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Optimal Lipschitz estimates for the $\overline{\partial}$ equation on a class of convex domains


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Optimal Lipschitz estimates for the $\bar{\partial}$ equation
on a class of convex domains (*)

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1. Introduction and statement of the main results

For every $m$-uplet of positive integers $N := (n_1, \ldots, n_m)$, we consider the following domain:

$$
\Omega_N := \left\{ Z = (Z_1, \ldots, Z_m) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_m} : \sum_{j=1}^{m} (|Z_j|^2 + |Z_j \cdot Z_j|) < 1 \right\},
$$

(1.1)

where $z \cdot w := \sum_{j=1}^{k} z_j w_j$ and $|z| := \sqrt{z \cdot \overline{z}}$, for all elements $z := (z_1, \ldots, z_k)$ and $w := (w_1, \ldots, w_k)$ of $\mathbb{C}^k$.

The euclidean ball of radius $\frac{\sqrt{2}}{2}$ in $\mathbb{C}^m$ and the minimal ball in $\mathbb{C}^{n_1}$ correspond respectively to the cases $n_1 = \cdots = n_m = 1$ and $m = 1$. The domains $\Omega_N$ were introduced by the second author in [?] where he computed their Bergman and Szegö kernels. We should point out that these domains are convex but they are neither strictly pseudoconvex nor piecewisely smooth except for the case of the euclidean balls.

Optimal estimates for the $\overline{\partial}$-equation were considered for the category of smooth domains by several authors. In [?], Krantz obtained the optimal Lipschitz and $L^p$ estimates for smooth strongly pseudoconvex domains. Later in [?], Chen, Krantz and Ma established that this kind of regularity holds for smooth complex ellipsoids. The general case of smooth convex domains of finite type was considered only recently in the works of Cumenge ([?],[?]), Diederich-Fischer-Fornæss [?], Fischer [?] and Hefer [?]. The aim of the present paper is to study the optimal Lipschitz regularity for the $\overline{\partial}$-equation in the class of convex domains $\Omega_N$. 

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To state the main results, we fix some notations and suppose without loss of generality that $n_1 \leq \cdots \leq n_m$. Since the case of the euclidean balls is well-known, we shall assume that $N \neq (1, \ldots, 1)$ and let $l$ denote the smallest nonnegative integer such that $n_{l+1} > 1$. We set $|N| := \sum_{j=1}^{m} n_j$.

The Lipschitz spaces we use herein are the classical ones and those given for $0 < \alpha \leq 1$, by

$$
\Lambda^{\alpha}_\infty(\Omega_N) := \left\{ f : \|f\|_{L^\infty(\Omega_N)} + \sup_{z, z + h \in \Omega_N, 0 < |h| < \frac{1}{2}} \frac{|f(z + h) - f(z)|}{|h|^\alpha \log |h|} \equiv \|f\|_{\Lambda^{\alpha}_\infty(\Omega_N)} < \infty \right\}.
$$

The first main result is the following. It generalizes our previous result [?]:

**THEOREM 1.1.**— Suppose that $N := (n_1, \ldots, n_m)$ is as above and the domain $\Omega_N$ is given by (1.1). Let

$$
\alpha = \alpha(N, p) = \begin{cases} \frac{1}{2} - \frac{|N| + m - l + 1}{p}, & \text{if } N \neq (2, \ldots, 2) \text{ and } p > 2(|N| + m - l + 1); \\ \frac{3m}{p}, & \text{if } N = (2, \ldots, 2) \text{ and } p > 6m. \end{cases}
$$

Then for every $\overline{\partial}$-closed $(0, 1)$-form $f$ with coefficients in $L^p(\Omega_N)$, there exists a function $u$ defined on $\Omega_N$ that satisfies $\overline{\partial} u = f$ (in the distribution sense) and the estimate

$$
\left\{ \begin{array}{ll}
\|u\|_{\Lambda^\alpha(\Omega_N)} \leq C_p \|f\|_{L^p(\Omega_N)}, & \text{if } p < \infty; \\
\|u\|_{\Lambda^{\frac{3m}{p}}(\Omega_N)} \leq C_\infty \|f\|_{L^\infty(\Omega_N)}, & \text{if } p = \infty.
\end{array} \right.
$$

The following result asserts that the regularity in Theorem 1.1 is sharp.

**THEOREM 1.2.**— Let $N, \Omega_N, p$, and $\alpha := \alpha(N, p)$ be as in the statement of Theorem 1.1. Then there exists a $\overline{\partial}$-closed $(0, 1)$-form $f$ with coefficients in $C^\infty(\Omega_N)$ that satisfies

$$
\left\{ \begin{array}{ll}
f \in L^s(\Omega_N), & \forall s < p, \quad \text{if } p < \infty; \\
f \in L^\infty(\Omega_N), & \text{if } p = \infty;
\end{array} \right.
$$

and if $u$ is a function satisfying $\overline{\partial} u = f$, then $u \notin \Lambda^{\alpha + \epsilon}(\Omega_N)$, $\forall \epsilon > 0$.

These results have been announced in [?].
Theorem 1.2 implies that if $N \neq (2, \ldots, 2)$ and $p \leq 2(|N| + m - l + 1)$ or if $N = (2, \ldots, 2)$ and $p \leq 6m$, then we cannot solve the $\partial$-equation on $\Omega_N$ under $L^p$ data with the Lipschitz regularity given above.

We observe that for $N = (2)$, the domain $\Omega(2)$ is linearly biholomorphic to the Reinhardt triangle $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$. The reduction of our Theorems 1.1 and 1.2 to this case, compared with the results obtained for domains of finite type ([?], [?], [?], [?], [?], [?]), shows that domain $\Omega(2)$ has the same gain of smoothness for the $\partial$-equation as strictly pseudoconvex smooth domains in $\mathbb{C}^2$. Our results show also that there exist smooth domains of finite type for which the gain of smoothness for the $\partial$-equation is worse than that of the singular domains $\Omega_N$.

The paper is organised as follows.

In Section 2 we introduce the main tools and prove preliminary results. The objects used are a complex manifold $\mathbb{H}_N$, its intersection $\mathbb{M}_N$ with the euclidean unit ball and a proper holomorphic mapping $F_N$ relating the $\partial$-equation on $\mathbb{M}_N$ to that on $\Omega_N$. We establish in this section Proposition 2.2 which gives an integral representation formula of Berndtsson type for the complex manifold $\mathbb{M}_N$. From this result we derive in Section 3 a formula of Martinelli-Bochner type (Theorem 3.1) and two formulas of Cauchy type (Theorems 3.2 and 3.3) for the complex manifold $\mathbb{M}_N$. These integral representations play a peculiar role in the construction of the $\partial$-solving operators on $\mathbb{M}_N$ and $\Omega_N$.

In Section 4 we give appropriate local coordinates on the complex manifold $\mathbb{H}_N$ which permit us to prove Theorem 5.1 in Section 5. The latter result will be called Theorem of reduction of estimates since from broad outlines, it reduces certain integral estimates on $\mathbb{M}_N$ to analogous integrals, but simpler, which are taken on some balls of $\mathbb{C}^{|N|}$. This result, combined with Section 6, allows us to establish integral estimates in Section 7.

An operator solution $T_1$ of the $\partial$-equation on $\mathbb{M}_N$ is constructed in Section 8 and related Lipschitz estimates are established there. The formula for $T_1$ is explicit and contains an integral term taken over the boundary $\partial\mathbb{M}_N$ of $\mathbb{M}_N$. In order to handle this term, we prove a sort of Stokes theorem in Section 9 which allows us to transform these integral estimates into analogous ones taken over $\mathbb{M}_N$ and then apply the Theorem of reduction of estimates.

Theorem 1.1 is proved in Section 10. By means of the operator $T_1$ and the proper holomorphic mapping $F_N$, we define an operator $T$, solution of the $\partial$-equation on the domain $\Omega_N$ and transfer the Lipschitz regularity for $T_1$
to that of the operator $T$. Finally, we prove Theorem 1.2 by giving concrete examples to show the sharpness of the results of Theorem 1.1. Then we conclude the paper by some remarks and open questions.

Throughout the paper, the letter $C$ denotes a finite constant that is not necessarily the same at each occurrence and that depends on $N$ and eventually other parameters.

2. The complex manifolds $\mathbb{H}_N$ and $\mathbb{M}_N$

In this section we fix the notations and prove some preliminary results. For the simplicity of calculations we only consider, without loss of generality, the case of the domain $\Omega_N$ with $N = (1, \ldots, 1, n, m)$, where $l, m, n$ are positive integers and $n, m > 1$. In this case we have $|N| = l + n + m$ and $\Omega_N$ can be written in the form

$$\Omega_N := \{ Z = (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^n \times \mathbb{C}^m : 2|x|^2 + |z|^2 + |z \cdot z| + |w|^2 + |w \cdot w| < 1 \}.$$

Consider the complex manifold $\mathbb{H}_N$ given by

$$\mathbb{H}_N := \{ Z = (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{m+1} \setminus \{0\} : z \cdot z = w \cdot w = 0 \}.$$

Let $\mathbb{B}_N$ be the euclidean open unit ball in $\mathbb{C}^{|N|+2}$ and $\partial \mathbb{B}_N$ its boundary. We set $\mathbb{M}_N := \mathbb{H}_N \cap \mathbb{B}_N$ and $\partial \mathbb{M}_N := \mathbb{H}_N \cap \partial \mathbb{B}_N$. We first point out that $\mathbb{H}_N$ and $\partial \mathbb{B}_N$ are transverse while the variety $\{ Z = (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} : z \cdot z = w \cdot w = 0 \}$ does not meet $\partial \mathbb{B}_N$ transversally. Denote by $dV, dV_1, dV_n$ and $dV_m$ the respective canonical measures on the complex manifolds $\mathbb{H}_N, \mathbb{C}^l, \mathbb{H}_n$ and $\mathbb{H}_m$. These measures are related by the following

**Proposition 2.1.** — For all compactly supported continuous functions $f$ on $\mathbb{H}_N$, we have

$$\int_{\mathbb{H}_N} f(Z) dV(Z) = C \int_{\mathbb{C}^l} \int_{\mathbb{H}_n} \int_{\mathbb{H}_m} f(x, z, w) dV_1(x) dV_n(z) dV_m(w).$$

**Proof.** — Observe that

$$dV(Z) := C \left( \sum_{p=1}^{l} dx_p \wedge d\bar{x}_p + \sum_{j=1}^{n+1} dz_j \wedge d\bar{z}_j + \sum_{k=1}^{m+1} dw_k \wedge d\bar{w}_k \right)^{l+n+m}.$$


In this formula the constant $C$ is equal to $\frac{1}{(l+n+m)!} \left( \frac{1}{2} \right)^{l+n+m}$. Therefore, a direct computing shows that

\[
d(V(Z)) = C \left( \sum_{p=1}^{l} dx_p \wedge d\bar{x}_p \right)^{l} \left( \sum_{j=1}^{n+1} dz_j \wedge d\bar{z}_j \right)^{n} \left( \sum_{k=1}^{m+1} dw_k \wedge d\bar{w}_k \right)^{m} \bigg|_{\mathbb{H}_N}.
\]

This completes the proof.

Let $\mathbb{E} := \{ t = (t_1, t_2, t_3) \in [0, 1]^3 : t_1^2 + t_2^2 + t_3^2 < 1 \}$ and $\partial \mathbb{E} := \{ t \in [0, 1]^3 : t_1^2 + t_2^2 + t_3^2 = 1 \}$ its boundary. Then the mapping $F : \mathbb{E} \times \partial \mathbb{B}_l \times \partial \mathbb{M}_n \times \partial \mathbb{M}_m \rightarrow \mathbb{M}_N$ given by $F(t, x, z, w) := tZ = (t_1x, t_2z, t_3w)$, where $t = (t_1, t_2, t_3)$ and $Z = (x, z, w)$, is a diffeomorphism. Moreover, it maps $\partial \mathbb{E} \times \partial \mathbb{B}_l \times \partial \mathbb{M}_n \times \partial \mathbb{M}_m$ onto $\partial \mathbb{M}_N$.

Let $d\sigma_n$ be the unique probability measure, $SO(n + 1, \mathbb{R})$-invariant on $\partial \mathbb{M}_n$. Similarly, let $d\sigma_m$ be the unique probability measure, $SO(m + 1, \mathbb{R})$-invariant on $\partial \mathbb{M}_m$. Finally, let $d\sigma_l$ be the surface measure on $\partial \mathbb{B}_l$. Combining Proposition 2.1 of [?] and Lemma 2.1 of [?], we obtain

**COROLLARY 2.2.** — For all compactly supported continuous functions $f$ on $\mathbb{H}_n$, we have

\[
\int_{\mathbb{H}_n} f(z) dV_n(z) = C \int_{\partial \mathbb{M}_n} t^{2n-1} \int_{\partial \mathbb{M}_n} f(t\zeta) d\sigma_n(\zeta) dt.
\]

There are obviously analogous integral formulas in polar coordinates with $\mathbb{M}_m$ and $\mathbb{B}_l$ in place of $\mathbb{H}_n$. We now define a natural measure $d\sigma$ on $\partial \mathbb{M}_N$ by setting $d\sigma := (F_*) (d\phi \wedge d\sigma_l \wedge d\sigma_n \wedge d\sigma_m)$, where $d\phi$ is the surface measure of the unit sphere $\partial \mathbb{E}$. Using this, Corollary 2.2 and integration in polar coordinates, one can establish the following

**LEMMA 2.3.** — For all compactly supported continuous functions $f$ on $\mathbb{H}_N$, we have

\[
\int_{\mathbb{H}_N} f(Z) dV(Z) = C(N) \int_{\partial \mathbb{M}_N} t^{2|N|-1} \int_{\partial \mathbb{M}_N} f(t\Theta) d\sigma(\Theta) dt.
\]

In what follows we shall establish some integral formulas on $\mathbb{M}_N$. To do so, we shall approximate $\mathbb{M}_N$ by appropriate regular varieties which are complete intersections. Then we apply to each of these varieties the results of Berndtsson in [?].
For $0 < r < 1$, let $\mathcal{D}_r$ be the domain of $\mathbb{C}^{\mid N\mid + 2}$ defined by

$$\mathcal{D}_r := \{ Z = (x, z, w) \in \mathbb{B}_N : |z| > r, |w| > r \}.$$ 

Note that the boundary of $\mathcal{D}_r$ is piecewisely smooth. We put $M_r := \mathbb{H}_N \cap \mathcal{D}_r$.

Let $s := (s_1, \ldots, s_{\mid N\mid + 2}) : \overline{\mathcal{D}_r} \times \mathcal{D}_r \longrightarrow \mathbb{C}^{\mid N\mid + 2}$ be a $C^1$ function that satisfies

$$|s(\Theta, Z)| \leq C|\Theta - Z| \quad \text{and} \quad |s(\Theta, Z) \cdot (\Theta - Z)| \geq C|\Theta - Z|^2 \quad (2.1)$$

uniformly for $\Theta \in \overline{\mathcal{D}_r}$ and for $Z$ in any compact subset of $\mathcal{D}_r$. We shall use the same symbol $s$ and set

$$s := \sum_{j=1}^{\mid N\mid + 2} s_j d\Theta_j.$$

In the sequel, we shall use simultaneously the following notations for $\Theta \in \mathbb{C}^{\mid N\mid + 2}$:

$$\Theta \equiv (\Theta_1, \ldots, \Theta_{\mid N\mid + 2}) \equiv (\xi, \zeta, \eta) \in \mathbb{C}^l \times \mathbb{C}^{n+1} \times \mathbb{C}^{m+1}.$$

We next set

$$\Phi := \left( \sum_{j=1}^{n+1} (\zeta_j + z_j) d\zeta_j \right) \wedge \left( \sum_{k=1}^{m+1} (\eta_k + w_k) d\eta_k \right).$$

For every $\epsilon > 0$, consider the differential form of bidegree $(\mid N\mid + 2, \mid N\mid + 1)$

$$K_{\epsilon}^s := \frac{s \wedge (\overline{\partial} s)^{\mid N\mid - 1} \wedge (\overline{\partial} Q_\epsilon)^2}{[s(\Theta, Z) \cdot (\Theta - Z)]^{\mid N\mid}}, \quad (2.2)$$

where $Q_\epsilon$ is the differential form of bidegree $(1, 0)$ given by

$$Q_\epsilon := \frac{\overline{\zeta} \cdot \zeta \left( \sum_{j=1}^{n+1} (\zeta_j + z_j) d\zeta_j \right) + \overline{\eta} \cdot \eta \left( \sum_{k=1}^{m+1} (\eta_k + w_k) d\eta_k \right)}{|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon}. \quad (2.3)$$

Denote by $d\Theta$ the canonical holomorphic form of $\mathbb{C}^{\mid N\mid + 2}$ given by

$$d\Theta_1 \wedge \ldots \wedge d\Theta_{\mid N\mid + 2} \equiv d\xi_1 \wedge \ldots \wedge d\xi_l \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_{n+1} \wedge d\eta_1 \wedge \ldots \wedge d\eta_{m+1}.$$
LEMMA 2.4. — Suppose that $0 < r < 1$.

1) If $u \in C^1(\overline{D}_r)$ and $Z \in \overline{M}_r$, then

$$u(Z) = C(N) \lim_{\epsilon \to 0} \left( \int_{\partial D_r} uK_s - \int_{D_r} \overline{\partial}u \wedge K_s^\epsilon \right).$$

2) If $u \in C(D_r)$, then

$$\lim_{\epsilon \to 0} \int_{D_r} \frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^{\frac{3}{2}}} d\Theta \wedge d\Theta = C(N) \int_{M_r} u(\Theta) dV(\Theta).$$

3) If $u \in C(\partial \mathbb{B}_N)$ and $\omega$ is the canonical volume form of $\partial \mathbb{B}_N$, then

$$\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N} \frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^{\frac{3}{2}}} \omega(\Theta) = C(N) \int_{\partial M_N} u(\Theta) d\sigma(\Theta).$$

Proof. — Part 1) follows from formulas (23) and (26) in the proof of Theorem 1 in [?]. Also, part 2) is an immediate consequence of identity (25) in [?].

To prove part 3), we may assume without loss of generality that the support of $u$ is contained in a sufficiently small open neighborhood $U \subset \mathbb{C}^{1|N|+2}$ of a point $\Theta_0 \in \partial M_N$. Using local coordinates and Lelong theory [?], we see that there exists a smooth $(2|N| - 1)$-volume form $d\mu$ defined on $U \cap \partial M_N$ such that

$$\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N \cap U} \frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^{\frac{3}{2}}} \omega(\Theta) = C \int_{\partial M_N \cap U} u(\Theta) d\mu(\Theta), \quad (2.4)$$

for all $u \in C_0(U)$. Therefore, part 3) is equivalent to the identity $d\mu = C d\sigma$.

Let $\psi$ be a function of class $C_0^\infty([0,1])$ supported in $[\frac{1}{2},1]$ such that

$$\int_0^1 \rho^{2|N|-1} \psi(\rho) d\rho = 1.$$ Consider the $C_0^\infty$ extension of $u$ given by

$$u(\rho Z) := \psi(\rho) u(Z), \quad \text{for } 0 \leq \rho \leq 1 \text{ and } Z \in \partial \mathbb{B}_N \cap U.$$ On the one hand, using (2.4), we have that

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}_N} \frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^{\frac{3}{2}}} d\Theta \wedge d\Theta$$
On the other hand, by part 2) and Lemma 2.3, we see that
\[ dp = C(N)du \]
and thereby completes the proof.

Next, set
\[ K_s := \frac{s \wedge (\bar{\partial} s)^{|N|-1} \wedge \Phi \wedge \bar{\partial}(\zeta \cdot \zeta) \wedge \bar{\partial}(\eta \cdot \eta)}{|\zeta|^2 |\eta|^2 |s(\Theta, Z) \cdot (\Theta - Z)|^{|N|}}. \]  

(2.5)

In view of (2.2), (2.3) and the equality which precedes Lemma 4 in [?], we see that \( K_s \) satisfies the identity
\[ K_s^{\epsilon} = \frac{\epsilon|\zeta|^2 |\eta|^2}{(|\zeta|^2 |\eta|^2 + |\eta \cdot \eta|^2 + \epsilon)^3} K_s. \]  

(2.6)

For every \( 1 \leq k \leq |N| + 2 \), denote by \( \omega_k(\bar{\Theta}) \) the \((0, |N| + 1)\)-form
\[ (-1)^{k-1} d\bar{\Theta}_1 \wedge \ldots \wedge d\bar{\Theta}_k \wedge \ldots \wedge d\bar{\Theta}_{|N|+2}. \]

We can write \( K_s \) in the form
\[ K_s = \sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\bar{\Theta}) \wedge d\Theta, \]  

(2.7)

where \( h_k \) are the component functions of \( K_s \) with respect to the forms \( \omega_1(\bar{\Theta}) \wedge d\Theta, \ldots, \omega_{|N|+2}(\bar{\Theta}) \wedge d\Theta \).

Let \( \bar{M}_N \) be the closure of \( M_N \) in \( \bar{B}_N \) and denote by \( C^k(\bar{M}_N) \), \( k \in \mathbb{N} \), the space of all \( C^k \) functions defined in a neighborhood of \( \bar{M}_N \) in \( \bar{B}_N \). If
Let $\mathcal{O}_{M_N}$ be the $\mathcal{O}$-operator on $M_N$. We end this section by the following

**Proposition 2.5.**  Consider a section $s$ satisfying (2.1), a function $u \in \mathcal{C}^1(M_N)$ and a $(0,1)$-form $f := \sum_{j=1}^{L} f_j d\Theta_j$ with coefficients in $\mathcal{C}(M_N)$ that satisfy $\mathcal{D}_{M_N} u = f|_{M_N}$ on $M_N$. Let $h_k$ be the functions defined in (2.7). Then for $Z \in M_N$,

$$u(Z) = C \int_{\partial M_N} \left( \sum_{k=1}^{L} \Theta_k h_k(\Theta, Z) \right) d\sigma(\Theta) + C \int_{M_N} \left( \sum_{k=1}^{L} f_k(\Theta) h_k(\Theta, Z) \right) dV(\Theta).$$

**Proof.**  For every $r \in [0,1[$ such that $Z \in M_r$, consider a $\mathcal{C}^1$ extension of $u|_{M_N}$ (which is also denoted by $u$) on $\overline{D}_r$ that satisfies $\mathcal{D} u = f$ on $M_r$. Suppose without loss of generality that $f = \mathcal{D} u$ on $\overline{D}_r$. Parts 1) and 2) of Lemma 2.4, combined with (2.6) and (2.7), imply that $u(z) = CI_r^1 + CI_r^2$, where

$$I_r^1 := \int_{\overline{D}_r} \left( \sum_{k=1}^{L} f_k(\Theta) h_k(\Theta, Z) \right) dV(\Theta),$$

$$I_r^2 := \lim_{\epsilon \to 0} \int_{\partial \overline{D}_r} \frac{|\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^3} \left( \sum_{k=1}^{L} h_k(\Theta, Z) \omega_k(\Theta) \wedge d\Theta \right).$$

The proof is a consequence of the following two equalities

$$\lim_{r \to 0} I_r^1 = \int_{M_N} \left( \sum_{k=1}^{L} f_k(\Theta) h_k(\Theta, Z) \right) dV(\Theta),$$

$$\lim_{r \to 0} I_r^2 = \int_{\partial M_N} u(\Theta) \left( \sum_{k=1}^{L} \Theta_k h_k(\Theta, Z) \right) d\sigma(\Theta).$$

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In order to prove these, fix a point \( Z \in \mathbb{M}_N \). By (2.1), (2.5) and (2.7), there is a constant \( C \) such that
\[
h_k(\Theta, Z) \leq \frac{C}{|\zeta|^2|\eta|^2}, \quad \text{for all } \Theta \in \mathbb{M}_N \setminus \mathbb{M}_r, \text{ with } 0 < r << 1. \tag{2.11}
\]
We deduce easily from (2.11) and the hypothesis \( \|f\|_{\mathbb{M}_N, \infty} < \infty \) that
\[
\lim_{r \to 0} \int_{\mathbb{M}_N \setminus \mathbb{M}_r} \left| \sum_{k=1}^{|N|+2} f_k(\Theta) h_k(\Theta, Z) \right| dV(\Theta) \leq \lim_{r \to 0} \int_{\mathbb{M}_N \setminus \mathbb{M}_r} \frac{C dV(\Theta)}{|\zeta|^2|\eta|^2} = 0,
\]
where the equality follows from Corollary 2.2. This proves (2.9).

Next, we prove (2.10). Appealing to Corollary 2.2, Lemma 2.3, (2.11) and the fact that the function \( u \) is bounded, we see that
\[
\lim_{r \to 0} \int_{\partial \mathbb{M}_N \setminus \partial \mathbb{M}_r} \left| \sum_{k=1}^{|N|+2} \Theta_k h_k(\Theta, Z) \right| |u(\Theta)| d\sigma(\Theta) \leq \lim_{r \to 0} \int_{\partial \mathbb{M}_N \setminus \partial \mathbb{M}_r} \frac{C d\sigma(\Theta)}{|\zeta|^2|\eta|^2} = 0.
\]
This, combined with part 3) of Lemma 2.4, implies that
\[
\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N \setminus \partial \mathbb{D}_r} \left( \frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^3} \left( \sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\Theta) \right) \right) \, d\sigma(\Theta) \to 0, \quad \text{as } r \to 0. \tag{2.12}
\]
from which it follows that (2.10) is a consequent of
\[
\lim_{r \to 0} \lim_{\epsilon \to 0} \int_{\partial \mathbb{D}_r \setminus \partial \mathbb{B}_N} \left( \frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta \cdot \zeta|^2 + |\eta \cdot \eta|^2 + \epsilon)^3} \left( \sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\Theta) \right) \right) \, d\sigma(\Theta) = 0. \tag{2.13}
\]

Next, we prove equality (2.13). We first make use of the following remark related to homogeneity properties of certain differential forms. Indeed, let \( \alpha, \beta > 0 \) and write the complex manifold \( \mathbb{M}_N \) as a complete intersection of \( \mathbb{B}_N \) and the two varieties given by the equations \( \alpha^2 \zeta \cdot \zeta = 0 \) and \( \beta^2 \eta \cdot \eta = 0 \). Applying Berndtsson’s formulas to these two equations and observing that (2.13) corresponds to the particular case \( \alpha = \beta = 1 \), then we obtain
We write \( \partial \mathcal{D}_r \setminus \partial \mathbb{B}_N \) as a union of the two smooth manifolds

\[
M^r_1 := \{ Z \in \mathbb{B}_N : |z| = r, |w| \geq r \};
\]
\[
M^r_2 := \{ Z \in \mathbb{B}_N : |z| \geq r, |w| = r \}.
\]

Let \( d\sigma_{r,j} \) be the canonical volume form on the manifold \( M^r_j, j = 1, 2. \) Applying equality (3) in Proposition 16.4.4 of Rudin \[?\] yields that on \( M^r_j, \)

\[
\omega_k(\Theta) \wedge d\Theta = C(N, j, k) d\sigma_{r,j}.
\] (2.15)

Choosing a function \( u \) and a section \( s \) appropriately and applying Lelong theory as in the proof of (2.4), it follows from (2.14) and (2.15) that on \( M^r_j, \) 0 < \( r < 1, \) \( j \in \{1, 2\}, \) we have

\[
\lim_{\epsilon \to 0} \frac{\epsilon \alpha^4 \beta^4 |\zeta|^2 |\eta|^2}{(\alpha^4 |\zeta| \cdot |\zeta|^2 + \beta^4 |\eta| \cdot |\eta|^2 + \epsilon)^3} d\sigma_{r,j}(\Theta)
\]

\[
= \lim_{\epsilon \to 0} \frac{\epsilon |\zeta|^2 |\eta|^2}{(|\zeta| \cdot |\zeta|^2 + |\eta| \cdot |\eta|^2 + \epsilon)^3} d\sigma_{r,j}(\Theta) = d\mu_{r,j}(\Theta),
\] (2.16)

in the distribution sense, where \( d\mu_{r,j} \) is a \( C^\infty \) differential form of maximal degree on the manifold \( M^r_j \cap \mathbb{H}_N. \) In view of (2.11) and (2.16), equality (2.13) will follow from the following equalities

\[
\lim_{r \to 0} \int_{M^r_j \cap \mathbb{H}_N} \frac{d\mu_{r,j}(\Theta)}{|\zeta|^2 |\eta|^2} = 0, \quad j = 1, 2.
\] (2.17)

We prove (2.17) for \( j = 1 \) which suffices to complete the proof. To do so, consider, for every \( \alpha, \beta > 0, \) the mapping \( F_{\alpha, \beta} \) given by:

\[
F_{\alpha, \beta}(x, z, w) := (x, \alpha z, \beta w), \quad \text{for } Z := (x, z, w) \in \mathbb{H}_N.
\]
We remark immediately that we have the following property of homogeneity:

\[ F_{r, s}(d\sigma_{r, s})(\Theta) = C(N) r^{2n-1} s^{2m} d\sigma_n(\zeta) \wedge dV_l(\xi) \wedge dV_m(\eta), \]

for \(0 < r, s \leq \frac{1}{2}\) and \(\Theta \equiv (\xi, \zeta, \eta) \in C^l \times \partial M_n \times M_m\). This, combined with equality (2.16), implies that

\[ F_{\alpha, \beta}(d\mu_{r, 1}) = C(N, r) \alpha^{2n-1} \beta^{2m} d\mu_{\xi, 1} \quad \text{on } M_1^\infty. \quad (2.18) \]

Take \(r_0 := \frac{1}{2}\). Since the differential form \(d\mu_{r_0, 1}\) is in \(C^\infty(M_1^{r_0})\), we see that

\[ \int_{|\zeta|=r_0, \frac{1}{2} < |\eta| < r_0} \frac{d\mu_{r_0, 1}(\Theta)}{|\zeta|^2 |\eta|^2} < \infty. \quad (2.19) \]

Using (2.18) and (2.19), it is easy to show that

\[ \int_{M_1^r \cap \Omega_N} \frac{d\mu_{r, 1}(\Theta)}{|\zeta|^2 |\eta|^2} \leq C \left( \frac{r}{r_0} \right)^{2n-3} \int_{|\zeta|=r_0, \frac{1}{2} < |\eta| < r_0} \frac{d\mu_{r_0, 1}(\Theta)}{|\zeta|^2 |\eta|^2} \to 0, \quad \text{as } r \to 0. \]

This implies (2.17) and thus completes the proof. \(\square\)

3. Integral formulas on the manifold \(\mathbb{M}_N\)

In this section we establish integral formulas of Martinelli-Bochner type (Theorem 3.1) and those of Cauchy type (Theorems 3.3 and 3.6). These formulas will allow us to construct the \(\bar{\partial}\)-solving operators.

**THEOREM 3.1.** — Suppose that \(u \in C^1(\overline{\mathbb{M}_N})\) and \(f := \sum_{k=1}^{[N]+2} f_k d\Theta_k\) is a (0, 1)-form with coefficients in \(C(\overline{\mathbb{M}_N})\) such that \(\bar{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}\). Then for every \(Z \in \mathbb{M}_N\),

\[
\begin{align*}
  u(Z) &= \left\{ \int_{\partial \mathbb{M}_N} \frac{A(\Theta, Z)}{|Z - \Theta|^{2[N]} |\zeta|^2 |\eta|^2} u(\Theta) \, d\sigma(\Theta) \right. \\
  & \quad + \int_{\mathbb{M}_N} \frac{1}{|Z - \Theta|^{2[N]}} \left( \sum_{k=1}^{[N]+2} B_k(\Theta, Z) f_k(\Theta) \right) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2} \left. \right\},
\end{align*}
\]
where

\[
A(\Theta, Z) := C(|\xi|^2 - \overline{\Theta} \cdot \xi)(|\zeta|^2 + z \cdot \overline{\zeta})(|\eta|^2 + \overline{\eta}w)
\]

\[
+ \quad C \left(-|z \cdot \zeta|^2 + |z \cdot \overline{\zeta}|^2 - |\zeta|^2(|\zeta|^2 + z \cdot \overline{\zeta} - \overline{z} \cdot \zeta)\right)(|\eta|^2 + w \cdot \overline{\eta})
\]

\[
+ \quad C \left(-|w \cdot \eta|^2 + |w \cdot \overline{\eta}|^2 - |\eta|^2(|\eta|^2 + w \cdot \overline{\eta} - \overline{w} \cdot \overline{\eta})\right)(|\zeta|^2 + z \cdot \overline{\zeta}),
\]

and \(B_k\) are polynomials given by the following formulas:

(i) if \(1 \leq k \leq l\), then

\[
B_k(\Theta, Z) := C(\overline{\xi}_k - \overline{x}_k)(|\zeta|^2 + z \cdot \overline{\zeta})(|\eta|^2 + \overline{\eta}w);
\]

(ii) if \(l < k \leq l + n + 1\) and \(j = k - l\), then

\[
B_k(\Theta, Z) := C \left(\overline{\xi}_j - \overline{\zeta}_j\right)(z \cdot \overline{\zeta} + |\zeta|^2 - (z_j + \zeta_j)z \cdot (\overline{\zeta} - \overline{z})\right)(|\eta|^2 + w \cdot \overline{\eta});
\]

(iii) if \(l + n + 1 < k < l + n + m + 2\) and \(i = k - l - n - 1\), then

\[
B_k(\Theta, Z) := C \left((\overline{w}_i - \overline{\eta}_i)(w \cdot \overline{\eta} + |\eta|^2) - (w_i + \eta_i)w \cdot (\overline{\eta} - \overline{w})\right)(|\zeta|^2 + z \cdot \overline{\zeta}).
\]

**Proof.** — Consider the Martinelli-Bochner section \(s_b(Z, \Theta) := \overline{\Theta} - \overline{Z}\).

In order to prove the theorem, we apply Proposition 2.5 to the section \(s_b\).

Using formulas (2.5), (2.7) and arguing as in the proof of Theorem 2.4 of [?], we compute explicitly the functions \(h_k\) associated to \(s_b\) and obtain the desired formula. \(\square\)

**Remark 3.2.** — If \(u \in C^1(\mathbb{M}_N)\) is bounded, then Proposition 2.5 and Theorem 3.1 hold for the dilated functions \(u_r(Z) := u(rZ), 0 < r < 1\). This shows that Theorem 3.1 remains true if we only assume that \(u \in C^1(\mathbb{M}_N)\) is bounded and

\[
\lim_{r \to 1^-} \int_{\partial\mathbb{M}_N} |u(\Theta) - u(r\Theta)|d\sigma(\Theta) = 0.
\]

Following Charpentier [?] let

\[
s_0(\Theta, Z) := \overline{\Theta}(1 - \Theta \cdot \overline{Z}) - \overline{Z}(1 - |\Theta|^2), \quad \text{and} \quad D(\Theta, Z) := s_0(\Theta, Z) \cdot (\Theta - Z).
\]

In what follows, \(\text{grad}_Z f\) denotes the gradient of a differentiable function \(f\) at a point \(Z\).
THEOREM 3.3. — There exist polynomials $R(\Theta, Z)$ and $P_k(\Theta, Z)$, $Q_k(\Theta, Z)$ for $1 \leq k \leq |N| + 2$, that satisfy the following properties:

(i) $R(\Theta, Z) = (C|\zeta|^2 + C|\zeta|^2 + C|\eta|^2)(|\zeta|^2 + z \cdot \overline{\zeta})(|\eta|^2 + w \cdot \overline{\eta})$.

(ii) For every $Z, \Theta \in \mathbb{B}_N$, and for every $1 \leq k \leq |N| + 2$,

$$P_k(\Theta, Z) = O((|\Theta - Z|(|\zeta|^2 + |z||\zeta|)(|\eta|^2 + |w||\eta|)),$$

$$Q_k(\Theta, Z) = O((|\Theta - Z|(|\zeta|^2 + |z||\zeta|)(|\eta|^2 + |w||\eta|)),$$

$$|\text{grad}_Z P_k(\Theta, Z)| = O((|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |w||\eta|)),$$

$$|\text{grad}_Z Q_k(\Theta, Z)| = O((|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |w||\eta|)).$$

(iii) Given a function $u \in C^1(\overline{M}_N)$ and a $(0, 1)$-form $\bar{\omega} := \sum_{k=1}^{N+2} f_k d\Theta_k \in C(\overline{M}_N)$ that satisfy $\bar{\partial} u = f|\overline{M}_N$, then for every $Z \in M_N$,

$$u(Z) = \int_{\partial M_N} \frac{R(\Theta, Z)}{(1 - Z \cdot \overline{\Theta})^{|N|}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2}$$

$$+ \int_{M_N} \frac{(1 - \Theta \cdot \overline{\Theta})^{|N|}}{D(\Theta, Z)^{|N|}} \sum_{k=1}^{N+2} [(1 - \Theta \cdot \overline{Z})P_k(\Theta, Z)]dV(\Theta)$$

$$+ (1 - |\Theta|^2)Q_k(\Theta, Z)\frac{dV(\Theta)}{|\zeta|^2|\eta|^2}.$$

Proof. — From the proof of Proposition 2.5 we may assume without loss of generality that there is a $C^1$ extension of $u|\overline{M}_N$, denoted again by $u$, such that $\partial u = f$ on $\mathbb{B}_N$. Let $K_0$ be the kernel associated to the section $s_0$ by formula (2.5). By virtue of (2.6), when we integrate $uk^\epsilon$ over $\partial \mathbb{B}_N$, all terms which contain $\bar{\partial} \Theta^2$ vanish. In addition we have $1 - |\Theta|^2 = 0$ and $D(\Theta, Z) = |1 - Z \cdot \Theta|^2$ so that

$$\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N} uK^\epsilon = \lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N} u(\Theta) \frac{\epsilon|\zeta|^2|\eta|^2}{(|\zeta|^2 + |\eta|^2 + \epsilon)^3} \cdot \frac{1}{|\zeta|^2|\eta|^2(1 - Z \cdot \overline{\Theta})^{|N|}}$$

$$\left\{ \left( \sum_{k=1}^{N+2} \bar{\Theta}_k d\Theta_k \right) \left( \sum_{k=1}^{N+2} d\overline{\Theta}_k \wedge d\Theta_k \right)^{|N| - 1} \wedge \Psi \wedge \bar{\partial}(\bar{\zeta} \cdot \zeta) \wedge \bar{\partial}(\bar{\eta} \cdot \eta) \right\}.$$
Rewriting the differential form in braces in the form $\sum_{k=1}^{N+2} h_k(\Theta, Z) \omega_k(\Theta) \wedge d\Theta$ and applying part 3) of Lemma 2.4, we obtain

$$\lim_{\epsilon \to 0} \int_{\partial B_N} u K_0^\epsilon = \int_{\partial M_N} \frac{\Theta_k h_k(\Theta, Z)}{|\zeta|^2 |\eta|^2 (1 - Z \cdot \Theta)^N} d\sigma(\Theta).$$

A straightforward calculation of the functions $h_k(\Theta, Z)$ shows that

$$R(\Theta, Z) := \sum_{k=1}^{N+2} \Theta_k h_k(\Theta, Z)$$

satisfies assertion (i) of the theorem.

Write the kernel $K_0$ in the form (2.7) as $K_0 = \sum_{k=1}^{N+2} h_k(\Theta, Z) \omega_k(\Theta) \wedge d\Theta$.

Then we have

$$I := \partial u \wedge K_0 = \sum_{k=1}^{N+2} f_k(\Theta) h_k(\Theta, Z) d\Theta \wedge d\bar{\Theta}.$$  \hspace{1cm} (3.2)

To finish the proof of the theorem, it suffices to prove the following lemma:

**Lemma 3.4.** — The functions $h_k$ in the formula (3.2) can be rewritten in the form

$$h_k(\Theta, Z) = \frac{(1 - \Theta \cdot \bar{Z})|N|^{-2}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^N} [(1 - \Theta \cdot \bar{Z}) P_k(\Theta, Z) + (1 - |\Theta|^2) Q_k(\Theta, Z)],$$

where $P_k$ and $Q_k$ are some polynomials that satisfy assertion (ii) of the theorem.

**End of the proof of Theorem 3.3.** — Suppose that the lemma above is proved. Applying Proposition 2.5 and using (3.1)-(3.3), the theorem follows.

**Proof of Lemma 3.4.** — By virtue of (2.5) and (3.2), we can write $I = I_1 + I_2$, where

$$I_1 := \left( \sum_{k=1}^{N+2} f_k d\bar{\Theta}_k \bigg\{ \frac{(1 - \Theta \cdot \bar{Z})|N|^{-1}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^N} \sum_{j=1}^{N+2} \left[ \bar{\Theta}_j (1 - \Theta \cdot \bar{Z}) - \bar{Z}_j (1 - |\Theta|^2) \right] \right) \wedge d\Theta \wedge \Psi \wedge \partial(\zeta \cdot \zeta) \wedge \partial(\eta \cdot \eta) \wedge \left[ \sum_{q=1}^{N+2} d\bar{\Theta}_q \wedge d\Theta_q \right]^{[N]-1},$$

$$I_2 := \left( \sum_{k=1}^{N+2} f_k d\bar{\Theta}_k \right) \wedge \left( \sum_{k=1}^{N+2} h_k(\Theta, Z) \omega_k(\Theta) \wedge d\Theta \right).$$
A straightforward computation shows that

\[ I_1 := \frac{(1 - \Theta \cdot \bar{Z})^{N-1}}{|\zeta|^2|\eta|^2 D(\Theta, Z)|^N} \left\{ C \sum_{k=1}^{l} f_k \left[ -\xi_k (1 - \Theta \cdot \bar{Z}) + \bar{z}_k (1 - |\Theta|^2) \right] \right\} \]

\[ + C \sum_{k=1}^{n+1} \left\{ \left[ -\eta_k (1 - \Theta \cdot \bar{Z}) + \bar{w}_k (1 - |\Theta|^2) \right] (w \cdot \bar{\eta} + |\eta|^2) \right\} d\Theta \wedge d\bar{\Theta}. \]
Then a simple calculation gives that
\[
\Psi \wedge \partial(\zeta \cdot \zeta) \wedge \partial(\eta \cdot \eta) \wedge \left[ \sum_{k=1}^{[N]+2} d\Theta_k \wedge d\Theta_k \right]^{[N]-2} = \Psi \wedge \partial(\zeta \cdot \zeta) \wedge \partial(\eta \cdot \eta)
\]
\[
\wedge \left( C\omega_{\xi}^{l-2} \wedge \omega_{\zeta}^{n} \wedge \omega_{\eta}^{m} + C\omega_{\xi}^{l} \wedge \omega_{\zeta}^{n-2} \wedge \omega_{\eta}^{m} + C\omega_{\xi}^{l} \wedge \omega_{\zeta}^{n} \wedge \omega_{\eta}^{m-2} + C\omega_{\xi}^{l-1} \right.
\]
\[
\wedge \omega_{\zeta}^{n-1} \wedge \omega_{\eta}^{m} + C\omega_{\xi}^{l} \wedge \omega_{\zeta}^{n-1} \wedge \omega_{\eta}^{m-1} + C\omega_{\xi}^{l-1} \wedge \omega_{\zeta}^{n} \wedge \omega_{\eta}^{m-1} \right)
\]
\[
\equiv \Psi \wedge \partial(\zeta \cdot \zeta) \wedge \partial(\eta \cdot \eta) \wedge \left( \sum_{k=1}^{6} J_k \right). \quad (3.5)
\]

To conclude the proof of Lemma 3.4, it suffices to prove the following lemma:

**Lemma 3.5.** — For every \(1 \leq k \leq 6\), the differential form
\[
I_{2k} := \left[ \sum_{j=1}^{[N]+2} f_j d\Theta_j \right] \wedge \left[ \sum_{j=1}^{[N]+2} \left[ \Theta_j (1 - \Theta \cdot \overline{Z}) - \overline{Z}_j (1 - |\Theta|^2) \right] d\Theta_j \wedge \overline{\Theta} |\Theta|^2 \right.
\]
\[
\wedge \left[ \sum_{j=1}^{[N]+2} \overline{Z}_j d\Theta_j \right] \wedge J_k \wedge \Psi \wedge \partial(\zeta \cdot \zeta) \wedge \partial(\eta \cdot \eta)
\]
can be expressed as the product of the canonical volume form \(d\Theta \wedge d\overline{\Theta}\) and a function of the form
\[
\sum_{j=1}^{[N]+2} f_j \left( (1 - \Theta \cdot \overline{Z}) P_j (\Theta, Z) + (1 - |\Theta|^2) Q_j (\Theta, Z) \right),
\]
where \(P_j, Q_j\) are some polynomials satisfying assertion (ii) of Theorem 3.3.

**End of the proof of Lemma 3.4.** — Suppose that Lemma 3.5 is proved. We deduce from the definition of \(I_2, I_{2k}\) and (3.5) that
\[
I_2 = \frac{(1 - \Theta \cdot \overline{Z})^{[N]-2}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^{[N]}} \cdot \left( \sum_{k=1}^{6} I_{2k} \right).
\]
Therefore Lemma 3.4 follows from Lemma 3.5. \(\Box\)

**Proof of Lemma 3.5.** — We break the proof into 6 cases according to the integer \(k, 1 \leq k \leq 6\).
Case 1: $J_1 = \omega_{\zeta}^{l-2} \wedge \omega_{\zeta}^{n} \wedge \omega_{\eta}^{m}$. In this case a direct computation shows that

$$I_{21} = (z \cdot \bar{\zeta} + |\zeta|^2)(w \cdot \bar{\eta} + |\eta|^2) \left[ \sum_{j=1}^{l} f_{j} d\xi_{j} \right]$$

$$\wedge \left[ \sum_{j=1}^{l} \left[ \bar{\xi}_{j} (1 - \Theta \cdot \bar{Z}) - \bar{x}_{j} (1 - |\Theta|^2) \right] d\xi_{j} \right] \wedge |\zeta|^2$$

$$\wedge \left[ \sum_{j=1}^{l} \bar{x}_{j} d\xi_{j} \right] \wedge \omega_{\zeta}^{l-2}$$

$$= (1 - \Theta \cdot \bar{Z})(z \cdot \bar{\zeta} + |\zeta|^2)(w \cdot \bar{\eta} + |\eta|^2) \left\{ \left[ \sum_{j=1}^{l} f_{j} d\xi_{j} \right] \right.$$}

$$\left. \wedge \left[ \sum_{j=1}^{l} \bar{\xi}_{j} d\xi_{j} \right] \wedge \left[ \sum_{j=1}^{l} \bar{x}_{j} d\xi_{j} \right] \wedge |\zeta|^2 \wedge \omega_{\zeta}^{l-2} \right\}.$$

Since

$$\left[ \sum_{j=1}^{l} \bar{\xi}_{j} d\xi_{j} \right] \wedge \left[ \sum_{j=1}^{l} \bar{x}_{j} d\xi_{j} \right] = \sum_{j,k=1,j<k}^{l} (\bar{\xi}_{j} \bar{x}_{k} - \bar{\xi}_{k} \bar{x}_{j}) d\xi_{j} \wedge d\xi_{k},$$

we see easily that $I_{21}$ satisfies the conclusion of the lemma.

Case 2: $J_2 = \omega_{\zeta}^{l} \wedge \omega_{\zeta}^{n-2} \wedge \omega_{\eta}^{m}$. In this case we can rewrite $I_{22}$ in the form

$$(w \cdot \bar{\eta} + |\eta|^2) \left\{ \left[ \sum_{j=1}^{n+1} f_{j+1} d\zeta_{j} \right] \wedge \left[ \sum_{j=1}^{n+1} \left[ \bar{\zeta}_{j} (1 - \Theta \cdot \bar{Z}) - \bar{z}_{j} (1 - |\Theta|^2) \right] d\zeta_{j} \right. \right.$$}

$$\left. \wedge \left[ \sum_{j=1}^{n+1} \bar{z}_{j} d\zeta_{j} \right] \wedge \left[ \sum_{k,p=1}^{n+1} (z_{k} + \zeta_{k} \bar{\zeta}_{p} d\zeta_{k} \wedge d\zeta_{p} \right] \wedge \omega_{\zeta}^{n-2} \wedge \omega_{\eta}^{m+1} \right\}.$$

In view of the proof of Lemma 2.7 in [?], the differential form in braces can be expressed as the product of $d\zeta \wedge d\bar{\zeta}$ and a function of the form

$$\sum_{j=1}^{n+1} f_{j+1} ((1 - \Theta \cdot \bar{Z})S_{j}(\Theta, Z) + (1 - |\Theta|^2)T_{j}(\Theta, Z)),$$

where $S_{j}, T_{j}$ are some polynomials such that

$$S_{j}(\Theta, Z) = O(|z - \zeta||\zeta|^2), \quad T_{j}(\Theta, Z) = O(|z - \zeta||\zeta|^2),$$

$$\text{grad}_{\zeta} S_{j}(\Theta, Z) = O(|\zeta|^2 + |\zeta||z|), \quad \text{grad}_{\zeta} T_{j}(\Theta, Z) = O(|\zeta|^2 + |\zeta||z|).$$
Combining what we have proved so far, we obtain that $I_{22}$ satisfies the conclusion of the lemma.

**Case 3:** $J_3 = \omega_i^l \land \omega_z \land \omega_m^{m-2}$. This case can be treated in the same way as the previous case.

**Case 4:** $J_4 = \omega_i^{l-1} \land \omega_z^1 \land \omega_m^m$. Then we have

\[
I_{24} = \left[ \sum_{j=1}^{l+n+1} f_j d\Theta_j \right] \land \left\{ \sum_{t=1}^{l+n+1} \left[ \Theta_t(1 - \Theta \bullet \overline{Z}) - \overline{Z}_t(1 - |\Theta|^2) \right] d\Theta_t \land \overline{\partial}(|\xi|^2 + |\zeta|^2) \right\}
\land \left[ \sum_{s=1}^{l+n+1} \overline{Z}_s d\Theta_s \right] \land \left\{ \sum_{k,p=1}^{n+1} (z_k + \zeta_k) \overline{\xi}_p d\xi_k \land d\overline{\xi}_p \right\} \land \omega_{\xi}^{n-1} \land \omega_{\xi}^{l-1} \right\} \land (w \bullet \overline{\eta} + |\eta|^2) \omega_{\eta}^{m+1}.
\]

By splitting $\sum_{j=1}^{l+n+1} f_j d\Theta_j$ into a sum of the two parts $\sum_{j=1}^{l} f_j d\xi_j$ and $\sum_{k=1}^{n+1} f_{k+l} d\overline{\xi}_k$, we also split $I_{24}$ into two corresponding parts as $I_{24} = I_{241} + I_{242}$. A little calculation gives that

\[
I_{241} = \sum_{j=1}^{l} \sum_{k,p=1}^{n+1} f_j (z_k + \zeta_k) \overline{\xi}_p d\xi_j \land d\xi_k \land d\overline{\xi}_p
\]

\[
\land \left\{ \left[ \sum_{t=1}^{l+n+1} \left[ \Theta_t(1 - \Theta \bullet \overline{Z}) - \overline{Z}_t(1 - |\Theta|^2) \right] d\Theta_t \right] \land \overline{\partial}|\xi|^2 \right\}
\land \left[ \sum_{s=1}^{l+n+1} \overline{Z}_s d\Theta_s \right] \land \omega_{\xi}^{l-1} \land \omega_{\xi}^{n-1} \land (w \bullet \overline{\eta} + |\eta|^2) \omega_{\eta}^{m+1}
\]

\[
= C(1 - \Theta \bullet \overline{Z})(w \bullet \overline{\eta} + |\eta|^2) \left\{ \sum_{j=1}^{l} f_j \left( \sum_{k,p=1}^{n+1} (z_k + \zeta_k) \overline{\xi}_p \zeta_k \left( \overline{\xi}_p \overline{\xi}_p - \overline{x}_j \overline{\xi}_p \right) \right) \right\} \land \omega_{\xi}^{l} \land \omega_{\xi}^{n+1} \land \omega_{\eta}^{m+1}.
\]

Similarly, since $\overline{\partial}|\xi|^2 = \sum_{s=1}^{l} \xi_s d\overline{\xi}_s$, we obtain
It can be checked that $I_{241}, I_{242}$ and $I_{24}$ satisfy the conclusion of the lemma.

Case 5: $J_5 = \omega_\xi^{l-1} \wedge \omega_\zeta^{n-1} \wedge (w \cdot \bar{\eta} + |\eta|^2) \omega_\eta^{m+1}$

$$I_{242} = \sum_{s=1}^{l} \sum_{k,p=1}^{n+1} f_{k+l}(z_k + \zeta_k)\bar{\xi}_p\xi_s d\zeta_k \wedge d\bar{\zeta}_p \wedge d\bar{\xi}_s$$

$$\wedge \left\{ \left[ \sum_{t=1}^{l+n+1} \left[ \Theta_t(1 - \Theta \cdot \bar{Z}) - \bar{Z}_t(1 - |\Theta|^2) \right] d\Theta_t \right] \wedge \left[ \sum_{t=1}^{l+n+1} \bar{Z}_t d\Theta_t \right] \right\}$$

$$\wedge \omega_\xi^{l-1} \wedge \omega_\zeta^{n-1} \wedge (w \cdot \bar{\eta} + |\eta|^2) \omega_\eta^{m+1}$$

$$= C(1 - \Theta \cdot \bar{Z})(w \cdot \bar{\eta} + |\eta|^2) \sum_{k=1}^{n+1} \sum_{s=1}^{l} \sum_{p=1}^{n+1} \left( (z_k + \zeta_k)\bar{\xi}_p\xi_s \right.$$}

$$\left. \left( \bar{x}_s\bar{\xi}_p - \bar{\xi}_s\bar{x}_p \right) - (z_p + \zeta_p)\bar{\xi}_p\xi_s \left( \bar{x}_s\zeta_k - \bar{\xi}_s\zeta_k \right) \right\} \wedge \omega_\xi^{l} \wedge \omega_\zeta^{n+1} \wedge \omega_\eta^{m+1}.$$

It can be checked that $I_{241}, I_{242}$ and $I_{24}$ satisfy the conclusion of the lemma.

Case 5: $J_5 = \omega_\xi^{l} \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^{m-1}$. Observe that

$$I_{25} = \sum_{j=l+1}^{l+N+2} f_j d\Theta_j$$

$$\wedge \left\{ \sum_{t=l+1}^{l+N+2} \left[ \Theta_t(1 - \Theta \cdot \bar{Z}) - \bar{Z}_t(1 - |\Theta|^2) \right] d\Theta_t \right\}$$

$$\wedge \bar{\Theta}(|\zeta|^2 + |\eta|^2) \wedge \sum_{s=l+1}^{l+N+2} \bar{Z}_s d\Theta_s \wedge \sum_{k,p=1}^{n+1} (z_k + \zeta_k)\bar{\xi}_p d\zeta_k \wedge d\bar{\zeta}_p$$

$$\left[ \sum_{r,s=1}^{m+1} (w_r + \eta_r)\bar{\eta}_s d\eta_r \wedge d\bar{\eta}_s \right] \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^{m-1} \wedge \omega_\xi^{l}.$$
We obtain in exactly the same way an explicit expression for $I_{252}$. Finally, we deduce from these expressions that $I_{251}$, $I_{252}$ and $I_{25}$ satisfy the conclusion of the lemma.

Case 6: $J_6 = \omega_1^{m-1} \land \omega_n^{m-1} \land \omega_\xi^l$. This last case can be treated in the same way as Case 4. The proof of Lemma 3.5 is therefore complete. □

We end this section with the study of the particular case $N = (2, 2)$. In this case we write for $Z, \Theta \in B_N$:

$$Z \equiv (z, w) \equiv (z_1, z_2, z_3, w_1, w_2, w_3), \quad \text{and} \quad \Theta \equiv (\zeta, \eta) \equiv (\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3).$$

To establish optimal Lipschitz estimates for the domain $\Omega_{(2,2)}$, we need a more precise formulation of the Cauchy type formula given in Theorem 3.3.

**Theorem 3.6.** — Let $N := (2, 2)$. There are polynomials $R(\Theta, Z)$ and $P_{jk}(\Theta, Z)$, $Q_{jk}(\Theta, Z)$, $1 \leq j \leq 2$, $1 \leq k \leq 4$, that satisfy the following properties:

(i) $R(\Theta, Z) = (C|\zeta|^2 + C|\eta|^2)(|\zeta|^2 + z \cdot \bar{\zeta})(|\eta|^2 + w \cdot \bar{\eta})$.

(ii) For every $Z, \Theta \in B_N$, and for every $1 \leq j \leq 2$ and $1 \leq k \leq 4$,

$$P_{1k}(\Theta, Z) = O\left(|\Theta - Z|(|\zeta_3| + |\eta_3| + |\Theta - Z|)(|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right),$$

$$Q_{1k}(\Theta, Z) = O\left(|\Theta - Z|(|\zeta_3| + |\eta_3| + |\Theta - Z|)(|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right),$$

We obtain in exactly the same way an explicit expression for $I_{252}$. Finally, we deduce from these expressions that $I_{251}$, $I_{252}$ and $I_{25}$ satisfy the conclusion of the lemma.
(iii) Let \( u \in C^1(\overline{M}_N) \) and \( f := f_1 d\bar{\zeta}_1 + f_2 d\bar{\zeta}_2 + f_3 d\bar{\eta}_1 + f_4 d\bar{\eta}_2 \) is a \((0,1)\)-form with coefficients in \( C(\overline{M}_N) \) that satisfy \( \bar{\partial}_{\overline{M}_N} u = f |_{\overline{M}_N} \), then for every \( Z \in \overline{M}_N \),

\[
\begin{align*}
|\text{grad}_Z P_{jk}(\Theta, Z)| &= O \left( (|\zeta|^2 + |\zeta||\eta|)(|\eta|^2 + |\eta||w|) \right), \\
|\text{grad}_Z Q_{jk}(\Theta, Z)| &= O \left( (|\zeta|^2 + |\zeta||\eta|)(|\eta|^2 + |\eta||w|) \right).
\end{align*}
\]

**Proof.** — We return to the arguments used in the proof of Theorem 3.3. By the hypothesis on \( f \) and (3.2), we have that

\[
I := au \wedge K_0 = (f_1 H_1 + f_2 H_2 + f_3 H_3 + f_4 H_4) d\Theta \wedge d\overline{\Theta},
\]

(3.7)

with \( H_1 := h_1, H_2 := h_2, H_3 := h_4 \) and \( H_4 := h_5 \). To complete the proof, it suffices to prove the following

**Lemma 3.7.** — The functions \( H_k \) in formula (3.7) can be expressed in the form

\[
H_k(\Theta, Z) = \sum_{j=1}^{2} \frac{(1 - \Theta \cdot Z)^{1+j}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^4} [(1 - \Theta \cdot Z) P_{jk}(\Theta, Z) + (1 - |\Theta|^2) Q_{jk}(\Theta, Z)],
\]

(3.8)

where \( P_{jk} \) and \( Q_{jk} \) are some polynomials satisfying assertion (ii) of the theorem.

**Proof of Lemma 3.7.** — Following the proof of Lemma 3.4, we write \( I = I_1 + I_2 \). By virtue of (3.4), the functions \( H_k \) associated to \( I_1 \) (similarly to those associated to \( I \) in (3.7)) are in the form (3.8) with \( j = 2 \).

Since \( l = 0 \) and \( \omega_\xi = 0 \), formula (3.5) becomes
Therefore, to conclude the proof of Lemma 3.7, it suffices to prove the following

**LEMMA 3.8.** For every $1 \leq k \leq 3$, the differential form

$$I_{2k} := \left[ f_1d\zeta_1 + f_2d\zeta_2 + f_3d\eta_1 + f_4d\eta_2 \right]$$

$$\wedge \sum_{r=1}^{6} \left[ \Theta_r(1 - \Theta \cdot Z) - Z_r(1 - |\Theta|^2) \right] d\Theta_r$$

$$\wedge |\Theta|^2 \wedge \left[ \sum_{s=1}^{6} Z_s d\Theta_s \right] \wedge J_k \wedge \Psi \wedge \overline{\partial(\zeta \cdot \zeta)} \wedge \overline{\partial(\eta \cdot \eta)}$$

can be expressed as the product of the canonical volume form $d\Theta \wedge d\overline{\Theta}$ and a function of the form

$$\sum_{t=1}^{4} f_t \left( (1 - \Theta \cdot \overline{Z})P_{1t}(\Theta, Z) + (1 - |\Theta|^2)Q_{1t}(\Theta, Z) \right),$$

where $P_{1t}, Q_{1t}$ are some polynomials satisfying assertion (ii) of the theorem.

*End of the proof of Lemma 3.7.* Suppose that Lemma 3.8 is proved. In view of (3.9) and the expression of $I_2$ given at the beginning of the proof of Lemma 3.4, we see that

$$I_2 = \frac{(1 - \Theta \cdot \overline{Z})^2}{\zeta^2|\eta|^2D(\Theta, Z)^4} \cdot \left( \sum_{k=1}^{3} I_{2k} \right).$$

Therefore, Lemma 3.7 follows from Lemma 3.8. 

**Proof of Lemma 3.8.** We first remark that the case $k = 1$ corresponds to the case 5 in the proof of Lemma 3.5. Hence, by virtue of identity (3.6), $I_{21}$ satisfies the conclusion of the lemma. Consider the case $k = 2$ which corresponds to case 2 in the proof of Lemma 3.5. Then we have

$$- 202 -$$
A simple calculation gives that
\[ I_{22} = \left\{ f_1 d\bar{\zeta}_1 + f_2 d\bar{\zeta}_2 \right\} \wedge \sum_{r=1}^{3} \left[ \zeta_r (1 - \Theta \cdot \overline{Z}) - \bar{z}_r (1 - |\Theta|^2) \right] d\zeta_r \wedge \overline{\partial}|\zeta|^2 \]
\[ \wedge \left[ \sum_{s=1}^{3} z_s d\zeta_s \right] \wedge \left[ \sum_{t, p=1}^{3} (z_t + \zeta_t) \bar{\zeta}_p d\zeta_t \wedge d\bar{\zeta}_p \right] \wedge (w \cdot \overline{\eta} + |\eta|^2) \omega_\eta^3. \]

A simple calculation gives that
\[ I_{22} = \left\{ f_1 (1 - \Theta \cdot \overline{Z}) (C\overline{\zeta}_3 \zeta_2 + C\zeta_3 \overline{\zeta}_2) \right\} \left[ \sum (-1)^{\epsilon(r, s, t)} \right] \]
\[ + (\overline{\zeta}_r z_s - \overline{\zeta}_s z_r) (z_t + \zeta_t) (1 - \Theta \cdot \overline{Z}) (C\overline{\zeta}_3 \zeta_1 + C\zeta_3 \overline{\zeta}_1) \]
\[ \left[ \sum (-1)^{\epsilon(r, s, t)} (\overline{\zeta}_r z_s - \overline{\zeta}_s z_r) (z_t + \zeta_t) \right] \cdot (w \cdot \overline{\eta} + |\eta|^2) \omega_\zeta^3 \wedge \omega_\eta^3, \]

where the sum is taken over all permutations \((r, s, t)\) of \(\{1, 2, 3\}\) such that \(r < s\) and where \(\epsilon(r, s, t)\) is the sign of such permutations. It follows from this that \(I_{22}\) satisfies the conclusion of the lemma. Similarly, we have the same conclusion for \(I_{23}\), which completes the proof. \(\square\)

4. Local coordinate systems
on the complex manifolds \(\mathbb{H}_n\) and \(\mathbb{H}_m\)

In the next theorem, we construct an open neighborhood \(U_n\) of \(\mathbb{H}_n\) in \(\mathbb{C}^{n+1}\), and for every \(z \in U_n\), a coordinate chart \(\Phi^z\) defined on a coordinate patch \(U(z)\) of \(\mathbb{H}_n\) that possess some interesting properties of homogeneity. The same construction will be applied to the complex manifold \(\mathbb{H}_m\). These local coordinate systems will allow us in the next section to reduce certain types of integral estimates over \(\mathbb{M}_N\) to simpler integral estimates over \(\mathbb{C}^{[N]}\).

**Theorem 4.1.** There are an open neighborhood \(U_n\) of \(\mathbb{H}_n\) in \(\mathbb{C}^{n+1}\) and constants \(C_1, C_2, C_3 > 1\) that satisfy the following properties:

1) If \(z \in \mathbb{C}^{n+1} \setminus U_n\) then \(\text{dist}(z, \mathbb{H}_n) > \frac{|z|}{C_1}\) with the understanding that \(\text{dist}(., .)\) is the euclidean distance.

2) If \(z \in U_n\) and if the open set \(U(z) := \left\{ \zeta \in \mathbb{H}_n : |\zeta - z| < \frac{|z|}{C_2} \right\}\) is non-empty, then there exists a diffeomorphism \(\Phi^z\) mapping \(U(z)\) into the open neighborhood \(\widetilde{U}(\tilde{z}) := \left\{ \tilde{\zeta} \in \mathbb{C}^n : |\tilde{\zeta} - \tilde{z}| < \frac{|\tilde{z}|}{C_3} \right\}\) of a point \(\tilde{z} \in \mathbb{C}^n\) which is exactly \(\Phi^z(z)\) in case \(z \in \mathbb{H}_n\) such that...
(i) $\zeta \cdot \overline{z} = \Phi^z(\zeta) \cdot \overline{\Phi^z(z)}$, for all $\zeta \in \mathcal{U}(z)$.

(ii) $|\Phi^z(z)| = |z|$ and $\frac{|\zeta|}{2} \leq |\Phi^z(\zeta)| \leq |\zeta|$, for all $\zeta \in \mathcal{U}(z)$.

(iii) For all $\zeta \in \mathcal{U}(z)$, we have $\frac{|\zeta|}{\lambda_{n+1}} \leq C_3 \frac{|\Phi^z(\zeta)|}{|\Phi^z(\zeta)|}$, where $\Phi^z := (\Phi^z_1, \ldots, \Phi^z_n)$.

(iv) For all compactly supported functions $f \in C_0(\mathcal{U}(z))$ such that $f \geq 0$, we have
\[
\int_{\mathcal{U}(z)} f dV_n \leq C_3 \int_{\tilde{\mathcal{U}}(\tilde{z})} (\Phi^z_*) f(\tilde{\zeta}) dV_n(\tilde{\zeta}),
\]
where $dV_n(\tilde{\zeta})$ denotes the Lebesgue measure on $\mathbb{C}^n$ and $\Phi^z_*$ is the pushforward of $f$ under the diffeomorphism $\Phi^z$.

**Remark 4.2.** — We construct in the same way an open neighborhood $\mathcal{U}_m$ of $\mathbb{H}_m$ in $\mathbb{C}^{m+1}$, and for every $w \in \mathcal{U}_m$, a coordinate chart $\Phi^w$ defined on a coordinate patch $\mathcal{U}(w)$ of $\mathbb{H}_m$ that possess the same properties as $\mathcal{U}_n$, $\Phi^z$ and $\mathcal{U}(z)$.

To prove Theorem 4.1, we need the following

**Lemma 4.3.** — There exists a constant $C_0 > 0$ such that $\max_{1 \leq j, k \leq n+1} |\text{Im}(z_j z_k)| > C_0$, for all $z := (z_1, \ldots, z_{n+1}) \in \partial \mathbb{M}_n$. Here $\text{Im}\lambda$ denotes the imaginary part of $\lambda \in \mathbb{C}$.

**Proof.** — Since the function $z \mapsto \max_{1 \leq j, k \leq n+1} |\text{Im}(z_j z_k)|$ is continuous on the compact set $\partial \mathbb{M}_n$, it attains its minimum at a point $z$. Therefore it suffices to prove that there exist $1 \leq j < k \leq n+1$ such that $\text{Im}(z_j z_k) \neq 0$. Suppose the contrary. Since $|z| = 1$, there is an $k$ such that $z_k \neq 0$. Hence for every $1 \leq j \leq n+1$, we have $z_j = \lambda_j z_k$ with $\lambda_j \in \mathbb{R}$, from which it follows that $0 = \left( \sum_{j=1}^{n+1} \lambda_j^2 \right) z_k^2$. Thus $z_k = 0$ and we obtain a contradiction. This completes the proof of the lemma. $\Box$

We now turn to the proof of Theorem 4.1.

**Proof of Theorem 4.1.** — The construction of the open neighborhood $\mathcal{U}_n$, the coordinate patches $\mathcal{U}(z)$ and the coordinate charts $\Phi^z : \mathcal{U}(z) \to \mathbb{C}^n$ for every $z \in \mathcal{U}_n$, will be done within two steps. First, by Lemma 4.3, we divide $\partial \mathbb{M}_n$ into $\frac{n(n+1)}{2}$ compact sets $E_{jk}$, $j < k$, where $E_{jk} := \{ z \in \partial \mathbb{M}_n : |\text{Im}(z_j z_k)| \geq C_0 \}$.

Fix a sufficiently small number $\delta > 0$. The exact value of $\delta$ will be clear in the course of the proof. Let $z$ be a point of $\mathbb{C}^{n+1}$.
Step 1: \( \text{dist}(z, \partial \mathbb{M}_n) < \delta. \)

According to the discussion above, suppose without loss of generality that there exist \( j < k \) and \( \hat{z} \in E_{jk} \) such that \( |z - \hat{z}| < \delta \). Define the diffeomorphism \( \Phi^z \) as follows: \( \Phi^z := (\Phi^z_1, \ldots, \Phi^z_n) \), where

\[
\begin{align*}
\Phi^z_{k-1}(\zeta) &= \frac{\zeta_j \overline{\zeta}_j + \zeta_k \overline{\zeta}_k}{\sqrt{|\zeta_j|^2 + |\zeta_k|^2}}, \\
\Phi^z_l(\zeta) &= \zeta_l, & \text{if } l < j; \\
\Phi^z_j(\zeta) &= \zeta_{l+1}, & \text{if } j \leq l < k - 1 \text{ or } k \leq l \leq n.
\end{align*}
\]

We can choose the functions \( \zeta_l, l \neq j \), as the \( n \)-local coordinate functions of \( \mathbb{H}_n \) at the point \( \hat{z} \). Substituting \( \zeta_j \) by \( i \sqrt{\sum_{l \neq j} \zeta_l^2} \) in the expression of \( \Phi^z \), straightforward computations show that the real Jacobian of \( \Phi^z \) at the point \( \zeta \) corresponding to this local coordinate system is equal to \( \frac{|\zeta_j \overline{\zeta}_j - \zeta_k \overline{\zeta}_k|^2}{|\zeta_j|^2(|\zeta_j|^2 + |\zeta_k|^2)} \). This quantity is uniformly bounded from above and from below by some positive constants as \( \zeta \in \mathbb{H}_n \) and \( z \) are very near to \( \hat{z} \in E_{jk} \). Therefore, when \( C_2 \) is sufficiently large, there exists a sufficiently small \( \delta \) so that for every \( \hat{z} \in E_{jk} \) and every \( z \) such that \( |z - \hat{z}| < \delta \), \( \Phi^z \) is a diffeomorphism from \( \{ \zeta \in \mathbb{H}_n : |\zeta - \hat{z}| < 2\delta \} \) to \( \{ \zeta \in \mathbb{C}^n : |\zeta - \Phi^z(z)| < \frac{1}{2C_2} \} \).

Taking \( C_1 > \frac{1}{2C_2} \) and observing that \( |\Phi^z(z)| = |z| \approx 1 \), it follows from the previous discussion that \( \Phi^z \) is a diffeomorphism from \( \mathcal{U}(z) \) onto an open neighborhood \( \hat{U}(\hat{z}) \) of the point \( \hat{z} := \Phi^z(z) \in \mathbb{C}^n \).

To finish part 2) of the theorem, it remains to prove assertions (i)-(iv).

Assertions (i) and (ii) can be checked directly. In particular, the estimate \( |\Phi^z(\zeta)| \leq |\zeta| \) follows from the Cauchy-Schwarz inequality.

We prove now assertion (iii). Consider two cases according to \( k \):

Case 1. \( k < n + 1 \). In this case, in view of the definition of \( \Phi^z \), we have \( \zeta_{n+1} = \Phi^z_n(\zeta) \). This, combined with (ii), implies assertion (iii).

Case 2. \( k = n + 1 \). If \( \zeta \in \mathcal{U}(z) \), then when \( C_1 \) is sufficiently large, we have \( 1 > |\zeta_{n+1}| \approx |\hat{z}_{n+1}| \geq C_0 \). Hence assertion (iii) is almost obvious.

It now remains to prove assertion (iv). By Proposition 2.1 in [?], for \( \zeta \in \mathcal{U}(z) \) we have the following identity:

\[
dV_n(\zeta) = C \frac{|\zeta|^2}{|\zeta_j|^2} d\zeta_1 \wedge d\overline{\zeta}_1 \wedge \ldots \wedge d\overline{\zeta}_j \wedge d\zeta_j \wedge \ldots \wedge d\zeta_{n+1} \wedge d\overline{\zeta}_{n+1}.
\]

Since \( 2 > |\zeta| > |\zeta_j| \approx |\hat{z}_j| \geq C_0 \), it follows that \( dV_n(\zeta) \approx (\Phi^z)^*(dV_n(\hat{\zeta})) \) for \( \hat{\zeta} = \Phi^z(\zeta) \). This implies assertion (iv).
Step 2: General case.

Set $U_n := \{rz : r > 0 \text{ and } \text{dist}(z, \partial \mathbb{M}_n) < \delta\}$.

If $z \in U_n$, then according to the definition above, there exist $\hat{z} \in \partial \mathbb{M}_n$ and $r > 0$ such that $|rz - \hat{z}| < \delta$. Therefore, the construction given in Step 1 can then be applied to the point $rz$. Hence, we can define

$$U(z) := \frac{1}{r} \cdot U(rz);$$

$$\Phi^z(\zeta) := \frac{1}{r} \cdot \Phi^{rz}(r\zeta), \quad \forall \zeta \in U(z).$$

Using the homogeneous invariance of the complex manifold $\mathbb{H}_n$ with respect to the dilations, we conclude that for every $z \in U_n$, the function $\Phi^z$ just defined satisfies part 2) of the theorem. To finish the proof of the theorem, it only remains to check part 1). Let $z \not\in U_n$. Then there exists a point $\hat{z} \in \mathbb{H}_n$ such that $|z - \hat{z}| = \text{dist}(z, \mathbb{H}_n)$. Since $z \not\in U_n$, we deduce that $|z - \hat{z}| \geq \delta|\hat{z}|$.

Hence

$$\left(\frac{1}{\delta} + 1\right) \text{dist}(z, \mathbb{H}_n) > |z - \hat{z}| + |\hat{z}| \geq |z|.$$

Thus, if we choose $C_1 > \frac{1}{\delta} + 1$, then part 1) is satisfied. This completes the proof of the theorem. $\square$

5. Reduction of estimates from $\mathbb{M}_N$ to $\mathbb{B}_|N|$,

This section proves the Theorem of reduction of estimates. We use the notations and the constants introduced in the previous section. In order to state this theorem, we need some more notations and definitions.

We denote by $\mathbb{B}_|N|$ the euclidean unit ball of $\mathbb{C}^{|N|}$. We often use the following notations for $\tilde{\Theta}, \tilde{Z} \in \mathbb{C}^{|N|}$:

$$\tilde{\Theta} \equiv (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in \mathbb{C}^i \times \mathbb{C}^n \times \mathbb{C}^m \quad \text{and} \quad \tilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w}) \in \mathbb{C}^i \times \mathbb{C}^n \times \mathbb{C}^m.$$

Let $dV(\tilde{\Theta})$ be the Lebesgue measure on $\mathbb{C}^{|N|}$. For every $i \in \{1, 2\}$, note $\mathbb{B}_{i,|N|}$ the euclidean ball of $\mathbb{C}^{|N|}$ centered at the origin with radius $i$. Thus $\mathbb{B}_|N| = \mathbb{B}_{1,|N|}$.

We shall define various notions of comparability.

**Definition 5.1. —** Consider two points $Z \equiv (x, z, w) \in \mathbb{B}_N$ and $\tilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w}) \in \mathbb{C}^{|N|}$. $Z$ is said to be comparable with $\tilde{Z}$ if the following conditions are true:
Remark 5.2. It should be noted that by this definition and Theorem 4.1 (ii), we have $|x| = |\tilde{x}|, |z| = |\tilde{z}|, |w| = |\tilde{w}|$. Hence $\tilde{Z} \in B_{|N|}$.

**DEFINITION 5.3.** Let $i \in \{1, 2\}$ and fix two comparable points $Z = (x, z, w) \in B_N$ and $Z = (x, \tilde{z}, \tilde{w}) \in C_{|N|}$.

We say that $\xi \in \mathcal{C}^i$ is $i$-comparable with $\tilde{\xi} \in \mathcal{C}^i$ if $\tilde{\xi} = \xi$.

We say that $\zeta \in \mathcal{H}_m$ is $i$-comparable with $\tilde{\zeta} \in \mathcal{C}^n$ if the following conditions are true:

1. If $|\zeta| > \sqrt{2}|z|$, then $|\tilde{\zeta}| = |\zeta|$.
2. If $|\zeta - z| < \frac{|\tilde{z}|}{C_1}$, then $\tilde{\zeta} = \Phi^i(\zeta)$.
3. If $|\zeta| \leq \sqrt{2}|z|$ and $|\zeta - z| \geq \frac{|\tilde{z}|}{C_1}$, then $|\tilde{\zeta}| \leq \sqrt{2} |\tilde{z}|$ and $|\tilde{\zeta} - \tilde{z}| > \frac{|\tilde{z}|}{C_2}$; if moreover $i = 1$, then we have $|\tilde{\zeta}| \leq |\zeta|$.

We can define in the same way the notion of $i$-comparability between $\eta \in \mathcal{H}_m$ and $\tilde{\eta} \in \mathcal{C}^m$ upon substituting $n$ by $m$ and $\Phi^i$ by $\Phi^w$.

Finally, two points $\Theta \equiv (\xi, \zeta, \eta) \in \overline{M}_N$ and $\tilde{\Theta} \equiv (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in \mathcal{C}^{|N|}$ are said to be $i$-comparable if $\xi$ (resp. $\zeta$ and $\eta$) is $i$-comparable with $\tilde{\xi}$ (resp. $\tilde{\zeta}$ and $\tilde{\eta}$).

**Remark 5.4.** We deduce easily from this definition and Theorem 4.1 (ii) that if $\Theta \in \overline{M}_N$ is $i$-comparable with $\tilde{\Theta} \in \mathcal{C}^{|N|}$, then $\tilde{\Theta} \in \overline{B}_{i,|N|}$.

**DEFINITION 5.5.** Let $i \in \{1, 2\}$ and fix two comparable points $Z \in B_N$ and $Z \in B_{|N|}$. Consider two non-negative measurable functions $K, \tilde{K}$ defined respectively on $\overline{M}_N$ and $\overline{B}_{i,|N|}$.

- We write $K \lesssim C \tilde{K}$ (respectively, $\tilde{K} \lesssim CK$) at $(Z, \tilde{Z})$ for a positive constant $C$ if for all points $\Theta \in \overline{M}_N$ i-comparable with $\tilde{\Theta} \in \overline{B}_{i,|N|}$,

  $$K(\Theta) \leq C \tilde{K}(\tilde{\Theta})$$

  (respectively, $\tilde{K}(\tilde{\Theta}) \leq CK(\Theta)$).

- We write $K \approx \tilde{K}$ at $(Z, \tilde{Z})$ if there exists $C > 0$ such that $K \lesssim C \tilde{K} \lesssim C^2 K$.
Now we are in a position to state the main theorem of this section.

**Theorem 5.6.** — Let \( i \in \{1,2\} \) and fix two comparable points \( Z \in \mathbb{B}_N \) and \( \tilde{Z} \in \mathbb{B}_{|N|} \). Let \( C \) be a positive constant. Consider non-negative measurable functions \( K, L \) defined on \( \overline{M}_N \) and \( \tilde{K}, \tilde{L} \) defined on \( \overline{B}_{i,|N|} \) such that

\[
K \lesssim C \tilde{K} \quad \text{and} \quad L \lesssim C \tilde{L} \lesssim C^2 L \quad \text{at} \ (Z, \tilde{Z}).
\]

For every \( \alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) such that \( 0 \leq \alpha_1 < 2n, 0 \leq \alpha_2 < 2m \) and \( 0 \leq \alpha_3, \alpha_4 < 2 \), we set

\[
K_\alpha(\Theta) := K(\Theta) \left( 1 + \frac{|z|}{|\zeta|} \right)^{\alpha_1} \left( 1 + \frac{|w|}{|\eta|} \right)^{\alpha_2} \left( \frac{\zeta}{\xi_{n+1}} \right)^{\alpha_3} \left( \frac{\eta}{\eta_{m+1}} \right)^{\alpha_4},
\]

\[
\tilde{K}_{1,\alpha}(\tilde{\Theta}) := \tilde{K}(\tilde{\Theta}) \left( 1 + \frac{|\tilde{z}|}{|\tilde{\zeta}|} \right)^{\alpha_1} \left( 1 + \frac{|\tilde{w}|}{|\tilde{\eta}|} \right)^{\alpha_2} \left( \frac{\tilde{\zeta}}{\tilde{\xi}_n} \right)^{\alpha_3} \left( \frac{\tilde{\eta}}{\tilde{\eta}_m} \right)^{\alpha_4},
\]

\[
\tilde{K}_{2,\alpha}(\tilde{\Theta}) := \tilde{K}(\tilde{\Theta}) \left| \frac{\tilde{\zeta}}{\xi_n} \right|^{\alpha_3} \left| \frac{\tilde{\eta}}{\eta_m} \right|^{\alpha_4}.
\]

Then there exists a constant \( C_4 \) that depends only on \( N, \alpha \) and \( C, C_1, C_2, C_3 \), (in particular this constant is independent of \( Z \) and \( \tilde{Z} \)), such that

1) \[
\int_{\overline{M}_N} K_\alpha(\Theta)dV(\Theta) \leq C_4 \int_{\overline{B}_{i,|N|}} \tilde{K}_{i,\alpha}(\tilde{\Theta})dV(\tilde{\Theta});
\]

2) for \( \delta > 0 \),

\[
\int_{\Theta \in \overline{M}_N, \ L(\Theta) \leq \delta} K_\alpha(\Theta)dV(\Theta) \leq C_4 \int_{\tilde{\Theta} \in \overline{B}_{i,|N|}, \ \tilde{L}(\tilde{\Theta}) \leq C_4 \delta} \tilde{K}_{i,\alpha}(\tilde{\Theta})dV(\tilde{\Theta});
\]

3) for \( 0 < \delta_1 < \delta_2 \),

\[
\int_{\Theta \in \overline{M}_N, \ \delta_1 \leq L(\Theta) \leq \delta_2} K_\alpha(\Theta)dV(\Theta) \leq C_4 \int_{\tilde{\Theta} \in \overline{B}_{i,|N|}, \ \frac{\delta_1}{C_4} \leq \tilde{L}(\tilde{\Theta}) \leq C_4 \delta_2} \tilde{K}_{i,\alpha}(\tilde{\Theta})dV(\tilde{\Theta}).
\]

**Proof.** — We shall only prove part 3). The two other assertions can be shown in exactly the same way. Firstly, we extend the domain of definition of the functions \( K, L, \tilde{K}, \tilde{L} \) by setting

\[
K(\Theta) = L(\Theta) := 0, \quad \text{if} \ \Theta \in \mathbb{H}_N \setminus \overline{M}_N;
\]

\[
\tilde{K}(\tilde{\Theta}) = \tilde{L}(\tilde{\Theta}) := 0, \quad \text{if} \ \tilde{\Theta} \in \mathbb{C}^{|N|} \setminus \overline{B}_{i,|N|}.
\]
By the hypothesis on $L$ and $\tilde{L}$, for every $\Theta \in \overline{M}_N$ such that $\delta_1 \leq L(\Theta) \leq \delta_2$ and for every $\tilde{\Theta} \in \overline{M}_{i,N}$ $i$-comparable with $\Theta$, we have

$$\frac{\delta_1}{C} \leq \tilde{L}(\tilde{\Theta}) \leq C \delta_2. \quad (5.1)$$

For every $\xi, \tilde{\xi} \in \mathbb{B}_l$ and $\eta, \tilde{\eta} \in \mathbb{H}_m$, consider the following integrals

$$R(\xi, \eta) := \int_{\zeta \in \mathbb{H}_n, \delta_1 \leq L(\xi, \zeta, \eta) \leq \delta_2} K(\xi, \zeta, \eta) \left(1 + \frac{|\xi|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta),$$

$$\tilde{R}_1(\xi, \tilde{\eta}) := \int_{\zeta \in \mathbb{H}_n, \delta_1 \leq L(\xi, \zeta, \tilde{\eta}) \leq \delta_2} \tilde{K}(\xi, \tilde{\zeta}, \tilde{\eta}) \left(1 + \frac{|\xi|}{|\tilde{\zeta}|}\right)^{\alpha_1} \left|\frac{\tilde{\zeta}}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\tilde{\zeta}),$$

$$\tilde{R}_2(\xi, \tilde{\eta}) := \int_{\zeta \in \mathbb{H}_n, \delta_1 \leq L(\xi, \zeta, \tilde{\eta}) \leq \delta_2} \tilde{K}(\xi, \tilde{\zeta}, \tilde{\eta}) \left|\frac{\tilde{\zeta}}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\tilde{\zeta}),$$

where $dV_n(\tilde{\zeta})$ denotes the Lebesgue measure on $\mathbb{C}_n$.

Next, consider the following integrals

$$S(\xi) := \int_{\mathbb{H}_m} R(\xi, \eta) \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} \left|\frac{\eta}{\eta_{m+1}}\right|^{\alpha_4} dV_m(\eta),$$

$$\tilde{S}_1(\tilde{\xi}) := \int_{\mathbb{H}_m} \tilde{R}_1(\tilde{\xi}, \tilde{\eta}) \left(1 + \frac{|\tilde{w}|}{|\tilde{\eta}|}\right)^{\alpha_2} \left|\frac{\tilde{\eta}}{\tilde{\eta}_{m+1}}\right|^{\alpha_4} dV_m(\tilde{\eta}),$$

$$\tilde{S}_2(\tilde{\xi}) := \int_{\mathbb{H}_m} \tilde{R}_2(\tilde{\xi}, \tilde{\eta}) \left|\frac{\tilde{\eta}}{\tilde{\eta}_{m+1}}\right|^{\alpha_4} dV_m(\tilde{\eta}),$$

where $dV_m(\tilde{\eta})$ is the Lebesgue measure on $\mathbb{C}_m$.

We outline the main ideas of the proof. Suppose that $\xi$ (resp. $\eta$) is $i$-comparable with $\tilde{\xi}$ (resp. $\tilde{\eta}$). Using the hypothesis that $K \leq C \tilde{K}$, we shall prove that

$$R(\xi, \eta) \leq C_4 \tilde{R}_i(\tilde{\xi}, \tilde{\eta}), \quad i = 1, 2. \quad (5.2)$$

Next, we shall establish in the same way as in the proof of (5.2) the following estimate : (note that $\xi = \tilde{\xi}$)

$$S(\xi) \leq C_4 \tilde{S}_i(\tilde{\xi}), \quad i = 1, 2. \quad (5.3)$$
Finally, an application of Fubini's theorem gives that

\[ \int_{\Theta \in \mathbb{M}_N, \, \delta_1 \leq L(\Theta) \leq \delta_2} K_\alpha(\Theta) dV(\Theta) = \int_{B_1} S(\xi) dV_1(\xi), \]

and

\[ \int_{\tilde{\Theta} \in B_{1,1,N_1}, \, \delta_1 \leq \tilde{L}(\tilde{\Theta}) \leq C\delta_2} \tilde{K}_{i,\alpha}(\tilde{\Theta}) dV(\tilde{\Theta}) = \int_{\tilde{B}_1} \tilde{S}_i(\xi) dV_1(\xi). \]

Part 3) now follows by combining (5.3) with the latter two estimates. It now remains to prove inequality (5.2).

To do so, divide the domain of integration \( \{ \zeta \in \mathbb{H}_n : \delta_1 \leq L(\xi, \zeta, \eta) \leq C\delta_2 \} \) of \( R(\xi, \eta) \) into the three subsets:

\[ E_1 := \left\{ |\zeta - z| < \frac{|z|}{C_1} \right\}; \quad E_2 := \left\{ |\zeta| > \sqrt{2}|z| \right\}; \]

\[ E_3 := \left\{ |\zeta| \leq \sqrt{2}|z| \text{ and } |\zeta - z| \geq \frac{|z|}{C_1} \right\}. \]

Also, divide the domain of integration \( \{ \tilde{\zeta} \in \mathbb{C}^n : \delta_1 \leq \tilde{L}(\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \leq C\delta_2 \} \) of \( \tilde{R}_i(\tilde{\xi}, \tilde{\eta}) \) into three corresponding subsets:

\[ \tilde{E}_1 := \left\{ |\tilde{\zeta} - \tilde{z}| < \frac{1}{C_2} \right\}; \quad \tilde{E}_2 := \left\{ |\tilde{\zeta}| > \sqrt{2}|\tilde{z}| \right\}; \]

\[ \tilde{E}_3 := \left\{ |\tilde{\zeta}| \leq \sqrt{2}|\tilde{z}| \text{ and } |\tilde{\zeta} - \tilde{z}| \geq \frac{1}{C_2} \right\}. \]

Estimate (5.2) will follow by combining three integral estimates of the form \( \int_{E_j} \leq C \int_{\tilde{E}_j} \) with some appropriate integrands and \( j = 1, 2, 3 \). Therefore, we may assume without loss of generality that \( E_j \neq \emptyset, j = 1, 2, 3 \).

Combining Theorem 4.1, definition 5.3 and estimate (5.1), we see that \( (\xi, \zeta, \eta) \) is \( i \)-comparable with \( (\tilde{\xi}, \Phi^*(\zeta), \tilde{\eta}) \in \tilde{E}_1 \), for every \( \zeta \in E_1 \). Hence, the hypothesis \( K \lesssim C \tilde{K} \) implies that \( K (\xi, \zeta, \eta) \leq C \tilde{K} (\tilde{\xi}, \Phi^*(\zeta), \tilde{\eta}) \). Moreover, the fact that \( \zeta \in E_1 \) gives that \( |\zeta| > \left( 1 - \frac{1}{C_1} \right) |z| \). Therefore, applying Theorem 4.1 (iii)-(iv) gives that
Next, we prove the estimate of the form \( \int_{E_2} \leq C_4 \int_{E_2} \). Set \( I := \{ |\zeta| : \zeta \in E_2 \} \).

We remark that \( t_{ct} I \leq 1 \) for every \( \zeta \in E_2 \). Therefore, by integration in polar coordinates (Corollary 2.2), we obtain

\[
\int_{E_2} K(\xi, \zeta, \eta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta) 
\leq C_4 \int_{E_2} \tilde{K} \left(\tilde{\xi}, \Phi^z(\zeta), \tilde{\eta}\right) \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\tilde{\zeta}).
\]

(5.4)

In summary, we have that

\[
\int_{E_2} K(\xi, \zeta, \eta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta) \leq C_4 \int_{E_2} \tilde{K} \left(\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}\right) dV_n(\tilde{\zeta}).
\]

(5.5)

It now remains to prove the estimate of the form \( \int_{E_3} \leq C_4 \int_{E_3} \). Consider two cases according to the value of \( i \):
Case $i = 1$. We set $R := \sup_{\zeta \in E_3} |\zeta|$. In view of definition 5.3, Remark 5.2 and estimate (5.1), we see that $(\xi, \zeta, \eta)$ is 1-comparable with $(\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in \tilde{E}_3$ for every $\zeta \in E_3$ and $\tilde{\zeta} \in \mathbb{C}^n$ such that $|\tilde{\zeta} - \tilde{\zeta}| \geq \frac{\tilde{\zeta}}{C_2}$ and $|\tilde{\zeta}| \leq |\zeta|$. Therefore, using integration in polar coordinates, we obtain

$$\int_{\tilde{E}_3} K(\xi, \zeta, \eta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta)$$

where on the third line, $\sigma_n(\tilde{\zeta})$ is the surface measure of the euclidean unit sphere $\partial \mathbb{B}_n$ of $\mathbb{C}^n$.

Case $i = 2$. We see easily that

Moreover, $(\xi, \zeta, \eta)$ is 2-comparable with $(\tilde{\xi}, \tilde{\zeta}, \tilde{\eta})$ for every $\zeta \in E_3$ and $\tilde{\zeta} \in \tilde{E}_3$. On the other hand, by Remark 5.2, we have $|\zeta| = |\tilde{\zeta}|$. Thus,

$$\int_{\tilde{E}_3} K(\xi, \zeta, \eta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta)$$

where on the third line, $\sigma_n(\tilde{\zeta})$ is the surface measure of the euclidean unit sphere $\partial \mathbb{B}_n$ of $\mathbb{C}^n$.
Now estimate (5.2) follows from (5.4)-(5.7). This completes the proof of part 3).

To conclude this section, we give without proof some properties of the relations \( \lesssim \) and \( \approx \).

**Proposition 5.7.** Let \( Z, \tilde{Z} \) and \( K, L, \tilde{K}, \tilde{L} \) be as in the statement of Theorem 5.6. Suppose that \( K \lesssim \tilde{K} \) and \( L \lesssim \tilde{L} \) at \((Z, \tilde{Z})\). Then \( K + L \lesssim \tilde{K} + \tilde{L} \) and \( K^\alpha L^\beta \lesssim \tilde{K}^\alpha \tilde{L}^\beta \), for every \( \alpha, \beta \geq 0 \).

If in addition \( K \approx \tilde{K} \) and \( L \approx \tilde{L} \) then \( K + L \approx \tilde{K} + \tilde{L} \) and \( K^\alpha L^\beta \approx \tilde{K}^\alpha \tilde{L}^\beta \), for every \( \alpha, \beta \in \mathbb{R} \).

### 6. Integral kernels

The pairs of integral kernels \( K, \tilde{K} \) satisfying the condition \( K \approx \tilde{K} \) that we shall use are studied here. Recall the function \( D \) introduced by Charpentier [?]:

\[
D(\Theta, Z) := |1 - \Theta \cdot \overline{Z}|^2 - (1 - |\Theta|^2)(1 - |Z|^2), \quad \text{for all } \Theta, Z \in \mathbb{C}^k \text{ and } k \in \mathbb{N}.
\]

**Theorem 6.1.** Let \( i \in \{1, 2\} \) and fix two comparable points \( Z \in \mathbb{B}_N \) and \( \tilde{Z} \in \mathbb{B}_{|N|} \). Consider two functions \( K, \tilde{K} \) defined respectively on \( \mathbb{M}_N \) and \( \mathbb{\overline{M}},|N| \) that correspond to one of the following three cases:

1. \( i = 2 \) and \( K(\Theta) := |\Theta - Z|, \quad \tilde{K}(\Theta) := |\Theta - \tilde{Z}|; \)
2. \( i = 1 \) and \( K(\Theta) := |1 - \Theta \cdot \overline{Z}|, \quad \tilde{K}(\Theta) := |1 - \Theta \cdot \overline{\tilde{Z}}|; \)
3. \( i = 1 \) and \( K(\Theta) := D(\Theta, Z), \quad \tilde{K}(\Theta) := D(\Theta, \tilde{Z}). \)

Then \( K \approx \tilde{K} \) at \((Z, \tilde{Z})\).

**Proof.** Using the definitions 5.1, 5.3 and 5.5, it can be easily checked that

\[
|z - \zeta| \approx |\tilde{z} - \tilde{\zeta}| \quad \text{and} \quad |w - \eta| \approx |\tilde{w} - \tilde{\eta}| \quad \text{at} \quad (Z, \tilde{Z}).
\]

Applying Proposition 5.7 to the latter two relations, assertion (1) follows.

To prove assertions (2) and (3), we need the following estimates of Bonami-Charpentier [?, p. 67]:

\[
|1 - \Theta \cdot \overline{Z}| \approx (1 - |Z|^2) + (1 - |\Theta|^2) + |\text{Im } \Theta \cdot \overline{Z}| + |\Theta - Z|^2, \quad (6.1)
\]
and
\[ D(\Theta, Z) \approx (1-|Z|^2)|\Theta - Z|^2 + (|\Theta|^2 - |Z|^2)^2 + |\text{Im } \Theta \cdot \overline{Z}|^2 + |\Theta - Z|^4, \tag{6.2} \]
for every $\Theta, Z \in \mathbb{B}_k$, where $\mathbb{B}_k$ is as usual the euclidean ball of $\mathbb{C}^k$.

Write $Z = (x, z, w) \in \mathbb{B}_N$ and $\tilde{Z} = (\tilde{x}, \tilde{z}, \tilde{w}) \in \mathbb{B}_{|N|}$. Let $\Theta = (\xi, \zeta, \eta) \in \overline{M}_N$ be 1-comparable with $\tilde{\Theta} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in \mathbb{B}_{|N|}$. We break the proof into four cases.

**Case 1:** $\tilde{z} = \Phi^z(z), \tilde{\zeta} = \Phi^z(\zeta)$ and $\tilde{w} = \Phi^w(w), \tilde{\eta} = \Phi^w(\eta)$.

In this case by Theorem 4.1 (i)-(ii), we have that
\[ \zeta \cdot \overline{z} = \tilde{\zeta} \cdot \overline{\tilde{z}}, \quad \eta \cdot \overline{w} = \tilde{\eta} \cdot \overline{\tilde{w}}, \quad \text{and } |z| = |\overline{z}|, |w| = |\overline{w}|. \tag{6.3} \]
We deduce easily from the first two equalities of (6.3) that $|1 - \Theta \cdot \overline{Z}| = |1 - \tilde{\Theta} \cdot \overline{\tilde{Z}}|$, which proves assertion (2).

On the other hand, we have the following identity:
\[ D(\Theta, Z) = (1-|Z|^2)|\Theta - Z|^2 + |\overline{Z} \cdot (\Theta - Z)|^2. \]
By assertion (1), we have $|\Theta - Z|^2 \approx |\tilde{\Theta} - \tilde{Z}|^2$. This, combined with (6.3), implies that $D(\Theta, Z) \approx D(\tilde{\Theta}, \tilde{Z})$, which completes the proof of assertion (3).

**Case 2:** $\tilde{z} = \Phi^z(z), \tilde{\zeta} = \Phi^z(\zeta)$ and $|w - \eta| \geq \frac{|w|}{C_1}, |\tilde{w} - \tilde{\eta}| \geq \frac{|\tilde{w}|}{C_2}$.

In this case it is easy to check that
\[ \max \{|w|, |\eta|\} \lesssim |w - \eta|, \quad \text{and } |\text{Im } \eta \cdot \overline{w}| \lesssim |w - \eta|^2. \tag{6.4} \]
Now we set
\[ Z' := (x, z), \quad \tilde{Z}' := (\tilde{x}, \tilde{z}), \quad \Theta' := (\xi, \zeta), \quad \tilde{\Theta}' := (\tilde{\xi}, \tilde{\zeta}). \]
Combining (6.1),(6.3) and (6.4), we obtain
\[ |1 - \Theta \cdot \overline{Z}| \approx |\eta - w|^2 + (1 - |Z'|^2) + (1 - |\Theta'|^2) \]
\[ + |\text{Im } \Theta' \cdot \overline{Z}'| + |\Theta' - Z'|^2 \approx |\eta - w|^2 + |1 - \Theta' \cdot \overline{Z}'|. \tag{6.5} \]
On the one hand, we have \(|\eta - w|^2 \approx |\tilde{\eta} - \tilde{w}|^2\). On the other hand, proceeding as in the first case, we get \(|1 - \Theta' \cdot \overline{Z}'| = |1 - \tilde{\Theta}' \cdot \overline{\tilde{Z}}'|\). This, combined with (6.5), shows that \(|1 - \Theta \cdot \overline{Z}| \approx |1 - \tilde{\Theta} \cdot \overline{\tilde{Z}}|\), which completes the proof of assertion (2).

We now come to the proof of assertion (3). Applying (6.2), (6.3) and (6.4), we see easily that

\[
D(\Theta, Z) \approx (1 - |Z|^2)|\Theta - Z|^2 + \left( |\Theta'|^2 - |Z'|^2 \right)^2 + |\text{Im } \Theta' \cdot \overline{Z}'|^2 + |\Theta' - Z'|^4 + |\eta - w|^4.
\]

Since \((1 - |Z|^2)|\Theta - Z|^2 = \left(1 - |Z'|^2 - |w|^2\right) \left( |\Theta' - Z'|^2 + |\eta - w|^2\right)\), we obtain

\[
D(\Theta, Z) \approx \left[(1 - |Z'|^2)|\Theta' - Z'|^2 + \left( |\Theta'|^2 - |Z'|^2 \right)^2 + |\text{Im } \Theta' \cdot \overline{Z}'|^2 \right.

+ \left. |\Theta' - Z'|^4 + |\eta - w|^2 \left(1 - |Z'|^2\right) \right].
\]

(6.6)

On the one hand, we have \(|\eta - w|^2 \approx |\tilde{\eta} - \tilde{w}|^2\) and \(1 - |Z'|^2 = 1 - |\overline{Z'}|^2\). On the other hand, proceeding as in the first case, we can show that \(D(\Theta', \overline{Z'}) \approx D\left(\tilde{\Theta}', \overline{\tilde{Z}}\right)\). This, combined with (6.6), shows that \(D(\Theta, Z) \approx D\left(\tilde{\Theta}, \overline{\tilde{Z}}\right)\) and the proof of assertion (3) is thereby completed.

**Case 3:** \(\tilde{w} = \Phi^w(w), \tilde{\eta} = \Phi^w(\eta)\) and \(|z - \zeta| \geq \frac{|z|}{C_1}, |\overline{z - \tilde{\zeta}}| \geq \frac{|\overline{z}|}{C_2}\).

This case can be treated analogously as the previous case.

**Case 4:** \(|z - \zeta| \geq \frac{|z|}{C_1}, |\overline{z - \tilde{\zeta}}| \geq \frac{|\overline{z}|}{C_2}\) and \(|w - \eta| \geq \frac{|w|}{C_1}, |\overline{w - \tilde{\eta}}| \geq \frac{|\overline{w}|}{C_2}\).

We repeat the arguments used in the proof of the second case. More precisely, proceeding as in the proof of (6.5) and (6.6), one can show that

\[
|1 - \Theta' \cdot \overline{Z'}| \approx |\zeta - z|^2 + |1 - \xi \cdot \overline{x}|D(\Theta', Z') \approx D(\xi, x) + |\zeta - z|^2 \left(1 - |x|^2\right).
\]

(6.7)

On the other hand, it is clear that \(|1 - \xi \cdot \overline{x}| = |1 - \tilde{\xi} \cdot \overline{\tilde{x}}|\) and \(D(\xi, x) = D(\tilde{\xi}, \overline{\tilde{x}})\).

These equalities, combined with (6.5), (6.6) and (6.7), imply that \(|1 - \Theta \cdot \overline{Z}| \approx |1 - \tilde{\Theta} \cdot \overline{\tilde{Z}}|\) and \(D(\Theta, Z) \approx D\left(\tilde{\Theta}, \overline{\tilde{Z}}\right)\). The proof of the theorem is complete in this last case. \(\square\)
7. Integral estimates

In this section, we prove, with the help of Theorem 5.6, two important integral estimates that will be used repeatedly throughout the paper.

For every $\lambda > 0$ and $Z \in \mathbb{B}_N$, consider the function

$$K_{\lambda,Z}(\Theta) := \frac{1}{|1 - \Theta \cdot \overline{Z}|^{N+1+\lambda}}, \quad \text{for all } \Theta \in \mathbb{B}_N.$$

The first result of this section is the following

**Theorem 7.1.** — For every $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}_4$ such that $0 \leq \alpha_3, \alpha_4 < 2$ and $0 \leq \alpha_1 + \alpha_3 < 2n$, $0 \leq \alpha_2 + \alpha_4 < 2m$, there exists a constant $C$ independent of $Z \in \mathbb{B}_N$ such that

$$\int_{\mathbb{M}_N} K_{\lambda,Z}(\Theta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} \left|\frac{\zeta}{\zeta_n+1}\right|^{\alpha_3} \left|\frac{\eta}{\eta_{m+1}}\right|^{\alpha_4} dV(\Theta) \leq C (1 - |Z|^2)^{-\left(\frac{\alpha_3 + \alpha_4}{2} + \frac{\alpha_1 + \alpha_2}{2}\right)}.$$

In order to state the second result of this section, we introduce some more notations. Let $\Theta := (\xi, \zeta, \eta)$, $Z := (x, z, w)$ and $Z' := (x', z', w')$ be three points of $\mathbb{M}_N$. Define

$$\Delta(\Theta, Z, Z') := \frac{1}{|\zeta|^2|\eta|^2} \sum_{j=1}^{[N]+2} \left|\frac{B_j(\Theta, Z)}{|\Theta - Z|^{2[N]}} - \frac{B_j(\Theta, Z')}{|\Theta - Z'|^{2[N]}}\right|,$$

where $B_j(\Theta, Z)$ are the polynomials given in the statement of Theorem 3.1.

**Theorem 7.2.** — Let $q$ be a real number such that $1 \leq q < \frac{2|N|+4}{2|N|+3}$. Then we have the estimate

$$\int_{\mathbb{M}_N} \Delta(\Theta, Z, Z')^q \left(\frac{|\zeta||\eta|}{|\zeta_n+1||\eta_{m+1}|}\right)^{2(q-1)} dV(\Theta) \leq \left\{ \begin{array}{ll} |Z - Z'|^{2[N]+4-(2|N|+3)q}, & \text{if } q > 1; \\ |Z - Z'| \log |Z - Z'|, & \text{if } q = 1. \end{array} \right.$$
LEMMA 7.3. — Given $0 < R_1 \leq R_2$, $\alpha < 1$ and $0 \leq \beta, \gamma$, then
\[
\int_0^{R_1} \frac{dx}{x^\alpha (x + R_1)^\beta (x + R_2)^\gamma} \leq C(\alpha, \beta, \gamma) \int_0^{R_2} \frac{dx}{(x + R_1)^{\alpha+\beta} (x + R_2)^\gamma}.
\]

LEMMA 7.4. — Consider $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathbb{R}^5$ such that $0 \leq \alpha_4, \alpha_5 < 2$, and $\alpha_2 < 2n, \alpha_3 < 2m$. For $a := (a_1, a_2) \in \mathbb{C}^2$, we set
\[
I_{a_1, a_2}(\tilde{\Theta}) := \frac{1}{|\tilde{\Theta}|^{\alpha_1} |\tilde{\zeta}|^{\alpha_2} |\tilde{\eta}|^{\alpha_3}} \left| \frac{\tilde{\zeta}}{\zeta_n - a_1} \right|^{\alpha_4} \left| \frac{\tilde{\eta}}{\eta_m - a_2} \right|^{\alpha_5}, \quad \tilde{\Theta} \equiv (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in \mathbb{C}^{|N|}.
\]
Then
\[
\int_{|\tilde{\Theta}| < \delta} I_{a_1, a_2}(\tilde{\Theta}) dV(\tilde{\Theta}) \leq C(N, \alpha) \delta^{2|N|-(\alpha_1+\alpha_2+\alpha_3)}, \quad \text{if } \alpha_1 + \alpha_2 + \alpha_3 < 2|N|;
\]
\[
\int_{\delta < |\tilde{\Theta}|} I_{a_1, a_2}(\tilde{\Theta}) dV(\tilde{\Theta}) \leq C(N, \alpha) \delta^{2|N|-(\alpha_1+\alpha_2+\alpha_3)}, \quad \text{if } \alpha_1 + \alpha_2 + \alpha_3 > 2|N|.
\]

Proof. — Consider the case $a = (0, 0)$. We use integration in polar coordinates and then apply Lemma 7.3 four times to obtain
\[
\int_{|\tilde{\Theta}| < \delta} I_{a_1, a_2}(\tilde{\Theta}) dV(\tilde{\Theta}) \leq \int_{0 < |\tilde{\Theta}| < \delta} \frac{1}{|\tilde{\Theta}|^{\alpha_1+\alpha_2+\alpha_3}} dV(\tilde{\Theta}) \leq C(N, \alpha) \delta^{2|N|-(\alpha_1+\alpha_2+\alpha_3)},
\]
which is the first estimate in the lemma. The second estimate can be proved in the same way. Now we consider the general case $a \in \mathbb{C}^2$. We write $\tilde{\Theta} \equiv (\tilde{\Theta}', \tilde{\eta}_m)$ and observe that if $|\tilde{\Theta}'| < \delta$ and $|\tilde{\eta}_m - a_2| < \frac{|a_2|}{2}$, then we have $|t - a_2| < 3\delta$, for every $t \in \mathbb{C}$ such that $|t - a_2| < \frac{|a_2|}{2}$. In addition, it is clear that
\[
\int_{|t - a_2| < \frac{|a_2|}{2}} \frac{dt \wedge d\tilde{t}}{|t - a_2|^\alpha_5} \leq C(\alpha_5) \int_{|t - a_2| < \frac{|a_2|}{2}} \frac{dt \wedge d\tilde{t}}{|t|^\alpha_5}.
\]
On the other hand, if $|\tilde{\eta}_m - a_2| \geq \frac{|a_2|}{2}$, then $|\tilde{\eta}_m| \leq 3|\tilde{\eta}_m - a_2|$. It follows from these considerations that
\[
\int_{|\tilde{\Theta}| < \delta} I_{a_1, a_2}(\tilde{\Theta}) dV(\tilde{\Theta}) \leq C(\alpha_5) \int_{|\tilde{\Theta}| < 3\delta} \frac{1}{|\tilde{\Theta}|^{\alpha_1} |\tilde{\zeta}|^{\alpha_2} |\tilde{\eta}|^{\alpha_3}} \left| \frac{\tilde{\zeta}}{\zeta_n - a_1} \right|^{\alpha_4} \left| \frac{\tilde{\eta}}{\eta_m} \right|^{\alpha_5}.
\]
The same reasoning, applied to the variables $\tilde{\zeta}$ and $\tilde{\zeta}_n$, shows that

$$\int_{|\tilde{\Theta}|<\delta} I_{\alpha,\alpha}(\tilde{\Theta})dV(\tilde{\Theta}) \leq C(\alpha_4, \alpha_5) \int_{|\tilde{\Theta}|<\delta} I_{\alpha,0}(\tilde{\Theta})dV(\tilde{\Theta}).$$

The proof of the first estimate is now complete. The second estimate can be established in a similar way. \hfill \Box

**Proof of Theorem 7.1.** — Let $\tilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w})$ be a point of $\mathbb{B}_{|N|}$ which is comparable with $Z$. Consider the function

$$K_{\lambda, Z}(\tilde{\Theta}) := \frac{1}{|1 - \tilde{\Theta} \cdot \tilde{Z}|^{N+1+\lambda}}, \quad \text{for all } \tilde{\Theta} \equiv (\tilde{\zeta}, \tilde{\zeta}_n, \tilde{\eta}) \in \mathbb{B}_{|N|}.$$

By Theorem 6.1(2) we have $K_{\lambda, Z} \approx K_{\lambda, \tilde{Z}}$. By Remark 5.2 we obtain $|Z| = |\tilde{Z}|$. Therefore, in view of Theorem 5.6 we see that Theorem 7.1 will follow from the estimate

$$\int_{\mathbb{B}_{|N|}} K_{\lambda, Z}(\tilde{\Theta}) \left(1 + \frac{|\tilde{Z}|}{|\tilde{\zeta}|}\right)^{\alpha_1} \left(1 + \frac{|\tilde{w}|}{|\tilde{\eta}|}\right)^{\alpha_2} \frac{\tilde{\zeta}}{|\tilde{\zeta}_n|} \frac{\tilde{\eta}}{|\tilde{\eta}_m|} dV(\tilde{\Theta}) \leq C \left(1 - |\tilde{Z}|^2\right)^{-(\lambda + \frac{\alpha_1}{2} + \frac{\alpha_2}{4})}. \quad (7.1)$$

Now we go back to the proof of Lemma 1.5 in the work of Bonami-Charpentier [?, p. 68-69]. We may assume without loss of generality that $|\tilde{w}_1| \geq \frac{1}{2\sqrt{|N|}}$ and set

$$\tilde{w}' := (\tilde{w}_2, \ldots, \tilde{w}_m) \quad \text{and} \quad A := 1 - |\tilde{Z}|^2.$$

As in [?, p. 68], observe that

$$u := 1 - |\tilde{\Theta}|^2, \quad v := \text{Im}(\tilde{\Theta} \cdot \tilde{Z}), \quad \tilde{\zeta}, \tilde{\zeta}_n \quad \text{and} \quad \tilde{\eta} := (\tilde{\eta}_2, \ldots, \tilde{\eta}_m)$$

form a system of coordinates whose jacobian is bounded from above and from below by positive constants uniformly in $\tilde{Z} \in \mathbb{B}_{|N|}$ and that satisfies $|\tilde{Z} - \tilde{\Theta}| \leq \frac{1}{4\sqrt{|N|}}$. Using the following estimate (see [?, p. 68]):

$$|1 - \tilde{\Theta} \cdot \tilde{Z}| \approx A + |u| + |v| + |\tilde{x} - \tilde{\xi}|^2 + |\tilde{z} - \tilde{\zeta}|^2 + |\tilde{w}' - \tilde{\eta}'|^2,$$

we see that in order to prove (7.1), it suffices to establish
To finish the proof we only need show that the last two integrals in the last line are finite. But this will follow from the following

**Lemma 7.5.** — For every real numbers $\alpha, \beta, \gamma$ such that $0 < \gamma$, $0 \leq \beta < 2$ and $0 \leq \alpha + \beta < 2n$, there is a constant $C := C(n, \alpha, \beta, \gamma)$ such that

$$A^{\lambda + \frac{3\alpha}{2} + \frac{\alpha}{4}} \int_{C^n} \frac{(1 + \frac{|z|}{|\zeta|})^{\alpha_1} \frac{1}{|\zeta|^m} \frac{1}{|\eta_m|^\alpha}}{A + |u| + |v| + |\bar{z} - \bar{\zeta}|^2 + |\bar{\zeta} - \bar{\zeta}|^2 + |\bar{w}' - \bar{\eta}'|^2} \cdot dudvdV(\zeta)dV(\zeta)dV(\eta') \left|\frac{N+1+\lambda}{N} \right|
$$

$$\leq C(N, \alpha, \lambda) \int_0^\infty \frac{du}{(1 + u)^{1+\frac{\alpha}{2}}} \int_0^\infty \frac{dv}{(1 + |v|)^{1+\frac{\alpha}{2}}} \int_{C^n} \frac{dV(\zeta)}{(1 + |x - \bar{\zeta}|^2)^{1+\frac{\alpha}{2}}}$$

$$\int_{C^n} \frac{(1 + \frac{|z|}{|\zeta|})^{\alpha_1} \frac{1}{|\zeta|^m} \frac{1}{|\eta_m|^\alpha}}{(1 + |z - \zeta|^2)^{n+\frac{\alpha}{2}}} \frac{dV(\zeta)}{C^{n-1}} \int_{C^{n-1}} \frac{1}{|\eta_m|^\alpha} \frac{dV(\eta')}{(1 + |\bar{w}' - \bar{\eta}'|^2)^{m-1+\frac{\alpha}{2}}}$$

To finish the proof we only need show that the last two integrals in the last line are finite. But this will follow from the following

**Lemma 7.5.** — For every real numbers $\alpha, \beta, \gamma$ such that $0 < \gamma$, $0 \leq \beta < 2$ and $0 \leq \alpha + \beta < 2n$, there is a constant $C := C(n, \alpha, \beta, \gamma)$ such that

$$I := \int_{C^n} \frac{(1 + \frac{|z|}{|\zeta|})^{\alpha} \frac{1}{|\zeta|^m} \frac{1}{|z|^\gamma}}{(1 + |z - \zeta|^2)^{n+\gamma}} dV(\zeta) < C, \quad \text{for all } z \in C^n,$$

where $dV$ is the Lebesgue measure of $C^n$.

**Proof.** — Dividing the domain of integration $C^n$ of $I$ into the three subsets $\{|\zeta| < \frac{|z|}{2}\}$, $\{|\zeta| > 2|z|\}$ and $\{|\zeta| \leq |\zeta| \leq 2|z|\}$, we thus divide $I$ into three corresponding terms $I_1, I_2$ and $I_3$. We now estimate each of these terms. On the one hand, we have

$$I_1 \lesssim \int_{\{|\zeta| < \frac{|z|}{2}\}} \frac{|z|^\alpha}{(1 + |z|^2)^{n+\gamma}} dV(\zeta) \lesssim \frac{|z|^\alpha}{(1 + |z|^2)^{n+\gamma}} \cdot |z|^{2n-\alpha-\beta} < C,$$

where the second inequality holds by applying a variant of Lemma 7.4. On the other hand,

$$I_2 \lesssim \int_{\{|\zeta| > 2|z|\}} \frac{dV(\zeta)}{|\zeta|^\beta} \lesssim \int_{|\zeta| < 1} \frac{dV(\zeta)}{|\zeta|^\beta} + \int_{|\zeta| \geq 1} \frac{dV(\zeta)}{|\zeta|^{2n+2\gamma}} |\zeta|^\beta < C,$$
where the last inequality follows from applying twice a variant of Lemma 7.4. Finally,

$$I_3 \leq \int_{\{\frac{1}{2} \leq |\zeta| \leq 2|z|\}} \frac{1}{(1 + |z - \zeta|^2)^{n+\gamma}} dV(\zeta)$$

$$\leq \int_{|z-\zeta|<1} \frac{dV(\zeta)}{|\zeta_n|^{\beta}} + \int_{|z-\zeta|\geq 1} \frac{dV(\zeta)}{|z-\zeta|^{2n+2\gamma}|\zeta_n|^{\beta}} < C,$$

where the last inequality holds by applying twice a variant of Lemma 7.4 and an obvious change of variable. The lemma is now proved. \qed

In order to prove Theorem 7.2, we need the following

**Lemma 7.6.** — There is a constant $C = C(n) > 1$ such that for all points $z, z' \in \mathbb{H}_n$, there is a smooth curve $\gamma = \gamma_{z,z'} : [0,1] \to \mathbb{H}_n$ satisfying

$$\gamma(0) = z, \quad \gamma(1) = z', \quad |\gamma(t)| \leq \max \{|z|, |z'|\}, \quad |\gamma'(t)| \leq C|z - z'|.$$

**Proof.** — Suppose without loss of generality that $|z| \geq |z'|$. We set $\hat{z} := |z'| \frac{z}{|z|}$. Then a little geometric argument shows that $|z' - \hat{z}| \leq |z - z'|$ and $|z - \hat{z}| \leq |z - z'|$. Since the group $SO(n + 1, \mathbb{R})$ acts transitively on $\partial \mathbb{M}_n$, there exists a curve $\gamma_1 : [0,1] \to \mathbb{H}_n$ satisfying

$$\gamma_1(0) = \hat{z}, \quad \gamma_1(1) = z', \quad |\gamma_1(t)| = |z'|, \quad |\gamma_1'(t)| \leq C(n)|z' - \hat{z}|.$$

Define

$$\gamma_2(t) := \begin{cases} (1 - 2t)z + 2tz', & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma_1(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is easy to see that for every $t \neq \frac{1}{2}$, the curve $\gamma_2(t)$ satisfies all the properties stated in the lemma. To conclude the proof, it suffices to approximate in $\mathbb{H}_n$ the curve $\gamma_2$ by a smooth curve $\gamma$. \qed

**Proof of Theorem 7.2.** — We only give the proof for the case $q > 1$. For every points $Z, Z' \in \mathbb{H}_N$, consider the smooth curve

$$\gamma = \gamma_{Z,Z'} := (\gamma_{x,x'}, \gamma_{z,z'}, \gamma_{w,w'}) : [0,1] \to \mathbb{H}_N = \mathbb{H}_1 \times \mathbb{H}_n \times \mathbb{H}_m,$$

where $\gamma_{x,x'}, \gamma_{w,w'}$ are given by Lemma 7.6 and $\gamma_{x,x'}(t) := (1 - t)x + tx'$. Then it follows from Lemma 7.6 that there is a constant $C := C(N)$ such that
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$$\gamma(0) = Z, \  \gamma(1) = Z', \ |\gamma'(t)| \leq C|Z - Z'|,$$

$$|\zeta| \leq \max \{|z|, |z'|\}, \ |\eta| \leq \max \{|w|, |w'|\},$$

where $\Theta \equiv (\xi, \zeta, \eta) = \gamma(t)$. Set

$$E := \left\{ \Theta \in \mathbb{H}_N : |\Theta - Z| \geq 2C|Z' - Z| \right\}.$$

On the one hand, for $\Theta \not\in E$, using the definition of $A$ and Theorem 3.1, we check easily that

$$I_1 := \int_{\mathbb{H}_N \setminus E} \Delta(\Theta, Z, Z') q \left( \frac{|\zeta||\eta|}{|\zeta_{n+1}| |\eta_{m+1}|} \right)^{2(q-1)} dV(\Theta)$$

$$\lesssim \int_{|\Theta - Z| \leq 2C|Z' - Z|} \left( \frac{1 + |z|}{|\Theta - Z|(2|N|-1)q} \right)^{2q} \left( \frac{|\zeta||\eta|}{|\zeta_{n+1}| |\eta_{m+1}|} \right)^{2(q-1)} dV(\Theta)$$

$$+ \int_{|\Theta - Z'| \leq 3C|Z' - Z|} \left( \frac{1 + |z'|}{|\Theta - Z'|(2|N|-1)q} \right)^{2q} \left( \frac{|\zeta||\eta|}{|\zeta_{n+1}| |\eta_{m+1}|} \right)^{2(q-1)} dV(\Theta)$$

$$= I_{11} + I_{12}.$$ 

To estimate $I_{11}$ and $I_{12}$, it suffices to apply part 2) of Theorem 5.6 with $i = 2$ and Theorem 6.6 (1). This can be reduced to majorizing $I_{11}$ and $I_{12}$ by

$$\int_{|\tilde{\Theta}| < C|Z - Z'|} I_{a, a}(\tilde{\Theta}) dV(\tilde{\Theta}),$$

where

$$\alpha := ((2|N|-1)q, 2(q-1), 2(q-1), 2(q-1), 2(q-1)).$$

An application of Lemma 7.4 shows that the latter integral is bounded from above by $C|Z - Z'|^{2|N|+4-(2|N|+3)q}$. Hence

$$I_1 \lesssim |Z - Z'|^{2|N|+4-(2|N|+3)q}. \quad (7.2)$$

On the other hand, if $\Theta \in E$, then for every $0 \leq t \leq 1$ and $\gamma := \gamma_{Z, Z'}$, we have that $|\gamma(t) - \Theta| \approx |\Theta - Z|$. Therefore, using the explicit formula of $B_j(\Theta, Z)$ and taking into account the properties of the curve $\gamma$ stated at the beginning of the proof, the Mean Value Theorem, applied to the functions of variable $Z : \frac{B_j(\Theta, Z)}{|\Theta - Z|(2|N|)}$, shows that
Proceeding exactly as in estimating $I_{11}$ and $I_{12}$, we get

$$I_{21} = |Z - Z'|^q \int_E \frac{(1 + \frac{|z|}{|z'|} + \frac{|w|}{|w'|})^{3q}}{|\Theta - Z'|^{2|\Theta|^1}} \left( \frac{\frac{|z||\eta|}{\zeta_{n+1}||\eta_{m+1}|}}{||\zeta||^2 + ||\eta||^2} \right)^{2(q-1)} dV(\Theta) \lesssim |Z - Z'|^{2|\Theta|^1 + 4 - (2|\Theta|^1 + 3)q}. \quad (7.4)$$

Also,

$$I_{22} = |Z - Z'|^q \int_E \frac{(1 + \frac{|z|}{|z'|} + \frac{|w|}{|w'|})^{3q}}{|\Theta - Z'|^{2|\Theta|^1}} \left( \frac{\frac{|z||\eta|}{\zeta_{n+1}||\eta_{m+1}|}}{||\zeta||^2 + ||\eta||^2} \right)^{2(q-1)} dV(\Theta) \lesssim |Z - Z'|^{2|\Theta|^1 + 4 - (2|\Theta|^1 + 3)q}. \quad (7.5)$$

Therefore, it follows from (7.3), (7.4) and (7.5) that

$$\int_E \Delta(\Theta, Z, Z')^q \left( \frac{\frac{|z||\eta|}{\zeta_{n+1}||\eta_{m+1}|}}{||\zeta||^2 + ||\eta||^2} \right)^{2(q-1)} dV(\Theta) \lesssim I_{21} + I_{22} \lesssim |Z - Z'|^{2|\Theta|^1 + 4 - (2|\Theta|^1 + 3)q}.$$

This, combined with estimate (7.2), completes the proof of the theorem.

\[\square\]

8. Lipschitz estimates on the complex manifold $\mathbb{M}_N$

Let $u$ be a function in $C^1(\mathbb{M}_N)$. For every $Z \in \mathbb{M}_N$, define

$$(\text{grad}_{\mathbb{M}_N} u)(Z) := \sup |(f \circ \gamma)'(0)|,$$

where the supremum is taken over all smooth curves $\gamma : [0, 1] \rightarrow \mathbb{M}_N$ such that $\gamma(0) = Z$ and $|\gamma'(t)| \leq 1$. 

\[\rightarrow - 222 -\]
We begin this section with the following Hardy-Littlewood type lemma.

**Lemma 8.1.** For every $0 < \alpha \leq 1$, there exists a constant $C = C(N, \alpha)$ with the following property: Suppose $u \in C^1(M_N)$ and $K$ is some finite constant such that

$$(\text{grad}_{M_N} u) (Z) \leq K (1 - |Z|)^{\alpha - 1} \quad \text{for all } Z \in M_N.$$ 

Then $|u(Z) - u(Z')| \leq CK|Z - Z'|^{\alpha}$ for all $Z, Z' \in M_N$.

**Proof.** First we make the following remark:

- **If** $|X'| \leq |X|$, by noticing that $|(X', Y)| \leq |Z|$, then we write

\[ |u(Z) - u(Z')| = |u(X, Y) - u(X', Y')| \leq |u(X, Y) - u(X', Y)| + |u(X', Y) - u(X', Y')|. \]

- **If** $|Y'| \leq |Y|$, by noticing that $|(X, Y')| \leq |Z|$, then we write

\[ |u(Z) - u(Z')| = |u(X, Y) - u(X', Y')| \leq |u(X, Y) - u(X', Y')| + |u(X, Y') - u(X', Y')|. \]

Let $Z, Z'$ be two points of $M_N$ such that $0 < |Z'| \leq |Z| < 1$ and set $\delta := |Z - Z'|$.

First assume that $\delta < 1 - |Z|$. Applying the previous remark three times, we only need prove the lemma in one of the following three cases:

1) $x = x', z = z'$;  
2) $x = x', w = w'$;  
3) $z = z', w = w'$.

Suppose for example we are in the first case $x = x', z = z'$. In this case, take the curve $\gamma = \gamma_{Z, Z'}$. According to the hypothesis of the lemma and the properties of the curve $\gamma$ given in the proof of Theorem 7.2, we have

$$(\text{grad}_{M_N} u) (\Theta) \leq K\delta^{\alpha - 1}, \quad \text{for all } \Theta \in \gamma([0, 1]).$$

Therefore,

$$|u(Z) - u(Z')| \leq CK\delta^{\alpha - 1}|Z - Z'| = CK|Z - Z'|^{\alpha}.$$
The remaining cases $1 - |Z| < \delta \leq 1 - |Z'|$ and $1 - |Z'| < \delta$ can be checked using the same argument as in Lemma 6.4.8 of [?].

In order to state the main result of this section, we consider, for $1 \leq p < \infty$, the space

$$L^p(M_N) := \left\{ f : \left( \int_{M_N} |f(\Theta)|^p \frac{|\kappa_{n+1}|^2 \eta_{n+1}^2}{|\zeta|^2 |\eta|^2} dV(\Theta) \right)^{\frac{1}{p}} \equiv \|f\|_{M_N,p} < \infty \right\}.$$ 

If $f := \sum_{j=1}^{N+2} f_j d\Theta_j$ is a $(0,1)$-form defined in a neighborhood of $M_N$ in $\mathbb{B}_N$, we set

$$\|f\|_{M_N,p} := \sum_{j=1}^{N+2} \|f_j\|_{M_N,p}. \tag{8.1}$$

Recall that the norm $\| \cdot \|_{M_N,\infty}$ was defined by formula (2.8).

Next, for every $0 < \alpha < \beta \leq 1$ and for $X = M_N$ or $\partial M_N$, we define

$$\Gamma_{\alpha,\beta}(X) := \left\{ f : \|f\|_{\Lambda_\alpha(X)} + \sup_{\gamma \in c^2(\mathbb{B}_N)} \|f \circ \gamma\|_{\Lambda_\beta([0,1])} \right\},$$

$$\equiv \|f\|_{\Gamma_{\alpha,\beta}(X)} < \infty \right\}.$$ 

We can say informally that $\Gamma_{\alpha,\beta}(X)$ is the trace of the non-isotropic Lipchitz space $\Gamma_{\alpha,\beta}(\mathbb{B}_N)$ (see Definition 1.1 in Krantz [?]) on the manifold $X$.

**Theorem 8.2.** — Suppose that $u \in C^1(M_N)$ and consider a $(0,1)$-form

$$f := \begin{cases} \sum_{k=1}^{N+2} f_k d\Theta_k, & \text{if } N \neq (0,2,2); \\
 f_1 d\zeta_1 + f_2 d\zeta_2 + f_3 d\eta_1 + f_4 d\eta_2, & \text{if } N = (0,2,2); \\
\end{cases}$$

with coefficients in $C(M_N)$ such that $\overline{\partial} u = f|_{M_N}$ on $M_N$. Define $T_1 f$ on $\partial M_N$ as follows:

- for $N \neq (0,2,2),$

$$(T_1 f)(Z) := \int_{M_N} \sum_{k=1}^{N+2} \left[ \frac{(1 - \Theta \cdot \overline{Z}) P_k(\Theta, Z) + (1 - |\Theta|^2) Q_k(\Theta, Z)}{(1 - \Theta \cdot \overline{Z})^{N+1}(1 - \Theta \cdot \overline{Z})^2} \right] f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2};$$
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• for $N = (0, 2, 2)$,

$$(T_1 f)(Z) := \int_{\mathbb{M}_N} \sum_{j=1}^{2} \sum_{k=1}^{4} \frac{(1 - \Theta \cdot \bar{Z})^{1+j}}{|1 - \Theta \cdot \bar{Z}|^8} \cdot [(1 - \Theta \cdot \bar{Z}) P_j k(\Theta, Z) + (1 - |\Theta|^2) Q_j k(\Theta, Z)] f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2},$$

where $P_k, Q_k$ and $P_{jk}, Q_{jk}$ are the polynomials given by Theorems 3.3 and 3.6.

Then the definition of $T_1 f$ can be extended to $\mathbb{M}_N$ by setting

$$(T_1 f)(Z) := J_1(Z) + J_2(Z), \quad (8.2)$$

where

$$J_1(z) := \int_{\partial \mathbb{M}_N} \frac{A(\Theta, Z)}{|Z - \Theta|^2 |N|} (T_1 f)(\Theta) \frac{d\sigma(\Theta)}{|z|^2 |\eta|},$$

$$J_2(z) := \int_{\mathbb{M}_N} \frac{1}{|Z - \Theta|^2 |N|} \left( \sum_{k=1}^{[N]+2} B_k(\Theta, Z) f_k(\Theta) \right) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2}$$

and the operator $T_1 f$ satisfies

(i) $\bar{\partial}_{\mathbb{M}_N} (T_1 f) = f|_{\mathbb{M}_N}$.

Moreover, for every $p > 0$, we set (as in the statement of Theorem 1.1):

$$\alpha = \alpha(N, p) := \begin{cases} \frac{1}{2} - \frac{|N|+3}{p}, & \text{if } N \neq (0, 2, 2) \text{ and } p > 2(|N| + 3); \\ \frac{1}{2} - \frac{6}{p}, & \text{if } N = (0, 2, 2) \text{ and } p > 12. \end{cases}$$

Then there exists a constant $C := C(N, p)$ such that

(ii) $T_1 f|_{\partial \mathbb{M}_N} \in \Gamma_{\alpha, 2\alpha}(\partial \mathbb{M}_N)$ and $\|T_1 f|_{\partial \mathbb{M}_N}\|_{\Gamma_{\alpha, 2\alpha}(\partial \mathbb{M}_N)} \leq C \|f\|_{\mathbb{M}_N, p}$, if $p < \infty$; $T_1 f|_{\partial \mathbb{M}_N} \in \Gamma_{1, 1}(\partial \mathbb{M}_N)$ and $\|T_1 f|_{\partial \mathbb{M}_N}\|_{\Gamma_{1, 1}(\partial \mathbb{M}_N)} \leq C \|f\|_{\mathbb{M}_N, \infty}$, if $p = \infty$;

(iii) $T_1 f \in \Lambda_{\alpha}(\mathbb{M}_N)$ and $\|T_1 f\|_{\Lambda_{\alpha}(\mathbb{M}_N)} \leq C \|f\|_{\mathbb{M}_N, p}$.

Proof. — We only give the proof in the case $N \neq (0, 2, 2)$ and $p < \infty$. The first remaining case $N = (0, 2, 2)$ can be proved in exactly the same way by applying Theorem 3.6 instead of Theorem 3.3. The second remaining case $p = \infty$ follows essentially along the same lines as in our previous work [?] basing on the work of Greiner-Stein [??].
We first introduce two new integral operators $T_2$ and $T_3$:

$$(T_2 f)(Z) := \int_{M_N} \sum_{k=1}^{[N]+2} \frac{(1 - \Theta \cdot \overline{Z})} {D(\Theta, Z)^{[N]}} \cdot [(1 - \Theta \cdot \overline{Z})P_k(\Theta, Z)$$

$$+(1 - |\Theta|^2)Q_k(\Theta, Z)]f_k(\Theta) \frac{dV(\Theta)} {(|\zeta|^2|\eta|^2)},$$

$$(T_3 f)(Z) := \int_{M_N} \sum_{k=1}^{[N]+2} \left[ \frac{(1 - \Theta \cdot \overline{Z})P_k(\Theta, Z) + (1 - |\Theta|^2)Q_k(\Theta, Z)} {(1 - \Theta \cdot Z)^{[N]}(1 - \Theta \cdot \overline{Z})^2} \right]f_k(\Theta) \frac{dV(\Theta)} {(|\zeta|^2|\eta|^2)},$$

for all $Z \in \mathbb{B}_N$.

Applying Theorem 3.3 to the function $u$ gives that

$$(T_2 f)(Z) = u(Z) - \int_{\partial M_N} \frac{R(\Theta, Z)} {1 - Z \cdot \overline{\Theta}}u(\Theta) \frac{d\sigma(\Theta)} {(|\zeta|^2|\eta|^2)},$$

for all $Z \in \mathbb{M}_N$. (8.3)

Moreover, we note that

$$(T_1 f)(Z) = (T_2 f)(Z) = (T_3 f)(Z), \quad \text{for all } Z \in \partial M_N. \quad (8.4)$$

Arguing as in the proof of Lemma 3.5 in [?] and using Theorems 5.6, 6.1 and 7.1, one can show that

$$\lim_{r \to 1^-} \int_{\partial M_N} |(T_2 f)(\Theta) - (T_2 f)(r\Theta)|d\sigma(\Theta) = 0.$$ 

Therefore, in view of Remark 3.2, we can apply Theorem 3.1 to the function $T_2 f$. Next observe that (8.2) is just the Martinelli-Bochner formula. Hence by virtue of (8.3), the hypothesis and the fact that $R(\Theta, Z)$ is holomorphic in the variable $Z$, we obtain

$$T_1 f = T_2 f|_{M_N} \quad \text{and} \quad \overline{\partial}_{M_N}(T_1 f) = \overline{\partial}_{M_N} u = f|_{M_N}. \quad (8.5)$$

This completes the proof of assertion (i). In view of (8.4), assertion (ii) will follow from the following lemma.

**Lemma 8.3.**

$$
\begin{cases}
\|T_3 f\|_{M_N} \|_{L^{1,2a}(M_N)} \leq C\|f\|_{M_N,p} \quad \text{if} \quad p < \infty; \\
\|T_3 f\|_{M_N} \|_{L^{1,1}(M_N)} \leq C\|f\|_{M_N,\infty} \quad \text{if} \quad p = \infty.
\end{cases}
$$
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Proof. — Using the properties of the polynomials $P_k(\Theta, Z)$ and $Q_k(\Theta, Z)$ in Theorem 3.3(ii), we see that

$$|(\text{grad } T_3f)(Z)| \lesssim \sum_{k=1}^{N+2} \int_{\mathcal{M}_N} \frac{|\Theta - Z|}{|1 - \Theta \cdot Z|^{N+1}} \left(1 + \frac{|z|}{|\zeta|} \right) \left(1 + \frac{|w|}{|\eta|} \right) f_k(\Theta) dV(\Theta),$$

where $q$ verifies $p + 1 = 1$. Now applying Theorem 7.1 yields

$$C\|f\|_{M,N,p} \left( \int_{\mathcal{M}_N} \frac{1}{|1 - \Theta \cdot Z|^{(N+2)/2}} \left(1 + \frac{|z|}{|\zeta|} \right)^{q} \left(1 + \frac{|w|}{|\eta|} \right)^{q} dV(\Theta) \right)^{1/q},$$

where $q$ verifies $\frac{1}{p} + \frac{1}{q} = 1$. Now applying Theorem 7.1 yields

$$|(\text{grad } T_3f)(Z)| \leq C\|f\|_{M,N,p} (1 - |Z|)^{-\frac{1}{2} - \frac{N+3}{p}}.$$  \quad (8.6)

so that by the classical Hardy-Littlewood lemma for the euclidean ball $\mathbb{B}_N$ we see that

$$|(T_3f)(Z) - (T_3f)(Z')| \leq C\|f\|_{M,N,p} |Z - Z'|^{\frac{1}{2} - \frac{N+3}{p}}, \quad \text{for all } Z, Z' \in \mathbb{B}_N.$$

Therefore, choosing $Z' = 0$, we obtain

$$\|T_3f\|_{L^\infty(\mathbb{B}_N)} \leq |(T_3f)(0)| + C\|f\|_{M,N,p} \leq C\|f\|_{M,N,p}. \quad (8.7)$$

For every $u \in C^1(\mathbb{B}_N)$, set

$$(\text{grad}^t u)(Z) := \sup_{\gamma \in C^1_\gamma(\mathbb{B}_N): \gamma(0) = Z} |(u \circ \gamma)'(0)|.$$

By the proof Lemma 4.8 in [7], we see that $(\text{grad}^t |1 - \Theta \cdot Z|)(Z) \leq C|\Theta - Z|$. Therefore, a straightforward calculation shows that
Hence, arguing as in the proof of (8.6), we see that

\[
(\text{grad}^t T_3 f)(Z) \leq \sum_{k=1}^{[N]+2} \int_{M_N} \frac{|\Theta - Z|^2}{|1 - \Theta \cdot \overline{Z}|^{[N]+2}} \left(1 + \frac{|z|}{|\zeta|} \right) \left(1 + \frac{|w|}{|\eta|} \right) f_k(\Theta) dV(\Theta)
\]

\[
+ \sum_{k=1}^{[N]+2} \int_{M_N} \frac{1}{|1 - \Theta \cdot \overline{Z}|^{[N]+1}} \left(1 + \frac{|z|}{|\zeta|} \right) \left(1 + \frac{|w|}{|\eta|} \right) f_k(\Theta) dV(\Theta).
\]

Hence, arguing as in the proof of (8.6), we see that

\[
(\text{grad}^t T_3 f)(Z) \leq C\|f\|_{M_N,p} (1 - |Z|)^{-\frac{[N]+3}{p}}. \quad (8.8)
\]

Combining (8.6), (8.7) and (8.8), the lemma follows from Lemma 4.7 in [?]. \(\square\)

To prove assertion (iii), we need the following

**Lemma 8.4.**

\[|J_1(Z) - J_1(Z')| \leq C\|f\|_{M_N,p} |Z - Z'|^{\frac{1}{2} - \frac{[N]+3}{p}}, \quad \text{for all } Z, Z' \in M_N.\]

**Proof.** Observe that the polynomial \(A(\Theta, Z)\) satisfies

\[
\left| \frac{1}{|\zeta|^2|\eta|^2} \left( \text{grad}_Z \frac{A(\Theta, Z)}{|\Theta - Z|^{2[N]}} \right) \right| \lesssim \frac{1}{|\Theta - Z|^{2[N]}} \left(1 + \frac{|z|}{|\zeta|} \right) \left(1 + \frac{|w|}{|\eta|} \right) + \frac{1}{|\zeta|^2|\eta|^2} \left(1 + \frac{1}{|\zeta|} \right) \left(1 + \frac{1}{|\zeta|} \right) \left(1 + \frac{|z|}{|\zeta|} \right) + \frac{1}{|\zeta|^2|\eta|^2} \left(1 + \frac{1}{|\zeta|} \right) \left(1 + \frac{1}{|\zeta|} \right).
\]

In addition, if we set \(u \equiv 1\) in Theorem 3.1, then we see that

\[
\int_{\partial M_N} \frac{A(\Theta, Z)}{|Z - \Theta|^{2[N]} |\zeta|^2|\eta|^2} d\sigma(\Theta) = 1.
\]

Setting \(Z := rZ'\), \(Z' \in \partial M_N\), this implies that

\[
(\text{grad}_{M_N} J_1)(Z) \lesssim \int_{\partial M_N} \left| \text{grad}_Z \frac{A(\Theta, Z)}{|\Theta - Z|^{2[N]}} \right| |(T_1 f)(\Theta) - (T_1 f)(Z')| \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2}.
\]
Combining (8.4) and Lemma 8.3, we obtain
\[
| (T_1 f)(\Theta) - (T_1 f)(Z') | \leq C\| f \|_{\mathcal{M}_{N,p}} | \Theta - Z' |^{1/2 - \frac{|N|+3}{p}}.
\]
Hence,
\[
\text{grad}_{\mathcal{M}_N} J_1 \ (Z) \leq C\| f \|_{\mathcal{M}_{N,p}} \left( \int_{\partial \mathcal{M}_N} \left( 1 + \frac{|z'|}{|z|} \right) \left( 1 + \frac{|w'|}{|\eta|} \right) \frac{|\Theta - Z'|^{3/2 - \frac{|N|+3}{p}}}{|\Theta - rZ'|^{2|N|-1}} d\sigma(\Theta) \right).
\]
We shall establish in Proposition 9.4 below that the latter three integrals are dominated by \( C \ (1 - |Z|)^{-\frac{1}{2} - \frac{|N|+3}{p}}. \) Taking for granted Proposition 9.4, it follows that
\[
\text{grad}_{\mathcal{M}_N} J_1 \ (Z) \leq C\| f \|_{\mathcal{M}_{N,p}} (1 - |Z|)^{-\frac{1}{2} - \frac{|N|+3}{p}}.
\]
Finally, applying Lemma 8.1 to this gives the desired conclusion. \( \Box \)

We now complete the proof of Theorem 8.2. By H"older's inequality and Theorem 7.2, we have
\[
| J_2 (Z) - J_2 (Z') | \leq C\| f \|_{\mathcal{M}_{N,p}} |Z - Z'|^{1 - \frac{2|N|+4}{p}}, \quad \text{for all } Z, Z' \in \mathcal{M}_N.
\]
This, combined with Lemma 8.4 gives that
\[
| (T_1 f)(Z) - (T_1 f)(Z') | \leq C\| f \|_{\mathcal{M}_{N,p}} |Z - Z'|^{1/2 - \frac{|N|+3}{p}}, \quad \text{for all } Z, Z' \in \mathcal{M}_N.
\]
Arguing as in the proof of (8.7), one can show that
\[
\| T_1 f \|_{L^\infty(\mathcal{M}_N)} \leq C\| f \|_{\mathcal{M}_{N,p}}.
\]
This proves assertion (iii). \( \Box \)
9. A Stokes type theorem on the manifold $\mathbb{M}_N$ and applications

The main result of this section is the following Stokes type theorem:

**Theorem 9.1.** — Consider for every function $v \in C^1(\mathbb{M}_N)$ and every real numbers $\lambda < 2n - 1$ and $\mu < 2m - 1$, the function $u$ given by $u(\Theta) := \frac{v(\Theta)}{|\xi|^\lambda |\eta|^\mu}$, for $\Theta \in \mathbb{M}_N$. Then there is a constant $C := C(N)$ such that

$$\int_{\partial \mathbb{M}_N} u d\sigma \leq C \int_{\mathbb{M}_N} \left( |\xi| |\text{grad}_\xi u(\Theta)| + |\xi| |\text{grad}_\eta u(\Theta)| + |\eta| |\text{grad}_\eta u(\Theta)| + |u(\Theta)| \right) dV(\Theta).$$

**Remark 9.2.** — We do not know whether it is possible to establish a theorem of reduction of estimates from $\partial \mathbb{M}_N$ to $\partial \mathbb{B}_{|N|}$, similar to Theorem 5.6. To overcome this, we use Theorem 9.1 to estimate difficult integrals taken over $\partial \mathbb{M}_N$ by simpler ones taken over $\mathbb{M}_N$ and then apply Theorem 5.6. We have already encountered this type of integral estimates in the proof of Lemma 8.4.

**Proof.** — Set $d\xi := d\xi_1 \wedge \ldots \wedge d\xi_l$ and

$$\alpha_n(\zeta) := \frac{1}{n+1} \sum_{j=1}^{n+1} (-1)^{j-1} \frac{d\zeta_1 \wedge \ldots \wedge \widehat{d\zeta_j} \wedge \ldots \wedge d\zeta_{n+1}}{\zeta_j},$$

$$\alpha_m(\eta) := \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^{k-1} \frac{d\eta_1 \wedge \ldots \wedge \widehat{d\eta_k} \wedge \ldots \wedge d\eta_{m+1}}{\eta_k}.$$ 

By Proposition 2.1 in [?] and Proposition 2.1 above, we see that

$$dV_l(\xi) = Cd\xi \wedge d\xi,$$

$$dV_n(\zeta) = C|\zeta|^2 \alpha_n(\zeta) \wedge \alpha_n(\zeta)|_{\mathbb{H}_n}^2, dV_m(\eta) = C|\eta|^2 \alpha_m(\eta) \wedge \alpha_m(\eta)|_{\mathbb{H}_m};$$

$$dV(\Theta) = dV_l(\xi) \wedge dV_n(\zeta) \wedge dV_m(\eta). \quad (9.1)$$

Next, put

$$\omega_j(\xi) := d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_l, \quad (1 \leq j \leq l);$$

$$\omega_k(\zeta) := d\zeta_1 \wedge \ldots \wedge \widehat{d\zeta_k} \wedge \ldots \wedge d\zeta_{n+1}, \quad (1 \leq k \leq n+1);$$

$$\omega_p(\eta) := d\eta_1 \wedge \ldots \wedge \widehat{d\eta_p} \wedge \ldots \wedge d\eta_{m+1}, \quad (1 \leq p \leq m+1).$$
Finally, we define $\omega_{jk}(\zeta), \tilde{\omega}_{jk}(\zeta)$ ($1 \leq j, k \leq n + 1$) and $\omega_{pq}(\eta), \tilde{\omega}_{pq}(\eta)$ ($1 \leq p, q \leq m + 1$) in just the same way as $\omega_{pj}(z), \tilde{\omega}_{jk}(z)$ in [?, p. 507–508].

Consider the mapping $g : [0, +\infty[ \times \mathbb{C}^{N+2} \to \mathbb{C}^{N+2}$ given by

$$g(t, \Theta) := t\Theta.$$  

Using (9.1) and proceeding as in the proof of Lemma 2.1 in [?], we see that

$$\begin{align*}
(g^*dV)(t, \Theta) &= t^{2|N|-1} dt \wedge [I_\xi \wedge dV_n(\zeta) \wedge dV_m(\eta) + I_\zeta \wedge dV_1(\xi) \wedge dV_m(\eta) \\
&\quad + I_\eta \wedge dV_1(\xi) \wedge dV_n(\zeta)] + t^{2|N|}dV(\Theta),
\end{align*}$$  

(9.2)

where

$$
I_\xi := C \sum_{p=1}^l (-1)^{p-1} (\xi_p d\xi \wedge \omega_p(\xi) + \xi_p \omega_p(\xi) \wedge d\xi),
$$

$$
I_\zeta := C|\zeta|^2 \sum_{j,k=1}^{n+1} \frac{(-1)^{j+k}}{\zeta_j \zeta_k} \tilde{\omega}_{jk}(\zeta), \quad \text{and} \quad I_\eta := C|\eta|^2 \sum_{p,q=1}^{m+1} \frac{(-1)^{p+q}}{\eta_p \eta_q} \tilde{\omega}_{pq}(\eta).
$$

Now set

$$
\omega(\Theta) := I_\xi \wedge dV_n(\zeta) \wedge dV_m(\eta) + I_\zeta \wedge dV_1(\xi) \wedge dV_m(\eta) + I_\eta \wedge dV_1(\xi) \wedge dV_n(\zeta).
$$

(9.3)

Since $g$ is a diffeomorphism from $[0, +\infty[ \times \partial M_N \to \mathbb{H}_N$, it follows from (9.2) and (9.3) that

$$
\int_{\mathbb{H}_N} u(\Theta)dV(\Theta) = \int_0^{\infty} t^{2|N|-1} \int_{\partial M_N} u(t\Theta)\omega(\Theta), \quad \text{for all } u \in C_0(\mathbb{H}_N),
$$

so that by Lemma 2.3, we obtain $d\sigma = C\omega|_{\partial M_N}$. Therefore, since by the hypothesis $\lambda < 2n - 1, \mu < 2m - 1$, $|u(\Theta)| \lesssim \frac{1}{|\xi|^\lambda |\eta|^\mu}$ for all $\Theta \in M_N$, the homogeneity properties of the differential form $\omega(\Theta)$ and the same arguments as in the proof of (2.12), (2.13) and (2.17) of Proposition 2.5 imply that

$$
\int_{\partial M_N} u d\sigma = \lim_{r \to 0} \int_{\partial M_r} u \omega.
$$

(9.4)

Stokes theorem gives that

$$
\int_{\partial M_r} u \omega = \int_{M_r} du \wedge \omega + \int_{M_r} u d\omega.
$$

(9.5)
We shall estimate \( \int_{\mathbb{M}_r} du \wedge \omega \). Let \( Z \) be a point of \( \mathbb{M}_r \). Choose \( j \) and \( k \) with \( 1 \leq j \leq n + 1, 1 \leq k \leq m + 1 \) so that in a sufficiently small neighborhood \( \mathcal{U} := \mathcal{U}(Z) \) in \( \mathbb{M}_r \), we have

\[
|\zeta_j| \geq \frac{1}{2} \max_{p \neq j} |\zeta_p| \quad \text{and} \quad |\eta_k| \geq \frac{1}{2} \max_{q \neq k} |\eta_q|.
\]

By (9.3) and (9.5), we obtain

\[
\left| \int_{\mathcal{U}} du \wedge \omega \right| \leq \left| \int_{\mathcal{U}} du \wedge I_\xi \wedge dV_n(\zeta) \wedge dV_m(\eta) \right| + \left| \int_{\mathcal{U}} du \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta) \right| + \left| \int_{\mathcal{U}} du \wedge I_\eta \wedge dV_l(\xi) \wedge dV_n(\zeta) \right|. \tag{9.7}
\]

We shall estimate for example \( \left| \int_{\mathcal{U}} du \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta) \right| \). It should be noted that the following identity is implicit in the proof of Lemma 2.1 of \([\ast]\): 

\[
\frac{1}{(n+1)^2} I_\zeta \bigg|_{\mathbb{H}_n} = \frac{(-1)^{j+k} |\zeta|^2}{\zeta_j \zeta_k} \bar{\omega}_{jk}(\zeta) \bigg|_{\mathbb{H}_n}, \quad \text{for} \ 1 \leq j, k \leq n + 1.
\]

Therefore, \( I_\zeta \big|_{\mathbb{H}_n} \) is equal to

\[
C \frac{|\zeta|^2}{|\zeta_j|^2} \bar{\omega}_{jj}(\zeta) \bigg|_{\mathbb{H}_n} = |\zeta|^2 \sum_{p=1}^{j-1} \frac{(-1)^{p+1} \zeta_p}{\zeta_j} \omega_{pj}(\zeta) \wedge \alpha_n(\bar{\zeta}) + |\zeta|^2 \sum_{q=1}^{j-1} \frac{(-1)^{n+q-1} \zeta_p}{\zeta_j} \omega_{qj}(\bar{\zeta}) \wedge \alpha_n(\zeta) + |\zeta|^2 \sum_{q=1}^{j+1} \frac{(-1)^{n+q+1} \zeta_p}{\zeta_j} \omega_{qj}(\bar{\zeta}) \wedge \alpha_n(\zeta).
\]

Combining the identity \( \alpha_n(\zeta) = (n+1) \frac{(-1)^{p-1}}{\zeta_p} \omega_p(\zeta), \ 1 \leq p \leq n+1 \), (see formula (2.6) in \([\ast]\)) and formula (9.1), a straightforward calculation gives that
By virtue of (9.6), we majorize easily the latter integral and obtain
\[
\int_{\tilde{U}} du \wedge I_\zeta \wedge dV_1(\xi) \wedge dV_m(\eta)
\]
\[
= \int_{\tilde{U}} d_\zeta u \wedge I_\zeta \wedge dV_1(\xi) \wedge dV_m(\eta)
\]
\[
= C \int_{\tilde{U}} \sum_{\substack{p=1, p \neq j \leq n+1}} \left( \frac{\partial u}{\partial \zeta_j} (-1)^p \zeta_p + \frac{\partial u}{\partial \zeta_p} (-1)^{p-1} \frac{\zeta_j^2}{\zeta_p} + \frac{\partial u}{\partial \zeta_j} (-1)^{p+n} \frac{\zeta_j^2}{\zeta_p} \right) dV(\Theta).
\]
By virtue of (9.6), we majorize easily the latter integral and obtain
\[
\left| \int_{\tilde{U}} du \wedge I_\zeta \wedge dV_1(\xi) \wedge dV_m(\eta) \right| \leq C(N) \int_{\tilde{U}} |\zeta||\text{grad}_\zeta u(\Theta)|dV(\Theta).
\]
Hence, in view of (9.7), it follows that
\[
\left| \int_{\tilde{U}} du \wedge \omega \right| & \leq C(N) \int_{\tilde{U}} (|\xi||\text{grad}_\xi u(\Theta)| + |\zeta||\text{grad}_\zeta u(\Theta)|
\]
\[
+ |\eta||\text{grad}_\eta u(\Theta)|dV(\Theta).
\]
On the other hand, we can prove in just the same way that
\[
\left| \int_{\tilde{U}} ud\omega \right| \leq C(N) \int_{\tilde{U}} |u|dV.
\]
These two estimates, combined with (9.4) and (9.5), complete the proof. \( \square \)

We now present two applications of Theorem 9.1.

**Proposition 9.3.** Let \( \lambda, \alpha_1, \alpha_2 \) be real numbers such that \( 0 < \lambda < 1 \), and \( 0 \leq \alpha_1 < 2n, 0 \leq \alpha_2 < 2m \). Then there exists a constant \( C := C(N, \lambda, \alpha_1, \alpha_2) \) such that for every \( Z \in \mathbb{B}_N \),
\[
\int_{\partial^N_M} \frac{1}{|1 - Z \cdot \Theta|^{N+1-\lambda}} \left( 1 + \frac{|\zeta|}{|\xi|} \right)^{\alpha_1} \left( 1 + \frac{|w|}{|\eta|} \right)^{\alpha_2} d\sigma(\Theta) \leq C \left( 1 - |Z|^2 \right)^{\lambda-1}.
\]
Proof. — Applying Theorem 9.1 gives that

$$\int_{\partial \mathbb{M}_N} \frac{1}{|1 - Z \cdot \Theta|^{N+1-\lambda}} \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} \, d\sigma(\Theta)$$

$$\leq C \int_{\mathbb{M}_N} \frac{1}{|1 - Z \cdot \Theta|^{N+2-\lambda}} \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} \, dV(\Theta)$$

$$\leq C \left(1 - |Z|^2\right)^{\lambda-1},$$

where the latter inequality follows from Theorem 7.1. □

The following proposition completes the missing point in the proof of Lemma 8.4 on page 51.

**Proposition 9.4.** — Suppose that $0 < \lambda < 1$. Then there is a constant $C := C(N, \lambda)$ such that for every $0 < r < 1$ and $Z \in \partial \mathbb{M}_N$,

$$I_1 := \int_{\partial \mathbb{M}_N} \frac{\left(1 + \frac{|z|}{|\zeta|}\right) \left(1 + \frac{|w|}{|\eta|}\right)}{|\Theta - rZ|^{2|N|}} |\Theta - Z|^\lambda d\sigma(\Theta) \leq C (1 - r)^{\lambda-1},$$

$$I_2 := \int_{\partial \mathbb{M}_N} \frac{\frac{1}{|\zeta|} \left(1 + \frac{|w|}{|\eta|}\right)}{|\Theta - rZ|^{2|N|-1}} |\Theta - Z|^\lambda d\sigma(\Theta) \leq C (1 - r)^{\lambda-1}.$$

**Proof.** — We only give the proof of the estimate for $I_2$. Starting from the elementary estimate $|\Theta - rZ| \approx (1 - r) + |\Theta - Z|$ for all $\Theta \in \partial \mathbb{M}_N$, we see that

$$I_2 \leq \int_{\partial \mathbb{M}_N} \frac{\frac{1}{|\zeta|} \left(1 + \frac{|w|}{|\eta|}\right)}{|(1 - r) + |\Theta - Z||^{2|N|-1}} |\Theta - Z|^\lambda d\sigma(\Theta)$$

$$\leq \int_{\mathbb{M}_N} \frac{\frac{1}{|\zeta|} \left(1 + \frac{|w|}{|\eta|}\right)}{|(1 - r) + |\Theta - Z||^{2|N|-1}} |\Theta - Z|^\lambda dV(\Theta)$$

$$\leq \int_{\mathbb{M}_N} \frac{\left(1 + \frac{|z|}{|\zeta|}\right) \left(1 + \frac{|w|}{|\eta|}\right)}{|(1 - r) + |\Theta - Z||^{2|N|-1}} |\Theta - Z|^\lambda dV(\Theta),$$
where the last two estimates follow respectively from Theorem 9.1 and the very elementary inequality \( \frac{1}{|\zeta|} \leq \left( 1 + \frac{|z|}{|\zeta|} \right) \frac{1}{|\zeta - z|} \). Hence, by part 3) of Theorem 5.6, the latter integral is dominated by \( C\tilde{I}_2 \), where

\[
\tilde{I}_2 := \int_{B_{2,|N|}} \frac{|	ilde{\Theta} - \tilde{Z}|^{\lambda-1}}{[1 - r] + |	ilde{\Theta} - \tilde{Z}|^{2|N|-1} |\tilde{\zeta} - \tilde{z}|} \, dV(\tilde{\Theta}).
\]

Dividing the domain of integration of \( \tilde{I}_2 \) into two regions

\[
\tilde{E}_1 := \left\{ \Theta \in B_{2,|N|} : |\tilde{\Theta} - \tilde{Z}| < 1 - r \right\}
\]

and

\[
\tilde{E}_2 := \left\{ \Theta \in B_{2,|N|} : |\tilde{\Theta} - \tilde{Z}| \geq 1 - r \right\},
\]

we thus break \( \tilde{I}_2 \) into two corresponding terms \( \tilde{I}_{21} \) and \( \tilde{I}_{22} \). We then apply Lemma 7.4 to estimate each of these terms and obtain

\[
\tilde{I}_{21} \leq \frac{1}{(1 - r)^{2|N|-1}} \int_{\tilde{E}_1} \frac{|\tilde{\Theta} - \tilde{Z}|^{\lambda-1}}{|\tilde{\zeta} - \tilde{z}|} \, dV(\tilde{\Theta}) \lesssim (1 - r)^{\lambda-1},
\]

\[
\tilde{I}_{22} \lesssim \int_{\tilde{E}_2} \frac{|\tilde{\Theta} - \tilde{Z}|^{\lambda-1}}{|\tilde{\Theta} - \tilde{Z}|^{2|N|-1} |\tilde{\zeta} - \tilde{z}|} \, dV(\tilde{\Theta}) \lesssim (1 - r)^{\lambda-1}.
\]

In summary, we have

\[
I_2 \lesssim \tilde{I}_2 = \tilde{I}_{21} + \tilde{I}_{22} \lesssim C(1 - r)^{\lambda-1},
\]

which completes the proof of the proposition. \( \square \)

10. Proof of the main results

In this section we prove Theorems 1.1 and 1.2. For this purpose, we first establish some preparatory results.

Consider the holomorphic mapping \( F_N : \overline{M}_N \rightarrow \overline{\Omega}_N \setminus \{0\} \) which maps every \( Z \equiv (x, z, w) \equiv (x_1, \ldots, x_l, z_1, \ldots, z_{n+1}, w_1, \ldots, w_{m+1}) \), element of \( \overline{M}_N \) to

\[
F_N(Z) := \tilde{Z} := \left( \frac{x_1}{\sqrt{2}}, \ldots, \frac{x_l}{\sqrt{2}}, \frac{z_1}{\sqrt{2}}, \ldots, \frac{z_n}{\sqrt{2}}, \frac{w_1}{\sqrt{2}}, \ldots, \frac{w_m}{\sqrt{2}} \right).
\]
Recall that \( dV(\Theta) \) is the canonical volume form of \( C^{[N]} \). It follows from formula (5.2) in [?] and formula (9.1) that

\[
\frac{\lvert \zeta_{n+1} \rvert^2 \eta_{n+1}^2}{\lvert \zeta \rvert^2 \eta^2} dV(\Theta) = CF_N^* \left( dV(\Theta) \right), \quad \text{for } \Theta \in \mathbb{M}_N \text{ and } \tilde{\Theta} = F_N(\Theta).
\]  

(10.1)

**Proposition 10.1.** Consider a \( \bar{\partial} \)-closed \((0, 1)\)-form \( f \) of class \( C^1 \) defined in a neighborhood of \( \partial \Omega_N \). Then the solution \( T_1(F_N^* f) \) given by Theorem 8.2 satisfies

\[
(T_1(F_N^* f))(Z) = (T_1(F_N f))(Z'),
\]

for all \( Z, Z' \in \mathbb{M}_N \) such that \( F_N(Z) = F_N(Z') \).

**Proof.** Suppose that \( f \in C^1_{0,1}(r\Omega_N) \) for some \( r > 1 \). Since \( r\Omega_N \) is pseudoconvex, there exists a function \( u \in C^1(\Omega_N) \) such that \( \bar{\partial} u = f \) in \( \Omega_N \). Therefore, it follows from (8.3) and (8.5) that for every \( Z \in \mathbb{M}_N \),

\[
(T_1(F_N^* f))(Z) = (u \circ F_N)(Z) - \int_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{(1 - Z \cdot \Theta)^{\lvert N \rvert}} (u \circ F_N)(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2}.
\]

Using this and the explicit formula of \( R(\Theta, Z) \), we see that the proof follows. \( \square \)

**Theorem 10.2.** For every \( 0 < \lambda \leq \frac{1}{2} \), there is a constant \( C := C(N, \lambda) \) such that

\[
\|u\|_{\Gamma_{\lambda, 2\lambda}(\mathbb{M}_N)} \leq C \|u\|_{\Lambda_{\lambda}(\partial \mathbb{M}_N)}
\]

for all functions \( u \) in \( C(\mathbb{M}_N) \) which are holomorphic in \( \mathbb{M}_N \).

**Proof.** Consider the holomorphic function \( U \in H(\mathbb{B}_N) \) defined by

\[
U(Z) := \int_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{(1 - Z \cdot \Theta)^{\lvert N \rvert}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2}, \quad \text{for all } Z \in \mathbb{B}_N.
\]  

(10.2)

Applying Theorem 3.3 to the function \( u \) yields

\[
U(Z) = u(Z), \quad \text{for all } Z \in \mathbb{M}_N.
\]

This shows that the theorem will follow from the estimate

\[
\|U\|_{\Gamma_{\lambda, 2\lambda}(\mathbb{B}_N)} \leq C \|u\|_{\Lambda_{\lambda}(\partial \mathbb{M}_N)}.
\]  

(10.3)

To prove this, observe by (10.2) and formula (2) in [?, Section 6.4.4] that the radial derivative \((RU)\) of \( U \) is given by

\[
(RU)(Z) =
\]

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Using this and arguing as in the proof of Theorem 6.4.9 of [?], it can be shown that

Therefore, by Proposition 9.3, so that by Theorem 6.4.10 of [?], inequality (10.3) follows and the proof is now complete. D

**THEOREM 10.3.** — For every $(0,1)$-form $f$ and real numbers $p, \alpha$ satisfying the hypothesis of Theorem 8.2, we have

$$\int_{\partial M_N} \frac{(C|\xi|^2 + C|\zeta|^2 + C|\eta|^2) ((|\eta|^2 + w \cdot \overline{\eta})z \cdot \overline{\zeta} + (|\zeta|^2 + z \cdot \overline{\zeta})w \cdot \overline{\eta})}{(1 - Z \cdot \overline{\Theta})^{N+1}} \, u(\Theta) \, d\sigma(\Theta) \cdot$$

Using this and arguing as in the proof of Theorem 6.4.9 of [?], it can be shown that

$$|(RU)(Z)| \leq C \|u\|_\Lambda(\partial M_N) \int_{\partial M_N} \frac{1}{1 - Z \cdot \overline{\Theta}^{N+1}} \left(1 + \frac{|z|}{|\zeta|}\right) \left(1 + \frac{|w|}{|\eta|}\right) \, d\sigma(\Theta).$$

Therefore, by Proposition 9.3,

$$|(RU)(Z)| \leq C \|u\|_\Lambda(\partial M_N) \left(1 - |Z|^2\right)^{\lambda-1}, \quad \text{for all } Z \in \mathbb{B}_N$$

so that by Theorem 6.4.10 of [?], inequality (10.3) follows and the proof is now complete. D

**Theorem 10.3.** — For every $(0,1)$-form $f$ and real numbers $p, \alpha$ satisfying the hypothesis of Theorem 8.2, we have

$$\begin{cases} T_1 f \in \Gamma_{\alpha,2\alpha}(M_N) \text{ and } \|T_1 f\|_{\Gamma_{\alpha,2\alpha}(M_N)} \leq C \|f\|_{M_{N,p}} \quad \text{if } p < \infty, \\ T_1 f \in \Gamma_{\frac{1}{2},1}(M_N) \text{ and } \|T_1 f\|_{\Gamma_{\frac{1}{2},1}(M_N)} \leq C \|f\|_{M_{N,\infty}} \quad \text{if } p = \infty, \end{cases}$$

where the constant $C := C(N,p)$.

**Proof.** Let $\frac{1}{2} < r < 1$ and set

$$rM_N := \{Z \in M_N : |Z| < r\} \quad \text{and} \quad r\partial M_N := \{rZ, \ Z \in \partial M_N\}.$$

Now we define the norm $\|f\|_{rM_{N,p}}$ in the same way as $\|f\|_{M_{N,p}}$ given in (8.1) by substituting the domain of integration $M_N$ by $rM_N$. It is obvious that $\|f\|_{rM_{N,p}} \leq \|f\|_{M_{N,p}}$.

Applying Theorem 8.2 to the complex manifold $rM_N$ gives an integral operator $T_r$ that satisfies the following properties:

$$\overline{\partial}_{\mathbb{H}_N}(T_r f) = f \quad \text{on } rM_N, \quad (10.4)$$

and

$$(T_r f)|_{r\partial M_N} \in \Gamma_{\alpha,2\alpha}(r\partial M_N) \text{ and } \|(T_r f)|_{r\partial M_N}\|_{\Gamma_{\alpha,2\alpha}(r\partial M_N)} \leq C \|f\|_{M_{N,p}}. \quad (10.5)$$
Setting
\[ u(Z) := (T_1 f)(Z) - (T_r f)(Z), \quad \text{for all } Z \in M_N, \quad (10.6) \]

Theorem 8.2 and (10.4) imply that \( u \) is holomorphic on \( r\partial M_N \). On the other hand, by Theorem 8.2(ii) and (10.5), (10.6), we get

\[ \|u\|_{r\partial M_N} \leq \|(T_r f)\|_{r\partial M_N} + \|(T_1 f)\|_{r\partial M_N} \leq C\|f\|_{M_{N,p}}. \]

Applying Theorem 10.2 to this estimate yields

\[ \|u\|_{r\partial M_N} \leq C\|f\|_{M_{N,p}}, \]

so that by (10.5), we get \( T_1 f = u + T_r f \in \Gamma_{\alpha,2\alpha}(r\partial M_N) \) and

\[ \|(T_1 f)\|_{r\partial M_N} \leq C\|f\|_{M_{N,p}}. \]

Since all admissible curves \( \gamma \in C^2(\mathbb{B}_N) \) such that \( \gamma \subset M_N \) lie on some manifold \( r\partial M_N \), the proof of the theorem is now complete. \( \square \)

**Proof of Theorem 1.1.** Consider first the case where \( f \) is a \( \overline{\partial} \)-closed \((0,1)\)-form of class \( C^1 \) defined in a neighborhood of \( \overline{\Omega}_N \). The general case will be treated later.

In view of (8.1) and (10.1), it can be checked that

\[ \|f\|_{L^p(\Omega_N)} = C(N,p)\|F_N^* f\|_{M_{N,p}}. \quad (10.7) \]

By Proposition 10.1, we can define the \( \overline{\partial} \)-solving operator \( T \) on \( \Omega_N \) as

\[ (T f)(\widetilde{Z}) := (T_1(F_N^* f))(Z), \quad (10.8) \]

for every \( \widetilde{Z} \in \overline{\Omega}_N \) and \( Z \in M_N \) such that \( F_N(Z) = \widetilde{Z} \).

Combining Proposition 10.1, Theorem 10.3 and equalities (10.7) and (10.8), we see that the operator \( T \) satisfies

\[ \overline{\partial}(T f) = f \quad \text{on } \Omega_N; \quad (10.9) \]

\[ \|F_N^*(T f)\|_{\Gamma_{\alpha,2\alpha}(M_N)} \leq C\|f\|_{L^p(\Omega_N)}. \quad (10.10) \]

Let \( \widetilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w}) \) and \( \widetilde{Z}' \equiv (\tilde{x}', \tilde{z}', \tilde{w}') \) be two elements of \( \Omega_N \). We shall show that there exists a constant \( C := C(N,p) \) such that

\[ \left| (T f)(\widetilde{Z}) - (T f)(\widetilde{Z}') \right| \leq C\|f\|_{L^p(\Omega_N)}|\widetilde{Z} - \widetilde{Z}'|^\alpha. \quad (10.11) \]
Using the remark made at the beginning of the proof of Lemma 8.1, we only need prove (10.11) in one of the following three cases:

1) \( \tilde{x} = \tilde{x}', \tilde{z} = \tilde{z}' \);  
2) \( \tilde{x} = \tilde{x}', \tilde{w} = \tilde{w}' \);  
3) \( \tilde{z} = \tilde{z}', \tilde{w} = \tilde{w}' \).

Consider for example the case \( \tilde{x} = \tilde{x}', \tilde{w} = \tilde{w}' \). In this case, estimate (10.11) becomes

\[
\left| (Tf)(\tilde{x}, \tilde{z}, \tilde{w}) - (Tf)(\tilde{x}', \tilde{z}', \tilde{w}') \right| \leq C\|f\|_{L^p(\Omega_N)}|\tilde{z} - \tilde{z}'|^\alpha,
\]

which can be proved by using (10.10) and arguing as in the proof of case 2 in Section 5 of [?].

It remains to treat the general case. If merely \( f \in L^p(\Omega_N) \), we can regularize \( f \) by convolution with a \( C_0^\infty \) function of sufficiently small support. Then the same limiting argument as in [?, p. 361-362] shows that the conclusion of the theorem holds also for such \( f \). This completes the proof of the theorem. \( \square \)

**Proof of Theorem 1.2.** — First suppose that \( p < \infty \). We break the proof into four cases. In the course of the proof, we shall see that the general case can be reduced to one of these four cases. In the sequel, we write for every \( Z \in \mathbb{C}^{\mid N \mid} \), \( Z \equiv (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^n \times \mathbb{C}^m \).

**Case 1:** \( n > 2 \) and \( m \geq 2 \).

For every real number \( \lambda_0 \) such that \( \frac{1}{4} < \lambda_0 < \frac{1}{2} \), consider two real numbers \( \lambda, \mu > 0 \) related by \( 2\mu^2 = \frac{1}{2} \left( \frac{1}{2} - \lambda \right) = \frac{1}{2} - \lambda_0 \). Let \( c \in \mathbb{C} \) such that \( |c| \leq 1 \), and consider the following elements of \( Q_N \):

\[
Z_{\lambda_0, c} := \left( \underbrace{0, \ldots, 0}_{l}, i\lambda, \mu c, \underbrace{0, \ldots, 0}_{n-3}, \frac{1}{2}, 2^{-l-1}, \underbrace{0, \ldots, 0}_{m-2} \right);
\]

\[
Z_{\lambda, c} := \left( \underbrace{0, \ldots, 0}_{l}, \lambda, \lambda_0, i\lambda_0, \mu c, \underbrace{0, \ldots, 0}_{n-3}, \frac{1}{2}, 2^{-l-1}, \underbrace{0, \ldots, 0}_{m-2} \right).
\]

Now we put \( f := \overline{\partial} u_0 \), where the function \( u_0 \) is given by

\[
u_0(\epsilon) := \frac{|z_3|^2}{(1 - \frac{z_1}{2} + \frac{iz_2}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2})^{\frac{3}{2} + \mid N \mid + \frac{1}{p}}}, \quad \text{for all } Z \in \Omega_N.
\]
Then we have
\[ f(Z) := \frac{z_3 \overline{z}_3}{(1 - \frac{z_1}{2} + i\frac{z_2}{2} - \frac{w_1}{2} + i\frac{w_2}{2})^{\frac{1}{2} + \frac{|N|+2}{p}}}. \]

Suppose that \( u \) is a solution of the equation \( \overline{\partial}u = f \) on \( \Omega_N \). Since \( u - u_0 \) is holomorphic on \( \mathbb{Q}^N \) and \( u_0(Z,0) = u_0(Z_{\lambda,0}) = 0 \), by Cauchy formula we have
\[
\frac{1}{2\pi} \int_{0}^{2\pi} u(Z_{\lambda,e^{i\theta}}) d\theta - u(Z_{\lambda,0}) = \frac{|\mu|^2}{(\frac{1}{2} - \lambda)^{\frac{1}{2} + \frac{|N|+2}{p}}},
\]
\[
\frac{1}{2\pi} \int_{0}^{2\pi} u(Z_{\lambda_0,e^{i\theta}}) d\theta - u(Z_{\lambda_0,0}) = \frac{|\mu|^2}{(\frac{1}{2} - \lambda_0)^{\frac{1}{2} + \frac{|N|+2}{p}}}.\]

If \( u \in \Lambda_{\alpha+\epsilon}(\Omega_N) \) for some \( \epsilon > 0 \), then the difference between the two left hand sides is \( O(|\lambda - \lambda_0|^\alpha+\epsilon) \). On the other hand, the difference between the two right sides is greater than \( C|\lambda - \lambda_0|^\alpha \). Letting \( \lambda_0 \) tend to \( \frac{1}{2} \), we reach a contradiction. Hence \( u \notin \Lambda_{\alpha+\epsilon}(\Omega_N) \).

It now remains to check that \( f \in L^{p-\epsilon}(\Omega_N) \) for all \( \epsilon > 0 \). Applying (10.7) and using the local coordinates \( \Phi^z \) and \( \Phi^w \) of Theorem 4.1 with \( z := (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \in \mathbb{H}^{n-1} \) and \( w := (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \in \mathbb{H}^{m-1} \), it follows that for every \( \epsilon \geq 0 \),
\[
\|f\|_{L^{p-\epsilon}(\Omega_N)}^{p-\epsilon} \approx \int_{\mathbb{C}^{n+1}} \frac{|z_3|^{p-\epsilon}|z_4|^2}{(1 - z_1)^{(p-\epsilon)(\frac{1}{2} + \frac{|N|+2}{p})}} dV(Z),
\]
where \( \mathcal{U} \) is a sufficiently small neighborhood of the point \( (1,0,\ldots,0) \in \mathbb{C}^{\lvert N \rvert} \) and \( dV(Z) \) is the Lebesgue measure of \( \mathbb{C}^{\lvert N \rvert} \). We now explain briefly how the estimate \( \approx \) in the latter line could be obtained. Indeed, using the local coordinates \( \Phi^z \) and \( \Phi^w \), the function \( \lvert \zeta_{n+1} \rvert \) (resp. \( \lvert \eta_{m+1} \rvert \)) appearing in the \( \| \cdot \|_{\mathbb{M}_{N,p}} \) norm in (8.1) becomes the function \( \lvert z_3 \rvert \) (resp. \( \lvert z_4 \rvert \)) defined in \( \mathbb{C}^{\lvert N \rvert} \).

By integration in polar coordinates, it is easy to reduce the estimate of the latter integral to that of the following one
\[
\int_{\mathbb{C}^{\lvert 1 - z \rvert < 1}} \frac{dz \wedge \overline{dz}}{|1 - z|^{2 - \frac{\epsilon (|N|+2)}{p}}}.\]

From this integral, we see that \( f \in L^{p-\epsilon}(\Omega_N) \) for all \( \epsilon > 0 \). This completes the proof in the first case. Furthermore, we remark that the method presented here can be applied to all domains \( \Omega_N \) where \( N := (n_1, \ldots, n_m) \) satisfies the condition \( n_m > 2 \).
Case 2: \( l \geq 1 \) and \( n, m \geq 2 \).

Choose \( \lambda_0, \lambda \) and \( \mu \) as in case 1. Let \( c \in \mathbb{C} \) such that \( |c| \leq 1 \) and consider the following points of \( \Omega_N \):

\[
Z_{\lambda,c} := \left( \mu c, 0, \ldots, 0, \lambda_1, 0, \ldots, 0, \frac{1}{2}, \frac{i}{2}, 0, \ldots, 0 \right);
\]

\[
Z_{\lambda_0,c} := \left( \mu c, 0, \ldots, 0, \lambda_0, 0, \ldots, 0, \frac{1}{2}, \frac{i}{2}, 0, \ldots, 0 \right).
\]

We set \( f := \overline{\partial} u_0 \), where the function \( u_0 \) is given by

\[
u_0(Z) := \frac{|x_1|^2}{\left(1 - \frac{z_1}{2} + \frac{i z_2}{2} - \frac{w_1}{2} + \frac{i w_2}{2}\right)^{\frac{1}{2} + \frac{\varepsilon}{p}}}, \quad \text{for all } Z \in \Omega_N.
\]

The rest of the proof follows along the same lines as that of case 1. Finally, we remark that the method used in this second case works also for all domains \( \Omega_N \) where \( N := (n_1, \ldots, n_m) \) satisfies the condition \( n_1 = 1 \) and \( n_m > 1 \).

Case 3: \( l = 0 \) and \( n = m = 2 \).

For every \( \lambda_0 \) such that \( \frac{1}{2\sqrt{2}} < \lambda_0 < \frac{1}{\sqrt{2}} \), let \( \lambda \) and \( \mu \) be two positive real numbers satisfying \( \mu^2 = \frac{1}{2} \left( \frac{1}{\sqrt{2}} - \lambda \right) = \frac{1}{\sqrt{2}} - \lambda_0 \). Let \( c \in \mathbb{C} \) such that \( |c| \leq 1 \) and consider the following elements of \( \Omega_N \)

\[
Z_{\lambda,c} := (\mu c, 0, \lambda, i\lambda), \quad \text{and} \quad Z_{\lambda_0,c} := (\mu c, 0, \lambda_0, i\lambda_0).
\]

We set \( f := \overline{\partial} u_0 \), where the function \( u_0 \) is defined by

\[
u_0(Z) := \frac{|z_1|^2}{\left(1 - \frac{w_1}{\sqrt{2}} + \frac{i w_2}{\sqrt{2}}\right)^{\frac{1}{2} + \frac{\varepsilon}{p}}}, \quad \text{for all } Z = (z_1, z_2, w_1, w_2) \in \Omega_N.
\]

Proceeding as in the proof of case 1, it can be checked that if a function \( u \) satisfies \( \overline{\partial} u = f \) then \( u \not\in \Lambda_{\alpha+\varepsilon}(\Omega_N), \ \forall \varepsilon > 0 \). It now remains to establish that \( f \in L^{p-\varepsilon}(\Omega_N) \) for all \( \varepsilon > 0 \).

We first apply (10.7), then use the local coordinates \( \Phi^w \) in Theorem 4.1 with \( w := \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0\right) \in \mathbb{H}_2 \), and conclude that for every \( \varepsilon \geq 0 \),
Here $\mathcal{U}$ is a sufficiently small neighborhood of the point $(1, 0, 0, 0)$ in $\mathbb{C}^4$ and $B_4$ (resp. $dV_4(Z)$) is the euclidean unit ball (resp. the Lebesgue measure) of $\mathbb{C}^4$.

By integration in polar coordinates, the estimate of the latter integral is reduced to that of the integral

$$
\int_{U \cap B_4} \frac{|z_2|^{p-\varepsilon}|z_3|^2}{|1 - z_1|^{(p-\varepsilon)(\frac{1}{2} + \frac{3}{p})}} dV_4(Z).
$$

From this integral we conclude that $f \in L^{p-\varepsilon}(\Omega_N)$. The proof of the theorem is complete in this third case. It should be noted that this method is applicable to all domains $\Omega_N$ where $N := (n_1, \ldots, n_m)$ satisfies the condition $n_1 = \cdots = n_m = 2$.

**Case 4:** $l = m = 0$ and $n = 2$.

In this case $\alpha(N, p) = \frac{1}{2} - \frac{3}{p}$. Let $z$ be a strongly convex point of the boundary $\partial \Omega_N$. It then follows from the work of Krantz in [?, Section 6] that there exists a $(0, 1)$-form $f \in C^\infty(\mathcal{U})$ that satisfies the conclusion of the theorem if $\Omega_N$ is replaced by $\mathcal{U}$. Here $\mathcal{U}$ is an open strongly convex neighborhood of $z$ in $\Omega_N \cup \{z\}$. In view of [?], we see easily that the form $f$ can be extended to a form of class $C^\infty(\Omega_N)$ satisfying the conclusion of the theorem. The proof is thus complete in this last case.

This argument also shows that the Lipschitz $(\frac{1}{2} + \varepsilon)$-estimates ($\varepsilon > 0$) do not hold for the case $p = \infty$. This completes the proof of Theorem 1.2. □

Finally, we conclude this paper by some remarks and open problems.

1. It seems to be of some interest to establish the $(L^p, L^q)$ type optimal regularity for the $\overline{\partial}$-equation on $\Omega_N$.

2. We conjecture that the Lipschitz $\frac{1}{2}$-regularity corresponding to the case $p = \infty$ in Theorem 1.1 is optimal. More precisely, this regularity can not be improved to Lipschitz $\frac{1}{2}$.

3. Does there exist a natural way to define the Nevanlinna class on the non-smooth domains $\Omega_N$ and find a related Blaschke type condition that characterizes the zeroes of the functions of this class?
Optimal Lipschitz estimates for the $\overline{\partial}$ equation on a class of convex domains

Bibliography