{c_n^2} \int_{\mathbb{C}} |z|^{2n-2} \left( |z|^4 - \frac{2c_n^2 |z|^2}{c_{n-1}^2} + \frac{c_n^4}{c_{n-1}^4} \right) e^{-|z|^m}\, d\lambda(z) \\ &= \frac{1}{c_n^2} \int_{\mathbb{C}} |z|^{2n+2}\, e^{-|z|^m}\, d\lambda(z) - \frac{2}{c_{n-1}^2} \int_{\mathbb{C}} |z|^{2n}\, e^{-|z|^m}\, d\lambda(z) \\ &+ \frac{c_n^2}{c_{n-1}^4} \int_{\mathbb{C}} |z|^{2n-2}\, e^{-|z|^m}\, d\lambda(z) \\ &= \frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \|S_m(u_n)\|^2 < \infty$$

if and only if

$$\lim_{n\to\infty} \frac{c_{n+1}^2}{c_n^2} < \infty.$$

By [12] , 16.8, $S_m$ is a Hilbert Schmidt operator if and only if

$$\sum_{n=0}^{\infty} \|S_m(u_n)\|^2 < \infty.$$

In our case we have

$$\frac{c_{n+1}^2}{c_n^2} = \Gamma\left(\frac{2n+4}{m}\right) / \Gamma\left(\frac{2n+2}{m}\right),$$

which, by Stirling's formula, implies that the corresponding canonical solution operator to $\overline{\partial}$ fails to be Hilbert Schmidt. $\square$

In the case of several variables the corresponding operator $S^*S$ is more complicated, nevertheless we can handle a sligthly different situation with different methods from functional analysis (see next section).

## 3. Weighted spaces of entire functions in several variables

In this part we show that the canonical solution operator to $\overline{\partial}$ is a Hilbert-Schmidt operator for a wide class of weighted spaces of entire functions.

The weight functions we are considering are of the form $z \mapsto \tau p(z)$, where $\tau > 0$ and $p : \mathbb{C}^n \longrightarrow \mathbb{R}$. We suppose that $p$ is a plurisubharmonic function satisfying

$$p^*(w) := \sup\{\Re < z, w > -p(z) \ : \ z \in \mathbb{C}^n\} < \infty.$$

Then $p^{**} = p$ and

$$\lim_{|z| \to \infty} \frac{p(z)}{|z|} = \infty$$

(see Lemma 1.1. in [6]). And it is easily seen that

$$\int_{\mathbb{C}^n} \exp[(\tau - \sigma)\, p(z)]\, d\lambda(z) < \infty,$$

whenever $\tau - \sigma < 0$.

We further assume that

$$\lim_{|z| \to \infty} \frac{\tilde{p}(z)}{p(z)} = 1,$$

where $\tilde{p}(z) = \sup\{p(z + \zeta) \ : \ |\zeta| \leq 1\}$.

It follows that the last property is equivalent to the following condition: for each $\sigma > 0$ and for each $\tau > 0$ with $\tau < \sigma$ there is a constant $C = C(\sigma, \tau) > 0$ such that

$$\tau\, \tilde{p}(z) - \sigma\, p(z) \leq C,$$

for each $z \in \mathbb{C}^n$.

Let $A^2(\mathbb{C}^n, \sigma p)$ denote the Hilbert space of all entire functions $h : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{C}^n} |h(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z) < \infty.$$

THEOREM 4. — *Suppose that $p$ is a weight function with the properties listed above. Then for each $\sigma > 0$ there exists a number $\tau > 0$ with $\tau < \sigma$ such that the canonical solution operator $S_1$ to $\overline{\partial}$ is a Hilbert-Schmidt operator as a mapping*

$$S_1 : A^2_{(0,1)}(\mathbb{C}^n, \tau p) \longrightarrow L^2(\mathbb{C}^n, \sigma p).$$

*Proof.* — By Lemma 28.2 from [12] we have to show that

$$\left[\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z)\right]^{1/2} \leq \int_{\mathcal{U}} |(f,g)|\, d\mu(g),$$

where $(.,.)$ denotes the inner product of the Hilbert space $A^2_{(0,1)}(\mathbb{C}^n, \tau p)$, $\mathcal{U}$ is the unit ball of $A^2_{(0,1)}(\mathbb{C}^n, \tau p)$ , $\mu$ is a Radon measure on the weakly compact set $\mathcal{U}$ and $f = \sum_{j=1}^n f_j\, d\overline{z}_j$ and $g = \sum_{j=1}^n g_j\, d\overline{z}_j$.

We first show that for $0 < \tau < \tau_1 < \tau_2 < \tau_3 < \sigma$ we have

$$\begin{aligned}
&\left[\int_{\mathbb{C}^n} |f_j(z)|^2 \exp(-2\tau_3 p(z))\, d\lambda(z)\right]^{1/2} \\
&\leq\ C_{\tau_3,\tau_2} \sup\{|f_j(z)| \exp(-\tau_2 p(z)) \ :\ z \in \mathbb{C}^n\} \\
&\leq\ C_{\tau_2,\tau_1} \int_{\mathbb{C}^n} |f_j(z)| \exp(-\tau_1 p(z))\, d\lambda(z) \\
&\leq\ C_{\tau_1,\tau} \left[\int_{\mathbb{C}^n} |f_j(z)|^2 \exp(-2\tau p(z))\, d\lambda(z)\right]^{1/2},
\end{aligned}$$

for each $f \in A^2_{(0,1)}(\mathbb{C}^n, \tau p)$.

To show this assertion we make use of the assumption that the coefficients of the $(0,1)$-form $f$ are entire functions:

The first inequality follows from the fact that

$$\int_{\mathbb{C}^n} \exp((2\tau_2 - 2\tau_3)p(z))\, d\lambda(z) < \infty.$$

For the second inequality use Cauchy's theorem for the coefficients $f_j$ of $f$ to show that for $B_z = \{\zeta \in \mathbb{C}^n \ :\ |z - \zeta| \leq 1\}$ we have

$$\begin{aligned}
|f_j(z)| &\leq C \int_{B_z} |f_j(\zeta)|\, d\lambda(\zeta) \\
&= C \int_{B_z} |f_j(\zeta)| \exp(-\tau_1 p(\zeta)) \exp(\tau_1 p(\zeta))\, d\lambda(\zeta)
\end{aligned}$$

$$\leq C \int_{B_z} |f_j(\zeta)| \exp(-\tau_1 p(\zeta))\, d\lambda(\zeta) \ \sup\{\exp(\tau_1 p(\zeta)) \,:\, \zeta \in B_z\}$$
$$\leq C' \int_{\mathbb{C}^n} |f_j(\zeta)| \exp(-\tau_1 p(\zeta))\, d\lambda(\zeta) \ \exp(\tau_2 p(z)),$$

where we used the properties of the weight function $p$.

The third inequality is a consequence of the Cauchy-Schwarz inequality:

$$\int_{\mathbb{C}^n} |f_j(\zeta)| \exp(-\tau_1 p(\zeta))\, d\lambda(\zeta)$$
$$= \int_{\mathbb{C}^n} |f_j(\zeta)| \exp(-\tau p(\zeta)) \exp((\tau - \tau_1) p(\zeta))\, d\lambda(\zeta)$$
$$\leq \left[\int_{\mathbb{C}^n} |f_j(\zeta)|^2 \exp(-2\tau p(\zeta))\, d\lambda(\zeta)\right]^{1/2}$$
$$\left[\int_{\mathbb{C}^n} \exp((2\tau - 2\tau_1) p(\zeta))\, d\lambda(\zeta)\right]^{1/2}.$$

By Hörmander's $L^2$-estimates ([8], Theorem 4.4.2]) we have for $\tau < \tau_3 < \sigma$ and the properties of the weight function

$$\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z)$$
$$\leq \int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\tau_3 p(z))\,(1 + |z|^2)^{-2}\, d\lambda(z)$$
$$\leq \int_{\mathbb{C}^n} |f(z)|^2 \exp(-2\tau_3 p(z))\, d\lambda(z).$$

Here we used the fact that the canonical solution operator $S_1$ can be written in the form $S_1(f) = v - P(v)$, where $v$ is an arbitrary solution to $\overline{\partial} u = f$ belonging to the corresponding Hilbert space and that $\|S_1(f)\| = \|v - P(v)\| = \min\{\|v - h\| \,:\, h \in A^2\} \leq \|v\|$.

Now choose $\tau_2$ such that $\tau < \tau_2 < \tau_3 < \sigma$, then we obtain from the above inequalities

$$\left[\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z)\right]^{1/2}$$
$$\leq D \int_{\mathbb{C}^n} |f(z)|_1 \exp(-\tau_2 p(z))\, d\lambda(z),$$

where $D > 0$ is a constant and $|f(z)|_1 := |f_1(z)| + \ldots + |f_n(z)|$.

Now define for $z \in \mathbb{C}^n$ and $\tau < \tau_1 < \tau_2$

$$\delta_z^{\tau_1,\tau}(f_j) := C_{\tau_1,\tau}^{-1}\, f_j(z)\, \exp(-\tau_1 p(z)).$$

Then

$$\begin{aligned} & C_{\tau_1,\tau}^{-1} \sup\{|f_j(z)| \exp(-\tau_1 p(z)) \, : \, z \in \mathbb{C}^n\} \\ \leq & \left[\int_{\mathbb{C}^n} |f_j(z)|^2 \exp(-2\tau p(z))\, d\lambda(z)\right]^{1/2}, \end{aligned}$$

which, by the Riesz representation theorem for the Hilbert space $A^2_{(0,1)}(\mathbb{C}^n, \tau p)$, means that each $\delta_z^{\tau_1,\tau}$ can be viewed as an element of $\mathcal{U}$.

For $\phi \in \mathcal{C}(\mathcal{U})$ the expression

$$\mu(\phi) = D \int_{\mathbb{C}^n} \phi(\delta_z^{\tau_1,\tau}) \exp((\tau_1 - \tau_2)p(z))\, d\lambda(z)$$

defines a Radon measure on the weakly compact set $\mathcal{U}$.

This follows from the fact that

$$\mu(\phi) \leq D \sup\{|\phi(g)| \, : \, g \in \mathcal{U}\} \int_{\mathbb{C}^n} \exp((\tau_1 - \tau_2)p(z))\, d\lambda(z).$$

Now take for $\phi$ the continuous functions $\phi_j(g_j) = |(f_j, g_j)|$, where $f_j$ is fixed. Then

$$\begin{aligned} & \int_{\mathcal{U}} |(f,g)|\, d\mu(g) \\ = \; & D \int_{\mathbb{C}^n} |f(z)|_1 \exp(-\tau_1 p(z)) \exp((\tau_1 - \tau_2)p(z))\, d\lambda(z) \\ = \; & D \int_{\mathbb{C}^n} |f(z)|_1 \exp(-\tau_2 p(z))\, d\lambda(z) \end{aligned}$$

and hence

$$\left[\int_{\mathbb{C}^n} |S_1(f)(z)|^2 \exp(-2\sigma p(z))\, d\lambda(z)\right]^{1/2} \leq \int_{\mathcal{U}} |(f,g)|\, d\mu(g). \qquad \square$$

## Bibliography


[1] AXLER (S.). — The Bergman space, the Bloch space, and commutators of multiplication operators, Duke Math. J. **53** (1986), 315–332.

[2] ARAZY (J.), FISHER (S.) and PEETRE (J.). — Hankel operators on weighted Bergman spaces, Amer. J. of Math. **110** (1988), 989–1054.

[3] ELSTRODT (J.). — *Maß - und Integrationstheorie,* Springer Verlag, Berlin 1996.

[4] FU (S.) and STRAUBE (E.J.). — Compactness of the $\overline{\partial}$–Neumann problem on convex domains, J. of Functional Analysis **159** (1998), 629–641.

[5] FU (S.) and STRAUBE (E.J.). — Compactness in the $\overline{\partial}$–Neumann problem, Complex Analysis and Geometry (J.McNeal, ed.), Ohio State Math. Res. Inst. Publ. **9** (2001), 141–160.

[6] HASLINGER (F.). — Weighted spaces of entire functions, Indiana Univ. Math. J. **35** (1986), 193–208.

[7] HASLINGER (F.). — The canonical solution operator to $\overline{\partial}$ restricted to Bergman spaces, Proc. Amer.Math. Soc. **129** (2001), 3321–3329.

[8] HÖRMANDER (L.). — *An introduction to complex analysis in several variables,* North-Holland Publishing Company, Amsterdam 1990 (3rd edition).

[9] JANSON (S.). — Hankel operators between weighted Bergman spaces, Ark. Mat. **26** (1988), 205–219.

[10] KRANTZ (St.). — *Function theory of several complex variables,* Wadsworth & Brooks/Cole, 1992 (2nd edition).

[11] KRANTZ (St.). — Compactness of the $\overline{\partial}$–Neumann operator, Proc. Amer. Math. Soc. **103** (1988), 1136–1138.

[12] MEISE (R.) und VOGT (D.). — *Einführung in die Funktionalanalysis,* Vieweg Studium **62**, Vieweg-Verlag 1992.

[13] SALINAS (N.), SHEU (A.) and UPMEIER (H.). — Toeplitz operators on pseudoconvex domains and foliation $C^*$– algebras, Ann. of Math. **130** (1989), 531–565.

[14] WALLSTEN (R.). — Hankel operators between weighted Bergman spaces in the ball, Ark. Mat. **28** (1990), 183–192.

[15] WEIDMANN (J.). — *Lineare Operatoren in Hilberträumen,* B.G. Teubner Stuttgart, Leipzig, Wiesbaden 2000.

[16] ZHU (K.H.). — Hilbert-Schmidt Hankel operators on the Bergman space, Proc. Amer. Math. Soc. **109** (1990), 721–730.