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Polynomial hulls and positive currents (*)

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ABSTRACT. — We extend the Wermer's theorem, to describe the polynomial hull of compact sets lying on the boundary of a smooth strictly convex domain of $\mathbb{C}^n$. We also extend the result to polynomial p-hulls and apply it to get properties of pluriharmonic or p.s.h. positive currents.

RÉSUMÉ. — Nous décrivons à la suite des travaux de Wermer, l'enveloppe polynomiale des ensembles compacts contenus dans le bord d'un domaine lisse strictement convexe de $\mathbb{C}^n$. Nous étendons aussi ce résultat aux p-enveloppes polynomiales et l'appliquons à l'étude de quelques propriétés des courants positifs pluriharmoniques ou p.s.h.

1. Introduction

In this paper, we are concerned with the following question – suppose $\Gamma \subset \mathbb{C}^n$ is compact and has finite 1-dimensional Hausdorff measure: is it true that the polynomial hull $\widehat{\Gamma}$ is an analytic variety or an union of varieties whose boundaries are included in Γ? The first result in this direction is due to Wermer [37], who proved that if Γ is a real analytic curve, then $\widehat{\Gamma} \setminus \Gamma$ is subvariety (possibly empty) of $\mathbb{C}^n \setminus \Gamma$. Using a tool developed by Bishop [10], Stolzenberg [35] was able to extend Wermer's theorem to the $\mathcal{C}^1$ case; Alexander [4] extended it to the case where Γ is connected. Alexander [5] has also shown that the hull of an arbitrary compact set of finite linear measure need not be a variety, although the hull in his example is a countable union

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of varieties. It is still an open question wherether the hull of an arbitrary compact set of finite linear measure is an union of varieties. The reader can find some applications of Wermer's theorem in Harvey-Lawson [21], Dolbeault-Henkin [14], Sarkis [29], Alexander-Wermer [7] and [11, 27].

In [11, 12, 25, 26], the authors extended Wermer's theorem for geometrically 1-rectifiable compact sets (*see* the definition in section 3). In this paper, we show that the Wermer's theorem is also true for every compact set of finite linear measure provided that this compact set is lying on the boundary of a smooth strictly convex domain. We give a notion of polynomial p-hull and prove an analogous result for p-hull of a compact set of finite $(2p-1)$-dimensional Hausdorff measure. Our proof use the case $p=1$ and a theorem of Shiffman on separately holomorphic functions. The polynomial 1-hull is the usual polynomial hull.

In the second part of this article, we use the generalized Wermer's theorem to study some properties of positive plurisubharmonic and plurisuperharmonic currents.

Let V be a complex manifold of dimension n. A current T of bidimension (p,p) or bidegree $(n-p,n-p)$ in V is called *plurisubharmonic* if $\mathrm{dd^c}T$ is a positive current, is called *plurisuperharmonic* if $-\mathrm{dd^c}T$ is a positive current and is called *pluriharmonic* if $\mathrm{dd^c}T=0$. In particular, every closed current is pluriharmonic.

Some analytic objects support interesting positive plurisubharmonic and plurisuperharmonic currents. In [19], Garnett proved that every laminated closed set supports a positive pluriharmonic current (see also [20, 9]). Duval and Sibony used positive plurisuperharmonic currents to describe polynomial hulls of compact sets in $\mathbb{C}^n$ [15]. Gauduchon, Harvey, Lawson, Michelson, Alessandrini, Bassanelli, etc. studied non-Kähler geometry using (smooth or not) pluriharmonic currents [1].

The reader can find others properties of theses classes of currents in Demailly [13], Skoda [34], Sibony, Berndtsson, Fornæss [31], [9], [17], Alessandrini and Bassanelli [2, 3, 8], etc. If T is a positive plurisubharmonic current, one can define a density $\nu(T,a)$ of T at every point a. This density is called the Lelong number of T at a.

We will prove that if the level set $\{\nu(T,a)\geqslant\delta\}$ is dense in the support of T for a suitable $\delta>0$, then the support X of T is a complex subvariety of pure dimension p and $T=\varphi[X]$, where φ is a weakly plurisubharmonic function on X. In particular, every rectifiable positive plurisubharmonic current is closed. Moreover, it has the form $c[X]$, where X is a complex subvariety and c is essentially equal to a positive integer in each component of X. We will also prove that if T is positive pluriharmonic (resp. plurisuperharmonic) and its support has locally finite $2p$-dimensional Hausdorff measure then the support X of T is a complex subvariety of pure dimension p and $T = \varphi[X]$, where φ is a weakly pluriharmonic (resp. plurisuperharmonic) function on X.

For closed positive currents, an analogous result has been proved by King [24] (see also [33, 32]). The proof of King does not work in the case of plurisubharmonic and plurisuperharmonic currents.

2. Some definitions

Let $\Gamma \subset \mathbb{C}^n$ be a compact set. *The polynomial hull* of Γ is the compact set $\widehat{\Gamma}$ defined by the following formula

$$\widehat{\Gamma} := \left\{x \in \mathbb{C}^n,\ |P(x)| \leqslant \max_{z\in\Gamma} |P(z)| \text{ for any polynomial } P\right\}.$$

We say that Γ is *polynomially convex* if $\widehat{\Gamma} = \Gamma$. We now introduce the notions of p-pseudoconcavity and polynomial p-hull.

Let V be a complex manifold of dimension $n \geqslant 2$ and X be a closed subset of V. Let $1 \leqslant p \leqslant n-1$ be an integer.

DEFINITION 2.1. — *We say that X is* p-pseudoconcave subset of V *if for every open set $U \subset\subset V$ and every holomorphic map f from a neighbourhood of $\overline{U}$ into $\mathbb{C}^p$ we have $f(X\cap U) \subset \mathbb{C}^p \setminus \Omega$ where Ω is the unbounded component of $\mathbb{C}^p \setminus f(X \cap bU)$.*

This means X has no "local peak point" for holomorphic maps into $\mathbb{C}^p$. In particular, X satisfies the maximum principle, i.e. locally, the modulus of any holomorphic function admits no strict maximum on X. Observe that if V is a submanifold of another complex manifold V', then X is p-pseudoconcave in V if and only if X is p-pseudoconcave in V'.

By the argument principle, every complex subvariety of pure dimension p of V is p-pseudoconcave. It is clear that if X is p-pseudoconcave, the $2p$-dimensional Hausdorff measure of X is strictly positive. If $1 \leqslant p \leqslant q \leqslant n-1$ and X is q-pseudoconcave then X is p-pseudoconcave. If $g : V \longrightarrow \mathbb{C}^{p-k}$ is a holomorphic map and $X \subset V$ is p-pseudoconcave then $g^{-1}(x) \cap X$ is k-pseudoconcave for every $x \in \mathbb{C}^{p-k}$. We have the following proposition.

PROPOSITION 2.2. — *Let T be a positive plurisuperharmonic current of bidimension (p,p) on a complex manifold V. Then the support* supp(T) *of T is p-pseudoconcave in V.*

Proof. — Since the problem is local we can assume that V is a ball in $\mathbb{C}^n$. Assume that $X := \mathrm{supp}(T)$ is not p-pseudoconcave. Then there exist an open set $U \subset\subset V$ and a holomorphic map $f : V' \longrightarrow \mathbb{C}^p$ such that $f(X \cap U) \not\subset \mathbb{C}^p \setminus \Omega$, where V' is a neighbourhood of $\overline{U}$ and Ω is the unbounded component of $\mathbb{C}^p \setminus f(X \cap bU)$.

Let $\Phi : V' \longrightarrow \mathbb{C}^p \times V'$ the holomorphic map given by $\Phi(z) := (f(z), z)$. Choose a bounded domain U' in $\mathbb{C}^p \times V'$ such that $U' \cap \Phi(V') = \Phi(U)$. Set $X' := \Phi(X \cap V')$ and $T' := 1_{U'}\Phi_*(T)$. Then T' is a positive plurisuperharmonic current in U'. Let $\pi : \mathbb{C}^{n+p} \longrightarrow \mathbb{C}^p$ be the linear projection on the first p coordinates. We have $\pi(\overline{X}' \cap bU') = f(X \cap bU)$ and $\pi(X' \cap U') = f(X \cap U) \not\subset \mathbb{C}^p \setminus \Omega$. The open set Ω is also the unbounded component of $\mathbb{C}^p \setminus \pi(\overline{X}' \cap bU')$.

Therefore $\pi_*(T')$ defines a positive plurisuperharmonic current of bidegree $(0,0)$ in Ω which vanishes in $\mathbb{C}^p \setminus \pi(\overline{X}')$. Hence there is a positive plurisuperharmonic function ψ on Ω, which is zero on $\mathbb{C}^p \setminus \pi(\overline{X}')$ such that $\pi_*(T') = \psi[\Omega]$ in Ω. By plurisuperharmonicity, this function vanishes identically. Fix a small open $W \subset \mathbb{C}^{n+p}$ such that $W \cap X' \neq \emptyset$ and $\pi(\overline{W}) \subset \Omega$. Set $\Psi := (i dz_1 \wedge d\overline{z}_1 + \cdots + i dz_p \wedge d\overline{z}_p)^p$. Then the positive measure $T' \wedge \pi^*(\Psi)$ vanishes in W since so does its push-forward by π. This still holds for every small linear pertubation π_ϵ of π. On the other hand, we can construct a strictly positive (p,p)-form ψ of $\mathbb{C}^{n+p}$ as a linear combination of $\pi_\epsilon^*(\Psi)$. Then we have $\langle T', 1_W \psi\rangle = 0$. Since T' is positive, $T' = 0$ on W. This is a contradiction. □

PROPOSITION 2.3. — *Let Γ be a compact subset of $\mathbb{C}^n$. Denote by $\mathcal{F}$ the family of p-pseudoconcave subsets of $\mathbb{C}^n \setminus \Gamma$ which are bounded in $\mathbb{C}^n$. Then the union Σ of elements of $\mathcal{F}$ is also belong to $\mathcal{F}$ (Σ is the biggest element of $\mathcal{F}$).*

Proof. — Let $S_1, S_2, \ldots$ be elements of $\mathcal{F}$ such that $S_i \subset S_{i+1}$. We show that $S := \overline{\cup S_i}$ belongs to $\mathcal{F}$. By the maximum principle, $S_i \subset \widehat{\Gamma}$. Then S is bounded in $\mathbb{C}^n$.

Assume that S does not belong to $\mathcal{F}$. Then there are a point $p \in S$, a neighbourhood $U \subset\subset \mathbb{C}^n \setminus \Gamma$ of p and a holomorphic map f from a neighbourhood of $\overline{U}$ into $\mathbb{C}^p$ such that $f(p)$ belongs to the unbounded component Ω of $\mathbb{C}^p \setminus f(S \cap bU)$. Fix small neighbourhoods V of p and W of $S \cap bU$ such that $f(\overline{V})$ is included in the unbounded component of $\mathbb{C}^p \setminus f(\overline{W})$.

For i large enough, we have $S_i \cap bU \subset W$ and $S_i \cap V \neq \emptyset$. Moreover, if p is a point in $S_i \cap V$, $f(p)$ belongs to the unbounded component of $\mathbb{C}^p \setminus f(S_i \cap bU)$. This contradicts the p-pseudoconcavity. Thus $S \in \mathcal{F}$.

Now, by Zorn's lemma, we can choose an element Σ' of $\mathcal{F}$ which is maximal for the inclusion. If $\Sigma' \neq \Sigma$ then there exists an element $S \in \mathcal{F}$ such that $S \not\subset \Sigma'$. The set $S \cup \Sigma'$ is p-pseudoconcave in $\mathbb{C}^n \setminus \Gamma$ and it belongs to $\mathcal{F}$. This contradicts the maximality of Σ'. □

DEFINITION 2.4. — *Let Γ and Σ be as in Proposition 2.3. We say that $\Sigma \cup \Gamma$ is the polynomial p-hull of Γ and we denote it by* $\mathrm{hull}(\Gamma, p)$.

The following proposition shows that we obtain the usual polynomial hull when $p = 1$.

PROPOSITION 2.5. — *We have* $\mathrm{hull}(\Gamma, 1) = \widehat{\Gamma}$ *for every compact subset Γ of $\mathbb{C}^n$.*

Proof. — By maximum principle, we have $\mathrm{hull}(\Gamma, 1) \subset \widehat{\Gamma}$. Now let $z \in \widehat{\Gamma} \setminus \Gamma$, we will prove that $z \in \mathrm{hull}(\Gamma, 1)$. By Duval-Sibony theorem [15], there are a positive current T of bidimension $(1,1)$ with compact support in $\mathbb{C}^n$ and a measure μ with support in Γ such that $\mathrm{dd^c} T = \mu - \delta_z$ where δ_z is the Dirac mass at z. By Proposition 2.2, $\mathrm{supp}(T)$ is 1-pseudoconcave in $\mathbb{C}^n \setminus \Gamma$. Thus $\mathrm{supp}(T) \subset \mathrm{hull}(\Gamma, 1)$ and $z \in \mathrm{hull}(\Gamma, 1)$. □

3. Hulls of sets of finite Hausdorff measure

Denote by $\mathcal{H}^k$ the Hausdorff measure of dimension k. A compact set $\Gamma \subset \mathbb{C}^n$ is called *geometrically k-rectifiable* if $\mathcal{H}^k(\Gamma)$ is finite and the geometric tangent cone of Γ is a real space of dimension k at $\mathcal{H}^k$-almost every point in Γ. If a compact set Γ is geometrically k-rectifiable, it is $(\mathcal{H}^k, k)$-*rectifiable* [28, p.208], i.e. there exist $\mathcal{C}^1$ manifolds V_1, V_2, ... such that $\mathcal{H}^k(\Gamma \setminus \cup V_m) = 0$ [16, 3.1.16, 3.2.18]. Now, suppose that V_1, V_2, ... are $\mathcal{C}^1$ oriented manifolds of dimension k in $\mathbb{C}^n$, $K_i \subset V_i$ are compact sets and n_1, n_2, ... are integers such that $\sum |n_i| \mathcal{H}^k(K_i) < +\infty$. Then we can define a current S of dimension k by

$$\langle S, \psi \rangle := \sum n_i \int_{K_i} \psi$$

for any test form ψ of degree k having compact support. Such a current is called *a rectifiable current.* We have the following theorem.

THEOREM 3.1. — *Let Γ be a compact subset of $\mathbb{C}^n$. Assume that Γ is geometrically $(2p-1)$-rectifiable with $1 \leqslant p \leqslant n-1$. Then* $\mathrm{hull}(\Gamma, p) \setminus \Gamma$ *is a complex subvariety of pure dimension p (possibly empty) of $\mathbb{C}^n \setminus \Gamma$. Moreover,* $\mathrm{hull}(\Gamma, p) \setminus \Gamma$ *has finite $2p$-dimensional Hausdorff measure and the boundary of the integration current* $[\mathrm{hull}(\Gamma, p) \setminus \Gamma]$ *is a rectifiable current* *of dimension* $2p-1$ *and has multiplicity* 0 *or* 1 $\mathcal{H}^{2p-1}$*-almost everywhere on* Γ.

The last property of the current $[\text{hull}(\Gamma,p)\setminus\Gamma]$ is called *the Stokes formula.* In the case $p=1$, this theorem is proved in [11, 12, 26] and it generalizes the results of Wermer [37], Bishop [10], Stolzenberg [35], Alexander [4] and Harvey-Lawson [21] (*see* also [14]).

In order to prove Theorem 3.1, we will slice Γ by complex planes of dimension $n-p+1$ and we will apply the known result in the case $p=1$. We will also use a theorem of Shiffman on separately holomorphic functions.

Set $X := \text{hull}(\Gamma,p)\setminus\Gamma$. Observe that if L is a linear complex $(n-p+1)$-plane then $X\cap L$ is 1-pseudoconcave in $L\setminus\Gamma$. Thus $X\cap L$ is included in the polynomial hull of $\Gamma\cap L$. Moreover, by Sard theorem, for almost every L the intersection $\Gamma\cap L$ is geometrically 1-rectifiable (*see* also [11]). Therefore $\text{hull}(\Gamma\cap L,1)\setminus\Gamma$ is a complex subvariety of pure dimension 1 of $L\setminus\Gamma$. We need the following lemma.

LEMMA 3.2.— *Let* V *be a complex manifold of dimension* $n\geqslant 2$ *and* $X\subset V$ *a* p*-pseudoconcave subset. If* X *is included in a complex subvariety* Σ *of pure dimension* p *of* V, *then* X *is itself a complex subvariety of pure dimension* p *of*

Proof. — Recall that every non empty p-pseudoconcave set has positive $\mathcal{H}^{2p}$ measure. Assume that X is not a complex subvariety of pure dimension p of V. Then there is a point $a\in X$ such that a is a regular point of Σ and $a\in\overline{\Sigma\setminus X}$. Let $V'\subset V$ be a small ball of centre a satisfying $V'\cap\Sigma\setminus X\neq\emptyset$. Since V' is small, we can choose a projection $\pi: V'\longrightarrow\mathbb{C}^p$ such that π is injective on $\Sigma\cap V'$. Then $\pi(X\cap W)$ meets the unbounded component of $\mathbb{C}^p\setminus\pi(X\cap bW)$ for every ball $W\subset\subset V'$ of center a such that $\partial W\cap\Sigma\setminus X\neq\varnothing$. This is impossible. □

Proof of Theorem 3.1. — By the latter lemma, for almost every L (such that $\Gamma\cap L$ is geometrically 1-rectifiable), $X\cap L$ is a complex subvariety of pure dimension 1 of $L\setminus\Gamma$. Let E be a complex $(n-p)$-plane which does not meet Γ. By Sard theorem, for almost every L passing through E, the intersection $\Gamma\cap L$ is geometrically 1-rectifiable. We deduce that $X\cap E=(X\cap L)\cap E$ is a finite set. We now use a Shiffman theorem on separately holomorphic functions in order to complete the proof of Theorem 3.1.

Let a be a point of X. We show that X is a complex variety of pure dimension p in a neighbourhood of a. Indeed, choose a coordinate system such that $\Pi(a) \notin \Pi(\Gamma)$, where $\Pi : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is the linear projection given by $\Pi(z) := (z_1, \ldots, z_p)$. Set $\pi_i : \mathbb{C}^n \longrightarrow \mathbb{C}^{p-1}$, $\pi_i(z) := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_p)$. Using a linear change of coordinates, we may suppose without loss of generality that for $\mathcal{H}^{2p-2}$-almost every $x \in \mathbb{C}^{p-1}$ and for every $1 \leqslant i \leqslant n$, $X \cap \pi_i^{-1}(x)$ is a complex subvariety of pure dimension 1 of $\pi_i^{-1}(x) \setminus \Gamma$.

Fix a small open neighbourhood V of $\Pi(a)$ such that $\Pi^{-1}(V) \cap \Gamma = \emptyset$. Put $z = (z', z'')$, $z' := (z_1, \ldots, z_p)$ and $z'' := (z_{p+1}, \ldots, z_n)$. Let g be a function which is defined in a subset V' of total measure of V. This function is called *separately holomorphic* if for every $1 \leqslant i \leqslant p$ and $\mathcal{H}^{2p-2}$-almost every $x \in \mathbb{C}^{p-1}$ the restriction of g on $V' \cap \{z_i = x\}$ can be extended to a holomorphic function on $V \cap \{z_i = x\}$. For any $k \geqslant 0$ and $p+1 \leqslant m \leqslant n$, we define the following measurable function on V

$$g_{m,k}(z') := \sum_{z \in \Pi^{-1}(z') \cap X} z_m^k.$$

The function

$$g_0(z') := g_{m,0}(z) = \#(\Pi^{-1}(z') \cap X)$$

is independent on m and takes only positive integer values.

By the choice of coordinates, $g_{m,k}(z')$ is $\mathcal{H}^{2p}$-almost everywhere equal to a separately holomorphic function. Thanks to Shiffman Theorem [30], $g_{m,k}(z')$ is equal $\mathcal{H}^{2p}$-almost everywhere to a function $\tilde{g}_{m,k}(z')$ which is holomorphic on V. In particular, $\tilde{g}_{m,0}(z')$ is equal to an integer r which does not depend on m and z'.

Now, consider the following equation system:

$$\sum_{i=1}^{r} \left[z_m^{(i)}\right]^k = \tilde{g}_{m,k}(z') \quad \text{with } p+1 \leqslant m \leqslant n \text{ and } 1 \leqslant k \leqslant r.$$

The set of points $\left(z', z_{p+1}^{(i_{p+1})}, \ldots, z_n^{(i_n)}\right)$ given by solutions of the system above is a complex subvariety Σ of pure dimension p of $\Pi^{-1}(V)$. We have $X \cap \Pi^{-1}(x) \subset \Sigma$ for $\mathcal{H}^{2p}$-almost every $x \in V$. We show that $X \cap \Pi^{-1}(V) \subset \Sigma$. Assume this is not true. Choose a point $b \in X \cap \Pi^{-1}(V) \setminus \Sigma$ and a neighbourhood $W \subset\subset \Pi^{-1}(V) \setminus \Sigma$ of b. The set $\Pi(X \cap \overline{W})$ has zero $\mathcal{H}^{2p}$ measure. But for almost every complex $(n-p+1)$-plane L passing through $\Pi^{-1}(\Pi(b))$, the intersection $X \cap L$ is a complex subvariety of pure dimension 1 of $L \setminus \Gamma$ which contains b. This implies that the set $\Pi(X \cap \overline{W})$ has positive $\mathcal{H}^{2p}$ measure. We reach a contradiction.

Now we have already shown that $X \cap \Pi^{-1}(V) \subset \Sigma$. By Lemma 3.2, $X \cap \Pi^{-1}(V)$ is a complex subvariety of pure dimension p of $\Pi^{-1}(V)$. Then

X is a complex subvariety of pure dimension p of $\mathbb{C}^n \setminus \Gamma$. The rest of the proof follows along the sames lines as the proof of Stokes formula given in [25, 11, 12]. $\square$

When Γ is included in the boundary of a smooth convex domain, its rectifiability in Theorem 3.1 is non necessary. We have the following result.

THEOREM 3.3. — *Let D be a strictly convex domain with smooth boundary of $\mathbb{C}^n$. Let Γ be a compact set in the boundary of D. Assume that the $(2p-1)$-dimensional Hausdorff measure of Γ is finite. Then* $\mathrm{hull}(\Gamma, p) \setminus \Gamma$ *is a complex subvariety of pure dimension p (possibly empty) of $\mathbb{C}^n \setminus \Gamma$. Moreover,* $\mathrm{hull}(\Gamma, p) \setminus \Gamma$ *has finite $2p$-dimensional Hausdorff measure and the boundary of the integration current* $[\mathrm{hull}(\Gamma, p) \setminus \Gamma]$ *is a rectifiable current of dimension $2p-1$ and has multiplicity* 0 *or* 1 *$\mathcal{H}^{2p-1}$-almost everywhere on Γ.*

We will prove Theorem 3.3 for $p = 1$. Using the case $p = 1$, we prove the general case exactly as in Theorem 3.1.

Given a closed set $F \subset \mathbb{C}^n$, $T_p(F)$ denotes the tangent cone of F at p. If L is a real line in $\mathbb{C}^n$, $p \in L$ and α is some angle, we denote by $C_\alpha^L(p)$ the cone of angle α around L, opening at the point p; i.e., if $p = 0$, and L is the x_1 axis, then $C_\alpha^L(p) = \{|y'| \leqslant \tan\alpha |x_1|\}$. Here y' denotes the other $2n-1$ standard coordinates. We first prove the theorem for *minimal subsets* of Γ– i.e., we assume that $a \in \widehat{\Gamma} \setminus \Gamma$ and that Γ is minimal with respect to this property. The existence of minimal subsets follows from Zorn's lemma.

The proof proceeds in three steps. First, we show that Γ is $(\mathcal{H}^1, 1)$-rectifiable; we then show that Γ possesses a tangent line $\mathcal{H}^1$-almost everywhere and we finish the proof using Theorem 3.1.

PROPOSITION 3.4. — *Let D be a strictly convex domain with smooth boundary and $\Gamma \subset bD$ a compact set of finite linear measure. If Γ is minimal, then Γ is geometrically* 1*-rectifiable.*

In order to proof this proposition, we need the following lemmas.

LEMMA 3.5 ([23]). — *Let X_1, X_2 be two polynomially convex subsets of $\mathbb{C}^n$ and $X = X_1 \cup X_2$. Assume there is a polynomial f such that $f(X_1) \cap \{\mathrm{Re} z \geqslant 0\} = \{0\}$, $f(X_2) \cap \{\mathrm{Re} z \leqslant 0\} = \{0\}$ and $f^{-1}(0) \cap X$ is polynomially convex. Then X is polynomially convex.*

Proof. — We choose two bounded domains Ω_1, Ω_2 in $\mathbb{C}$ such that

1. $\Omega_i \cup \{0\}$ contains $f(X_i)$.
2. $\overline{\Omega}_1 \cap \{\mathrm{Re}z \geqslant 0\} = \{0\}$ and $\overline{\Omega}_2 \cap \{\mathrm{Re}z \leqslant 0\} = \{0\}$.
3. $\mathbb{C} \setminus \overline{\Omega}_1 \cup \overline{\Omega}_2$ is connected.

In [23], Kallin considers the case where Ω_i are angular sectors. According to Mergelyan's theorem, the point 0 is peak for $\Omega_1 \cup \Omega_2$. Then, Kallin's proof works in our case. □

LEMMA 3.6.— *For a point $p \in \Gamma$, let H_p denote the real tangent plane to bD at p. Let l_p be a real linear functional with $H_p = \{l_p = 0\}$ and $D \subset \{l_p < 0\}$. Then for small $\epsilon > 0$, the set $\Gamma \cap \{l_p = -\epsilon\}$ contains at least two points.*

Proof. — Let α_p be the complex linear functional with $\mathrm{Re}(\alpha_p) = l_p$. Choose δ so that for ϵ, $0 < \epsilon < \delta$, $\{l_p < -\epsilon\} \cap \Gamma$ and $\{l_p > -\epsilon\} \cap \Gamma$ are both nonempty. Suppose that $\Gamma \cap \{l_p = -\epsilon\}$ contains 1 or zero points. By Lemma 3.5, $(\{l_p \geqslant -\epsilon\} \cap \Gamma)\widehat{\ } = \{l_p \geqslant -\epsilon\} \cap \widehat{\Gamma}$ and $(\{l_p \leqslant -\epsilon\} \cap \Gamma)\widehat{\ } = \{l_p \leqslant -\epsilon\} \cap \widehat{\Gamma}$. This contradicts the minimality of Γ. □

LEMMA 3.7.— *Γ is $(\mathcal{H}^1, 1)$-rectifiable.*

Proof. — The rectifiability follows from an application of Eilenberg's inequality [16, 2.10.25]: for $\Gamma \subset \mathbb{R}^m$,

$$\int_0^r \mathcal{H}^0(\Gamma \cap S(p,\rho))d\rho \leqslant \mathcal{H}^1(\Gamma \cap \mathrm{B}(p,r))$$

where $S(p,r)$ and $\mathrm{B}(p,r)$ denote the sphere and the ball of center p and radius r. Let $p \in \Gamma$ be arbitrary and let π_p be the projection onto the tangent plane of bD at p. After a linear change of coordinates, we can arrange that the projections of the sets $\{l_p = -\epsilon\} \cap bD$ will be approximately spheres of radius $\sqrt{\epsilon}$. We can find a C^1 diffeomorphism ϕ_p of $T_p(bD)$ which sends these surfaces to actual spheres of radius $\sqrt{\epsilon}$. Each of these spheres intersects $\phi_p \circ \pi_p(\Gamma)$ in two points; therefore Eilenberg's inequality shows that the lower density is 1 at p. We can do the same construction at nearby points, and since for points close enough to p, the projection map distorts lengths by a factor arbitrarily close to 1, we see that for $\delta > 0$, there is a neighborhood of p in which the lower density of Γ is larger than $1 - \delta$. By the generalization of Besicovitch's 3/4-theorem [28, Theorem 17.6], this shows that Γ is $(\mathcal{H}^1, 1)$-rectifiable. □

Proof of Proposition 3.4. — Recall that $(\mathcal{H}^1, 1)$-rectifiable sets admit approximate tangent line at $\mathcal{H}^1$-almost every point. We have shown that there are $\nu > 0$ and $r_0 > 0$ such that, every point $w \in \Gamma$ will have the property that $\mathcal{H}^1(\Gamma \cap \mathrm{B}(w,r)) > \nu r$, for $r < r_0$. Let $w \in \Gamma$, and suppose that Γ has an approximate tangent line L at w but not a geometric tangent line. Since there is no tangent line at w, there is an angle $\alpha > 0$ such that for all $r > 0$,

$$\mathrm{B}(w,r) \cap \Gamma \setminus C^L_\alpha(w) \neq 0.$$

From the definition of approximate tangent line we obtain that $\mathcal{H}^1(\mathrm{B}(w,r) \cap \Gamma \setminus C^L_{\alpha/2}(w)) = \mathrm{o}(r)$. We can find arbitrarily small r such that there exists $z \in \mathrm{B}(w,r) \cap \Gamma \setminus C^L_\alpha(w)$. We may assume without loss of generality that $|z-w| > r/2$. This, together with the choice of r, implies that for $\delta = \alpha r/8$ we have $\mathcal{H}^1(\Gamma \cap \mathrm{B}(z,\delta)) = \mathrm{o}(\delta)$. If r is small enough, this contradicts the property that $\mathcal{H}^1(\Gamma \cap \mathrm{B}(z,\delta)) > \nu\delta$. Therefore w has a geometric tangent line. □

Proof of Theorem 3.3 for $p = 1$. — Theorem 3.1 applies to show that the minimal components are varieties. The fact that the entire hull is a variety can be deduced from uniqueness theorem [11, 26]: the multiplicity bound immediately implies that there cannot be three disjoint varieties which meet on a set of positive length in Γ. Now, by Stokes formula (*see* also Forstneric [18]) there is a lower bound $\delta(d)$ on the length of bV, if V is a subvariety of dimension 1 of D which contains a point of distance at least d from bD. Putting these two facts together gives that the hull of Γ is a variety. The rest of Theorem is proved exactly as in [25, 11]. □

Remark 1. — Under the hypotheses of Theorems 3.1 and 3.3, by Lemma 3.2, every p-pseudoconcave subset of $\mathbb{C}^n \setminus \Gamma$ which is bounded in $\mathbb{C}^n$, is a complex subvariety of pure dimension p of $\mathbb{C}^n \setminus \Gamma$ and is included in $X = \mathrm{hull}(\Gamma, p) \setminus \Gamma$. By Proposition 2.2, every positive current T with compact support satisfying $\mathrm{dd^c}T \leqslant 0$ in $\mathbb{C}^n \setminus \Gamma$ has the form $\varphi[X]$, where φ is a weakly plurisuperharmonic function in X (*see* Definition in Section 4).

Remark 2. — Let K be a convex compact set such that $K \cap \overline{D}$ is convex. If $\Gamma \subset bD \setminus K$ has finite $\mathcal{H}^{2p-1}$ measure, then $\mathrm{hull}(\Gamma \cup K, p) \setminus (\Gamma \cup K)$ is a complex variety. The proof uses the same argument as in the case of polynomial hull (*see*, for example, [35, 12]). This version of Wermer's theorem implies that Theorem 3.3 also is valid for D strictly polynomially convex.

COROLLARY 3.8. — *Let V be a complex manifold of dimension $n \geqslant 2$ and X a p-pseudoconcave subset of V. Let K be a compact subset of V which admits a Stein neighbourhood. Assume that the $2p$-dimensional Hausdorff measure of $X \setminus K$ is locally finite in $V \setminus K$. Then X is a complex subvariety of pure dimension p of V.*

Proof. — Fix a point $a \in X \setminus K$ and $\mathrm{B} \subset\subset V \setminus K$ an open ball contening a such that $X \cap b\mathrm{B}$ has finite $\mathcal{H}^{2p-1}$ measure. Then $X \cap \mathrm{B}$ is p-pseudoconcave in B. We can consider B as an open ball in $\mathbb{C}^n$. Then $X \cap \mathrm{B} \subset \mathrm{hull}(X \cap b\mathrm{B}, p)$. By Remark 1, $X \cap \mathrm{B}$ is a complex subvariety of pure dimension p of B. This implies that $X \setminus K$ is a complex subvariety of pure dimension p of $V \setminus K$.

Since we can replace V by a Stein neighbourhood of K, suppose without loss of generality that V is a submanifold of $\mathbb{C}^N$. Observe that X is also p-pseudoconcave in $\mathbb{C}^N$. Let B' be a ball containing the compact K such that $X \cap \mathrm{B}'$ has finite $\mathcal{H}^{2p-1}$ measure. Then we obtain $X \cap \mathrm{B}' \subset \mathrm{hull}(X \cap b\mathrm{B}', p)$. By Remark 1, $X \cap \mathrm{B}'$ is a complex subvariety of pure dimension p of B'. This completes the proof. □

Remark 3. — We can also deduce this corollary from results of Oka, Nishino and Tadokoro [36].

4. Positive currents

Let V be a complex manifold of dimension n. Let X be a complex subvariety of pure dimension p of V. A function $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(X)$ is called *weakly plurisubharmonic* if it is locally bounded from above and $\mathrm{dd^c}(\varphi[X])$ is a positive current. In the regular part of X, φ is equal almost everywhere to an usual plurisubharmonic function. We say that φ is *weakly plurisuperharmonic* if $-\varphi$ is weakly plurisubharmonic and that φ is *weakly pluriharmonic* if both φ and $-\varphi$ are weakly plurisubharmonic.

If φ is a positive weakly plurisubharmonic function on X, we can define a positive plurisubharmonic current $T := \varphi[X]$ as follows:

$$\langle T, \psi \rangle := \int_X \varphi\psi$$

for every test form ψ of bidegree (p,p) compactly supported in V. According to Bassanelli [8, 1.24, 4.10], every positive plurisubharmonic current with support in X can be obtained by this way. This is also true for positive plurisuperharmonic and pluriharmonic currents.

Now, let T be a positive plurisubharmonic current of bidimension (p,p) of V. Thanks to a Jensen type formula of Demailly [3, 34, 13], if V is an open set of $\mathbb{C}^n$ which contains a ball $\mathrm{B}(a, r_0)$, the function

$$\frac{1}{2^p \pi^p r^{2p}} \langle T, 1_{\mathrm{B}(a,r)} \omega^p \rangle, \text{ for } 0 < r < r_0,$$

is non-decreasing, where $\omega := i\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + i\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$. Therefore, we can define

$$\nu(T,a) := \lim_{r\to 0} \frac{1}{2^p \pi^p r^{2p}} \langle T, 1_{\mathrm{B}(a,r)}\omega^p\rangle.$$

This limit is called *the Lelong number* of T at the point a. In [3], Alessandrini and Bassanelli proved that $\nu(T,a)$ does not depend on coordinates. Thus, the Lelong number is well defined for any manifold V. We have the following theorem.

THEOREM 4.1. — *Let T be a positive plurisubharmonic current of bidimension (p,p) in a complex manifold V of dimension n. Assume that there exists a real number $\delta > 0$ such that the level set $\{z \in V,\ \nu(T,z) \geqslant \delta\}$ is dense in the support* $\mathrm{supp}(T)$ *of T. Then* $\mathrm{supp}(T)$ *is a complex subvariety of pure dimension p of V and there exists a weakly plurisubharmonic function φ on $X := \mathrm{supp}(T)$ such that $T = \varphi[X]$.*

It is sufficient to prove that the support $X = \mathrm{supp}(T)$ of T is a complex subvariety of pure dimension p of V. Since the problem is local, we can suppose that V is a ball in $\mathbb{C}^n$. By Corollary 3.8, we have to prove that X has locally finite $\mathcal{H}^{2p}$ measure and X is p-pseudoconcave. Consider now the *trace measure* $\sigma := T \wedge \omega^p$ of T.

PROPOSITION 4.2. — *Let T be a plurisubharmonic current of bidimension (p,p) in a complex manifold V. Then for every $\delta > 0$ the level set $\{\nu(T,a) \geqslant \delta\}$ is closed and has locally finite $\mathcal{H}^{2p}$ measure. Moreover, under the hypothesis of Theorem 4.1, we have $\nu(T,a) \geqslant \delta$ for every $a \in X$ and X has locally finite $\mathcal{H}^{2p}$ measure.*

Proof. — We may suppose without loss of generality that V is an open subset of $\mathbb{C}^n$. Set $Y := \{\nu(T,a) \geqslant \delta\}$. Let $(a_n) \subset Y$ be a sequence which converges to a point $a \in V$. Fix an $r > 0$ such that $\mathrm{B}(a,r) \subset V$. We have

$$\sigma(\mathrm{B}(a,r)) \geqslant \sigma(\mathrm{B}(a_n, r - |a_n - a|)) \geqslant 2^p \pi^p \delta (r - |a_n - a|)^{2p},$$

which implies that

$$\sigma(\mathrm{B}(a,r)) \geqslant 2^p \pi^p \delta r^{2p}$$

and

$$\nu(T,a) = \lim_{r\to 0} \frac{\sigma(\mathrm{B}(a,r))}{2^p \pi^p r^{2p}} \geqslant \delta.$$

Therefore, $a \in Y$ and hence Y is closed. We have shown that $\nu(T,.)$ is u.s.c. The last inequality also implies that Y has locally finite $\mathcal{H}^{2p}$ measure [16, 2.10.19(3)]. □

LEMMA 4.3.— *Let T be a positive current of bidimension (p,p) in $\mathbb{C}^n$ with compact support. Let $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be the linear projection on the first p coordinates, $a \in \mathbb{C}^n$ and $b := \pi(a)$. Assume that T is plurisubharmonic on $\pi^{-1}(\mathrm{B}(b,r))$ for some $r > 0$. Then $\langle T, 1_{\pi^{-1}(\mathrm{B}(b,r))}\Psi\rangle \geqslant 2^p\pi^p r^{2p}\nu(T,a)$, where $\Psi := (i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \cdots + i\mathrm{d}z_p \wedge \mathrm{d}\overline{z}_p)^p$. In particular, the Lelong number of $\pi_*(T)$ at b is greater or equal to $\nu(T,a)$.*

Proof. — We may suppose without loss of generality $a = 0$ and $b = 0$. Set $z' := (z_1, \ldots, z_p)$, $z'' := (z_{p+1}, \ldots, z_n)$, $\omega := i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \cdots + i\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n$ and fix an $\epsilon > 0$. Let $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the linear map given by $\Phi_\epsilon(z) := (z', \epsilon z'')$ and set $T' := (\Phi_\epsilon)_* T$. Since the Lelong number is independent on coordinates [3], we have $\nu(T',0) = \nu(T,0)$. By the Jensen type formula of Demailly, we have

$$\langle T', 1_{\pi^{-1}(\mathrm{B}(0,r))}\omega^p\rangle \geqslant 2^p\pi^p r^{2p}\nu(T',0)$$

since $\pi^{-1}(\mathrm{B}(0,r))$ contains the ball $\mathrm{B}(0,r)$ in $\mathbb{C}^n$. From this inequality, it follows that

$$\langle T, 1_{\pi^{-1}(\mathrm{B}(0,r))}\Phi_\epsilon^*(\omega^p)\rangle \geqslant 2^p\pi^p r^{2p}\nu(T,0).$$

On the other hand, when $\epsilon \to 0$, $\Phi_\epsilon^*(\omega^p)$ converges uniformly to Ψ. Thus,

$$\langle T, 1_{\pi^{-1}(\mathrm{B}(0,r))}\Psi\rangle \geqslant 2^p\pi^p r^{2p}\nu(T,0).$$

This implies

$$\nu(\pi_*(T),0) = \lim_{r\to 0}\frac{1}{2^p\pi^p r^{2p}}\langle T, 1_{\pi^{-1}(\mathrm{B}(0,r))}\Psi\rangle \geqslant \nu(T,0),$$

which completes the proof. □

We deduce Theorem 4.1 from Corollary 3.8 and the following lemma.

LEMMA 4.4.— *The set X is p-pseudoconcave.*

Proof. — Assume that X is not p-pseudoconcave. Then there exist an open set $U \subset\subset V$ and a holomorphic map $f : V' \longrightarrow \mathbb{C}^p$ such that $f(X \cap U) \not\subset \mathbb{C}^p \setminus \Omega$, where V' is a neighbourhood of $\overline{U}$ and Ω is the unbounded component of $\mathbb{C}^p \setminus f(X \cap bU)$.

Consider $\Phi : V' \longrightarrow \mathbb{C}^p \times V'$ the holomorphic map given by $\Phi(z) := (f(z), z)$. We next choose a domain $U' \subset\subset \mathbb{C}^p \times V'$ such that $\Phi(U)$ is a submanifold of U'. Then $T' := 1_{U'}\Phi_*(T)$ is a positive plurisubharmonic current in U'. Moreover it is easy to see that $\nu(T',a) \geqslant \delta'$ for some $\delta' > 0$ and for every $a \in X' := \mathrm{supp}(T')$. Let $\pi : \mathbb{C}^{p+n} \longrightarrow \mathbb{C}^p$ be the linear projection on the first p coordinates. We have $\pi(\overline{X}' \cap bU') = f(X \cap bU)$ and

$\pi(X' \cap U') = f(X \cap U) \not\subset \mathbb{C}^p \setminus \Omega$. The open set Ω is also the unbounded component of $\mathbb{C}^p \setminus \pi(\overline{X}' \cap bU')$.

Therefore $\pi_*(T')$ defines a positive plurisubharmonic current in Ω which vanishes in $\mathbb{C}^p \setminus \pi(\overline{X}')$. Hence there is a positive plurisubharmonic function ψ on Ω, which vanishes on $\mathbb{C}^p \setminus \pi(\overline{X}')$ such that $\pi_*(T') = \psi[\Omega]$ in Ω. Thus for every $x \in \Omega$ we have $\nu(\pi_*(T'), x) = \psi(x)$. By Lemma 4.3, we have $\psi(x) \geqslant \delta'$ for every $x \in \pi(X') \cap \Omega$.

Now fix a point x in the boundary Σ of $\pi(X')$ in Ω. Let $c \in \Omega \setminus \pi(\overline{X}')$ be a point close to x. Let $b \in \pi(X')$ such that $\mathrm{dist}(b, c) = \mathrm{dist}(\pi(X'), c)$. Since c is close to x, we have $b \in \Sigma$ and $\mathrm{B}(c, |b-c|) \subset \Omega$. By the submean value property of plurisubharmonic functions and the fact that $\psi = 0$ on $\mathrm{B}(c, |b-c|)$, we obtain

$$\psi(b) = \lim_{r\to 0} \frac{1}{2^p \pi^p r^{2p}} \int_{\mathrm{B}(b,r)} \psi(z)\Psi = \lim_{r\to 0} \frac{1}{2^p \pi^p r^{2p}} \int_{\mathrm{B}(b,r)\setminus \mathrm{B}(c,|b-c|)} \psi(z)\Psi.$$

By upper semicontinuity, the last limit is smaller or equal to

$$\lim_{r\to 0} \frac{1}{2^p \pi^p r^{2p}} \int_{\mathrm{B}(b,r)\setminus \mathrm{B}(c,|b-c|)} \psi(b)\Psi = \frac{\psi(b)}{2}.$$

Thus $\psi(b) \leqslant \frac{1}{2}\psi(b)$ and $\psi(b) \leqslant 0$. This contradicts the fact that $\psi(b) \geqslant \delta'$. □

Corollary 4.5. — *Let T be a positive plurisubharmonic current of bidimension (p,p) in a complex manifold V of dimension n. If T is locally rectifiable, then there exists a locally finite family of complex subvarieties $(X_i)_{i\in I}$ of pure dimension p of V and positive integers n_i such that $T = \sum_{i\in I} n_i [X_i]$. In particular, T is closed.*

Remark 4. — King has proved the same result for rectifiable closed positive current [24]. Harvey, Shiffman and Alexander [6, 22] proved it for rectifiable closed currents (in this case, the positivity is not necessary).

Proof. — By [16, 4.1.28(5)], $\nu(T, a)$ is a strictly positive integer for a in a dense subset of $X := \mathrm{supp}(T)$. This, combined with Theorem 4.1, implies that X is a complex subvariety of pure dimension p of V and $T = \varphi[X]$, where φ is a weakly plurisubharmonic function on X. Since T is rectifiable, the function φ has integer values $\mathcal{H}^{2p}$-almost everywhere. Therefore φ is essentially equal to a positive integer in each irreducible component of X. □

The following theorem is a variant of Theorem 4.1 (*see* also Example 1).

THEOREM 4.1$'$.— *Let T be a positive plurisubharmonic current of bidimension (p,p) in a complex manifold V of dimension n. Assume that* $\mathrm{supp}(T)$ *has locally finite $\mathcal{H}^{2p}$ measure and the set $\mathcal{E} := \{z \in \mathrm{supp}(T), \nu(T,z) = 0\}$ has zero $\mathcal{H}^{2p-1}$ measure. Then* $\mathrm{supp}(T)$ *is a complex subvariety of pure dimension p of V and there exists a weakly plurisubharmonic function φ on $X := \mathrm{supp}(T)$ such that $T = \varphi[X]$.*

Proof. — We only need to prove Lemma 4.4. More precisely, we have to find a point $b \in \Sigma \setminus \pi(\mathcal{E})$ such that the upper-density

$$\overline{\Theta}(b) := \limsup_{r\to 0} \frac{1}{2^p\pi^p r^{2p}} \int_{\mathrm{B}(b,r)\cap\Omega} \Psi$$

of Ω at b is strictly positive. Following [38, 5.8.5, 5.9.5], the set $\Sigma' := \{b \in \Sigma, \overline{\Theta}(b) > 0\}$ has positive $\mathcal{H}^{2p-1}$ measure. It is sufficient to take b in $\Sigma' \setminus \pi(\mathcal{E})$. The last set is not empty since $\mathcal{H}^{2p-1}(\mathcal{E}) = 0$. □

For plurisuperharmonic currents, we have the following theorem.

THEOREM 4.6. — *Let T be a positive plurisuperharmonic current of bidimension (p,p) in a complex manifold V of dimension n. Let K be a compact subset of V which admits a Stein neighbourhood. Assume that in $V \setminus K$ the support* $\mathrm{supp}(T)$ *of T has locally finite $2p$-dimensional Hausdorff measure. Then* $\mathrm{supp}(T)$ *is a complex subvariety of pure dimension p of V and there exists a weakly plurisuperharmonic function φ on $X := \mathrm{supp}(T)$ such that $T = \varphi[X]$. Moreover, if T is pluriharmonic, then φ is weakly pluriharmonic on X.*

Proof. — By Proposition 2.2, $X := \mathrm{supp}(T)$ is p-pseudoconcave subset of V. Moreover, by Corollary 3.8, X is a complex subvariety of pure dimension p of V. The theorem follows. □

We remark here that Theorem 4.6 is false for positive plurisubharmonic currents. Consider an example.

Example 1.— Let ψ be a positive subharmonic function in $\mathbb{C}$ which vanishes in the unit disk $\mathrm{B}(0,1)$. We can take for example $\psi(z) := \log|z|$ if $|z| > 1$ and $\psi(z) := 0$ if $|z| \leqslant 1$. Let f be a holomorphic function in $\mathbb{C} \setminus \mathrm{B}(0,1/2)$ which cannot be extended to a meromorphic function on $\mathbb{C}$. Denote by $Y \subset \mathbb{C}^2$ the graph of f over $\mathbb{C} \setminus \mathrm{B}(0,1/2)$. Let φ be the subharmonic function on Y given by $\varphi(z) := \psi(z_1)$. Then $T := \varphi[Y]$ is a positive plurisubharmonic current of $\mathbb{C}^2$. Its support $X \subset Y$ is not a complex subvariety of $\mathbb{C}^2$ and cannot be extended to a complex subvariety. We can construct more complicated examples by taking countable combinations of such currents.

COROLLARY 4.7. — *Under the hypothesis of Theorem 4.1, Theorem 4.1′ or Theorem 4.6, if V is compact, then there are complex subvarieties X_1, …, X_k of pure dimension p of V and positive real numbers c_1, …, c_k such that $T = c_1[X_1] + \cdots + c_k[X_k]$. In particular, T is closed.*

Proof. — By Theorems 4, 4.1′ and 4.6, X is a complex subvariety of pure dimension p. Since V is compact, X has a finite number of irreducible components and φ is essentially constant in each component. □

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