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# Weighted Csiszár-Kullback-Pinsker inequalities and applications to transportation inequalities (*)

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**Abstract.** — We strengthen the usual Csiszár-Kullback-Pinsker inequality by allowing weights in the total variation norm; admissible weights depend on the decay of the reference probability measure. We use this result to derive transportation inequalities involving Wasserstein distances for various exponents: in particular, we recover the equivalence between a $T_1$ inequality and the existence of a square-exponential moment. Then we give a variant of the results obtained by Djellout, Guillin and Wu [5] about transportation inequalities for random dynamical systems, in which a sufficient condition is expressed in terms of exponential moments. An unpublished result by Blower [1] about the perturbation of a $T_2$ inequality is also recovered and generalized.

**Résumé.** — Nous généralisons l'inégalité bien connue de Csiszár-Kullback-Pinsker, en introduisant des fonctions de poids dans la variation totale ; les poids admissibles dépendent de la décroissance à l'infini de la mesure de référence. De cette nouvelle inégalité nous déduisons des inégalités de transport faisant intervenir certaines distances de Wasserstein, retrouvant en particulier l'équivalence d'une inégalité $T_1$ et de l'existence d'un moment carré-exponentiel. En application de ces résultats, nous établissons une variante des résultats de Djellout, Guillin et Wu [5] sur des inégalités de transport pour des systèmes dynamiques aléatoires, sous une condition plus naturelle et plus générale portant désormais sur des moments exponentiels. Par la même occasion nous retrouvons et généralisons un résultat non publié de Blower [1] relatif à la perturbation d'une inégalité $T_2$.

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# Contents



## 1. Introduction

Let $X$ be an abstract Polish space, and let $P(X)$ be the set of all Borel probability measures on $X$; let $d$ be a lower semi-continuous metric on $X$, and let $p$ belong to $[1, +\infty)$. Whenever $\mu, \nu$ belong to $P(X)$, we define

- the **Wasserstein distance of order** $p$ between $\mu$ and $\nu$ by

$$W_p(\mu, \nu) = \inf \left( \iint d(x,y)^p d\pi(x,y) \right)^{1/p}$$

where $\pi$ runs over the set of probability measures on $X \times X$ with marginals $\mu$ and $\nu$;

- the **Kullback information** of $\mu$ with respect to $\nu$ by

$$H(\mu|\nu) = \int f \log f \, d\nu, \qquad f = \frac{d\mu}{d\nu};$$

by convention $H(\mu|\nu) = +\infty$ if $\mu$ is not absolutely continuous with respect to $\nu$.

Both objects play an important role in a number of problems in probability theory, where they may be encountered under the names of Monge-Kantorovich distances, or minimal distances, and relative entropy, or relative $H$ functional. More information can be found, together with many references, in [11]. For various purposes it is of interest to investigate whether they can be compared to each other. The most famous such inequality is the **Csiszár-Kullback-Pinsker inequality**, which we shall denote CKP inequality for short: if $d$ is the trivial distance, i.e. $d(x,y) = 1_{x\neq y}$, then

$$2W_1(\mu,\nu) = \|\mu - \nu\|_{TV} \leqslant \sqrt{2H(\mu|\nu)},$$

where "$TV$" stands for the total variation norm.

Another class of inequalities which has been studied at length is encountered under the names of **Talagrand inequalities**, **transportation inequalities**, or **transportation cost-information inequalities**; we shall just denote it by $T_p$. By definition, a reference probability measure $\nu$ satisfies the $T_p(\lambda)$ inequality for some $\lambda > 0$ if

$$\forall \mu \in P(X), \quad W_p(\mu,\nu) \leqslant \sqrt{\frac{2H(\mu|\nu)}{\lambda}};$$

and it satisfies $T_p$ if it satisfies $T_p(\lambda)$ for some $\lambda > 0$. In particular, CKP inequality means that *any* reference probability measure satisfies $T_1(4)$ when $d$ is the trivial distance.

We note right away that $W_p \leqslant W_{p'}$ for $p \leqslant p'$, so that $T_p$ inequalities become stronger and stronger as $p$ becomes larger. The cases $p = 1$ and $p = 2$ are of particular interest.

The study of $T_p$ inequalities is a rather old topic [9], which recently received a new impulse. First, it was pointed out by Marton [7] and Talagrand [10] that these inequalities are a handy tool in the study of *concentration of measure* [6]; in particular, Talagrand showed how to take advantage of the good tensorization properties of inequality $T_2$, to establish concentration in product spaces. At the same time, he established the validity of $T_2$ for the Gaussian measure, which justifies the terminology of "Talagrand inequalities". On the other hand, recent developments of the theory of optimal transportation led to new connections between these inequalities and other classes of functional inequalities with a geometric content, in particular **logarithmic Sobolev inequalities**. For instance, the main result in [8] is that *a logarithmic Sobolev inequality implies a $T_2$ inequality* (and the converse is also true under some convexity assumption). Various proofs and variants of these results, together with a detailed discussion, can be found in [2, 8, 11].

On the other hand, the works by Bobkov and Götze [3], and Djellout, Guillin and Wu [5] suggest that there is still room for investigation in an abstract Polish space setting, without any underlying geometric structure. More precisely, given a reference probability measure $\nu$, one of the main results proven in these references is the equivalence between

1. $\nu$ satisfies a $T_1$ inequality;

2. there exists $\lambda$ such that $\int e^{t(f(x)-\int f(x)\,d\nu(x))}\,d\nu(x) \leqslant e^{\frac{t^2}{2\lambda}}$ for any real $t$ and Lipschitz function $f$ with Lipschitz norm 1;

3. $\nu$ admits a square-exponential moment, i.e. $\int e^{\alpha d(x,y)^2}\,d\nu(x)$ is finite for some $\alpha > 0$ and some (and thus any) $y$.

Notice how tractable is this criterium for $T_1$: for instance, the validity of a logarithmic Sobolev inequality depends on subtle properties of the reference measure, which imply not only the existence of a square-exponential moment, but also – among other features which are still poorly identified – strict positivity, in a quantitative way which has not been made precise so far (see however [4] for important progress in that direction). Djellout, Guillin and Wu explored various applications of their result, including $T_1$ inequalities in path space for solutions of stochastic differential equations, or $T_1$ inequalities in large dimension for random dynamical systems under adequate assumptions of weak dependence.

The purpose of this paper is twofold.

On one hand, we shall establish a generalization of the CKP inequality, allowing for a weight in the total variation. How much weight is allowed will depend on the decay of the reference measure. In that generalization, the optimal constant 4 will be lost, but this will be more than compensated by the gain of precision brought by the weight. In view of the large range of applications of the usual CKP inequality, we do hope that this generalization can be of interest in various contexts.

On the other hand, we shall point out that, instead of considering CKP inequality as just a particular case of $T_1$, it is possible to establish many general comparison results between $W_p$ and $H$ by studying the weighted CKP inequality. In particular, we shall recover in a straightforward way (and with improved constants) the above-mentionned result according to which a square-exponential moment implies $T_1$. Then we shall establish a variant of the result by Djellout, Wu and Guillin [5] about random dynamical systems, in which assumptions are only expressed in terms of exponential moments. Not only are these conditions easier to check, but they also allow for more generality. Also, we shall establish weakened versions of $T_1$ and $T_2$ inequalities, in which the square-root on the right-hand side is replaced by a combination of powers, and which are satisfied with quite a bit of generality, under just decay assumptions on the reference measure. Among them is a generalization of an unpublished partial result by Blower [1] about the perturbation of $T_2$ inequalities.

The plan of the paper is as follows. In section 2, we state our weighted CKP inequality and derive from it various applications to the study of $T_p$ inequalities and their variants. In section 3, we give a detailed proof of the weighted CKP inequality. Finally, in section 4, we show how our results can be applied to the study of discrete-time processes.

## 2. Main results

Working in a Polish space is a natural assumption when handling Wasserstein distances, because it is sufficient to derive all the well-known and useful properties of these distances, in particular their relation with the weak topology [11]. However, for all the results in this section, no use will be made of completeness or separability, and so we state the results with more generality.

In the sequel, the notation $\varphi(\mu-\nu)$ is a shorthand for the signed measure $\varphi\mu - \varphi\nu$.

THEOREM 2.1 (weighted CKP inequalities). — *Let $X$ be a measurable space, let $\mu$, $\nu$ be two probability measures on $X$, and let $\varphi$ be a nonnegative measurable function on $X$. Then*

$$(i)\ \|\varphi\,(\mu-\nu)\|_{TV} \leqslant \left(\frac{3}{2} + \log \int e^{2\varphi(x)}\, d\nu(x)\right)\left(\sqrt{H(\mu|\nu)} + \frac{1}{2}H(\mu|\nu)\right);$$

$$(ii)\ \|\varphi\,(\mu-\nu)\|_{TV} \leqslant \sqrt{2}\left(1 + \log \int e^{\varphi(x)^2}\, d\nu(x)\right)^{1/2}\sqrt{H(\mu|\nu)}.$$

*Remark 2.2.* — 1. The assumption $\int_X e^{\varphi^2}\, d\nu < +\infty$ is always stronger than the assumption $\int_X e^{2\varphi}\, d\nu < +\infty$, so the inequality (i) above always applies in more generality than (ii). Further note that if we choose $\varphi \equiv 1$ in (ii), we recover the usual CKP inequality

$$\|\mu-\nu\|_{TV} \leqslant c\sqrt{H(\mu|\nu)}$$

with the non-optimal constant $c = 2$ instead of $\sqrt{2}$. This shows that the constants on the right-hand side of (ii) cannot be improved by more than a factor $\sqrt{2}$. Although we worked quite a bit to decrease this numerical constant, it is likely that one can still do better, at least by replacing $\int e^{\varphi^2}$

with $\int e^{\lambda\varphi^2}$. Note though that the optimal constant $\sqrt{2}$ can be recovered by writing our proof again in the particular case $\varphi \equiv 1$, as it shall be pointed out in section 3.

2. Let us discuss very briefly the sharpness of the orders of magnitude in the above inequalities. When $\mu$ is very close to $\nu$, the Kullback information can be approximated by a weighted squared $L^2$ norm, which shows that it is natural to expect a term in $\sqrt{H(\mu|\nu)}$ (as opposed to another power of $H$) in the right-hand side. On the other hand, consider the situation when $X = \mathbb{R}^n$, and the reference measure $\nu$ is the standard Gaussian distribution; choose $\varphi(x) = \delta|x|$ for $\delta < 1/\sqrt{2}$. Then the left-hand side of inequality (ii) will be typically $O(\sqrt{n})$ as $n \to \infty$, while the right-hand side will be typically $O(n)$. If $\varphi(x) = \delta \sum |x_i|/\sqrt{n}$, then the left-hand side will be typically $O(n)$, while the right-hand side will be typically $O(n^{3/2})$. These examples suggest that Theorem 2.1 still leaves room for improvement for problems set in large dimension. As we shall see in Section 4, this loss of a $O(\sqrt{n})$ factor will put limitation on the validity of measure concentration inequalities that can be deduced from Theorem 2.1 in large dimension.

We postpone the proof of Theorem 2.1 to the next section, and now list two consequences.

COROLLARY 2.3. — *Let $X$ be a measurable space equipped with a measurable distance $d$, let $p \geqslant 1$ and let $\nu$ be a probability measure on $X$. Assume that there exist $x_0 \in X$ and $\alpha > 0$ such that $\int e^{\alpha d(x_0,x)^p}\, d\nu(x)$ is finite. Then*

$$\forall \mu \in P(X), \qquad W_p(\mu,\nu) \leqslant C\Big[H(\mu|\nu)^{\frac{1}{p}} + \Big(\frac{H(\mu|\nu)}{2}\Big)^{\frac{1}{2p}}\Big],$$

*where*

$$C := 2 \inf_{x_0\in X,\, \alpha>0} \left(\frac{1}{\alpha}\Big(\frac{3}{2} + \log \int e^{\alpha d(x_0,x)^p}\, d\nu(x)\Big)\right)^{\frac{1}{p}} < +\infty.$$

COROLLARY 2.4. — *Let $X$ be a measurable space equipped with a measurable distance $d$, let $p \geqslant 1$ and let $\nu$ be a probability measure on $X$. Assume that there exist $x_0 \in X$ and $\alpha > 0$ such that $\int e^{\alpha d(x_0,x)^{2p}}\, d\nu(x)$ is finite. Then*

$$\forall \mu \in P(X), \qquad W_p(\mu,\nu) \leqslant C\, H(\mu|\nu)^{\frac{1}{2p}},$$

*where*

$$C := 2 \inf_{x_0 \in X,\ \alpha>0} \left( \frac{1}{2\alpha} \left( 1 + \log \int e^{\alpha d(x_0,x)^{2p}} \, d\nu(x) \right) \right)^{\frac{1}{2p}} < +\infty.$$

*Particular case 2.5.* — When $X$ is bounded, a simpler bound holds:

$$\forall \mu \in P(X), \qquad W_p(\mu,\nu) \leqslant 2^{\frac{1}{2p}} \operatorname{diam}(X)\, H(\mu|\nu)^{\frac{1}{2p}},$$

where $\operatorname{diam}(X) := \sup\{d(x,y);\ x,y \in X\}$.

Since the proofs of these results are very similar, we only give the proof of Corollary 2.4.

*Proof of Corollary* 2.4. — On one hand it is known [11, Proposition 7.10] that

$$W_p^p(\mu,\nu) \leqslant 2^{p-1} \| d(x_0,.)^p \, (\mu-\nu) \|_{TV};$$

on the other hand the second part of Theorem 2.1 yields

$$\left\| \sqrt{\alpha}\, d(x_0,.)^p \, (\mu-\nu) \right\|_{TV} \leqslant \sqrt{2} \left( 1 + \log \int e^{\alpha d(x_0,x)^{2p}} \, d\nu(x) \right)^{1/2} \sqrt{H(\mu|\nu)}.$$

This concludes the argument. □

We now focus on some particular cases of interest, namely for $p=1$ and $p=2$ under assumptions of exponential moments of order 1, 2 and 4.

COROLLARY 2.6. — *Let $X$ be a measurable space equipped with a measurable distance $d$, let $\nu$ be a reference probability measure on $X$, and let $x_0$ be any element of $X$. Then*

*(i) If $\int_X e^{\alpha d(x_0,x)} \, d\nu(x) < +\infty$ for some $\alpha > 0$, then there is a constant $C$ such that*

$$\forall \mu \in P(X), \qquad W_1(\mu,\nu) \leqslant C \left( H(\mu|\nu) + \sqrt{H(\mu|\nu)} \right).$$

*(ii) If* $\int_X e^{\alpha d(x_0,x)^2}\, d\nu(x) < +\infty$ *for some* $\alpha > 0$, *then there is a constant* $C$ *such that*

$$\forall \mu \in P(X), \qquad W_1(\mu,\nu) \leqslant C\,\sqrt{H(\mu|\nu)};$$

$$\forall \mu \in P(X), \qquad W_2(\mu,\nu) \leqslant C\,\left[\sqrt{H(\mu|\nu)} + H(\mu|\nu)^{\frac{1}{4}}\right].$$

*In particular,* $\nu$ *satisfies* $T_1$.

*(iii) If* $\int_X e^{\alpha d(x_0,x)^4}\, d\nu(x) < +\infty$ *for some* $\alpha > 0$, *then there is a constant* $C$ *such that*

$$\forall \mu \in P(X), \qquad W_2(\mu,\nu) \leqslant C\, H(\mu|\nu)^{\frac{1}{4}}.$$

*Remark 2.7.* — Part (ii) of this corollary contains the result that the existence of an exponential moment of order 2 implies a $T_1$ inequality; according to [3], the converse is true, so this criterion is optimal. To compare these various results in practical situations, it is good to keep in mind the following elementary lemma:

LEMMA 2.8. — *Let* $X$ *be a measurable space equipped with a measurable distance* $d$, *let* $p \geqslant 1$ *and let* $\nu$ *be a probability measure on* $X$. *Then the following three statements are equivalent:*

1. *there exist* $x_0 \in X$ *and* $\alpha > 0$ *such that* $\int e^{\alpha d(x_0,x)^p}\, d\nu(x)$ *is finite;*
2. *for any* $x_0 \in X$, *there exists* $\alpha > 0$ *such that* $\int e^{\alpha d(x_0,x)^p}\, d\nu(x)$ *is finite;*
3. *there exists* $\alpha > 0$ *such that* $\iint e^{\alpha d(x,y)^p}\, d\nu(x)\, d\nu(y)$ *is finite.*

*Moreover,*

$$\begin{aligned} \inf_{x_0\in X} \int e^{\alpha d(x_0,x)^p}\, d\nu(x) &\leqslant \iint e^{\alpha d(x,y)^p}\, d\nu(x)\, d\nu(y) \\ &\leqslant \left( \inf_{x_0\in X} \int e^{\alpha 2^{p-1} d(x_0,x)^p}\, d\nu(x) \right)^2. \end{aligned}$$

*Remark 2.9.* — The following two results can be deduced from the equivalence between the existence of an exponential moment of order 2 and a $T_1$ inequality:

1. Let $\mu$ be a probability measure on a Polish space $X$, satisfying $T_1$. Then so does any probability measure $\nu = h\mu$, where $h$ is a $\mu$-almost surely bounded measurable function on $X$.

2. Let $\mu$ be a probability measure on $\mathbb{R}^d$ satisfying $T_1$. Then so does its marginal (via orthogonal projection) on any hyperplane of $\mathbb{R}^d$.

*Remark 2.10.* — Part (ii) also generalizes the perturbation result proven by Blower, who showed in [1] that an inequality of the form $W_2 \leqslant C(H^{1/2} + H^{1/4})$ holds true when $\nu$ is bounded from above and below by constant multiples of a reference measure $\nu_0$ satisfying $T_2$. Indeed if $\nu_0$ satisfies $T_2$, then it also satisfies $T_1$, so it has a finite square-exponential moment, and so does $\nu$ if it is bounded above by a constant multiple of $\nu_0$.

*Remark 2.11.* — Let $\nu$ be a reference probability measure having finite exponential moments of order $p$; how far is it from satisfying $T_p$? The preceding results indicate that the answer is very different for $p = 1$ and $p = 2$. If $T_1$ is not satisfied, this means that the decay of $\nu$ at infinity is not fast enough, and the $T_1$ inequality usually fails for *large* values of the Kullback information. On the contrary, if $T_2$ is not satisfied, this is not necessarily just for a question of fast decay (remember that $T_2$ implies strict positivity), and the $T_2$ inequality usually fails for *small* values of the Kullback information. In particular, it is no wonder that we did not manage to recover $T_2$ inequalities with our arguments taking into account only the decay of $\nu$.

## 3. Proof of the main inequalities

We shall now present detailed proofs of the main inequalities in Theorem 1.

*Proof of Theorem 2.1.* — Without loss of generality, we assume that $\mu$ is absolutely continuous with respect to $\nu$, with density $f$. We set $u := f - 1$, so that

$$\mu = (1+u)\nu;$$

we note that $u \geqslant -1$ and $\displaystyle\int u\, d\nu = 0$. We also define

$$h(v) := (1+v)\log(1+v) - v, \qquad v \in [-1, +\infty)$$

so that

$$H(\mu|\nu) = \int_X h(u)\, d\nu. \tag{3.1}$$

We note that $h \geqslant 0$.

We start with the **proof of inequality (i)**, splitting the weighted total variation as

$$\int \varphi\, d|\mu - \nu| = \int \varphi\, |u|\, d\nu = \int_{\{-1 \leqslant u \leqslant 4\}} \varphi\, |u|\, d\nu + \int_{\{u > 4\}} \varphi\, u\, d\nu. \qquad (3.2)$$

We shall estimate both terms separately, first bounding the **first term** ($u \leqslant 4$) in (3.2). By Cauchy-Schwarz inequality,

$$\int_{u \leqslant 4} \varphi\, |u|\, d\nu \leqslant \left( \int_{u \leqslant 4} \varphi^2\, d\nu \right)^{1/2} \left( \int_{u \leqslant 4} u^2\, d\nu \right)^{1/2}.$$

On the other hand, from the elementary inequality

$$-1 \leqslant v \leqslant 4 \quad \Longrightarrow \quad v^2 \leqslant 4h(v)$$

(a consequence of the fact that $h(v)/v$ is nondecreasing) we deduce

$$\int_{u \leqslant 4} u^2\, d\nu \leqslant 4 \int_{u \leqslant 4} h(u)\, d\nu.$$

Combining this with the nonnegativity of $h$ and (3.1), we find

$$\int_{u \leqslant 4} \varphi\, |u|\, d\nu \leqslant 2 \left( \int_X \varphi^2\, d\nu \right)^{1/2} \left( \int_X h(u)\, d\nu \right)^{1/2} = 2 \left( \int_X \varphi^2\, d\nu \right)^{1/2} H(\mu|\nu)^{1/2}. \qquad (3.3)$$

Since the function $t \longmapsto e^{2\sqrt{t}}$ is increasing and convex on $[1/4, +\infty)$ we can write

$$\begin{aligned} \exp\left( 2\sqrt{\int_X \varphi^2\, d\nu} \right) &\leqslant \exp\left( 2\sqrt{\int_X (\varphi + 1/2)^2\, d\nu} \right) \\ &\leqslant \int_X \exp\left( 2\sqrt{(\varphi + 1/2)^2} \right) d\nu = \int_X e^{2\varphi + 1}\, d\nu. \end{aligned}$$

In other words,

$$2\sqrt{\int_X \varphi^2\, d\nu} \leqslant 1 + \log \int e^{2\varphi}\, d\nu;$$

if we plug this into (3.3), we conclude that

$$\int_{u\leqslant 4} \varphi\,|u|\,d\nu \leqslant \left(1 + \log \int_X e^{2\varphi}\,d\nu\right) H(\mu|\nu)^{1/2}. \tag{3.4}$$

We now turn to the estimate of the **second term** ($u > 4$) in (3.2). By applying the Young-type inequality

$$w\xi \leqslant w\log w - w + e^{\xi} \qquad (w \geqslant 0, \xi \in \mathbb{R}) \tag{3.5}$$

with $w = u(x)$ and $\xi = \varphi(x) - Z$, where $Z$ is a nonnegative constant to be chosen later, we find

$$\begin{aligned} u(x)\varphi(x) &\leqslant u(x)\log u(x) - u(x) + e^{\varphi(x)-Z} + Zu(x) \\ &\leqslant h(u(x)) + \left(\inf_{v>4}\sqrt{h(v)}\right)^{-1} e^{\varphi(x)-Z}\sqrt{h(u(x))} + Zu(x) \end{aligned}$$

on $\{u(x) > 4\}$. By integration, we deduce

$$\int_{u>4} u\varphi\,d\nu \leqslant \int_{u>4} h(u)\,d\nu + \sqrt{k}\int_{u>4} e^{\varphi-Z}\sqrt{h(u)}\,d\nu + Z\int_{u>4} u\,d\nu,$$

where

$$k := \left(\inf_{v>4} h(v)\right)^{-1} = \frac{1}{h(4)} < \frac{1}{4}.$$

By Cauchy-Schwarz inequality again,

$$\begin{aligned} \int_{u>4} e^{\varphi-Z}\sqrt{h(u)}\,d\nu &\leqslant \sqrt{\int_X e^{2(\varphi-Z)}\,d\nu}\sqrt{\int_{u>4} h(u)\,d\nu} \\ &= \sqrt{\int_X e^{2(\varphi-Z)}\,d\nu}\,\sqrt{H(\mu|\nu)}. \end{aligned}$$

Finally, from the inequality

$$v \geqslant 4 \implies v \leqslant 4k\,h(v)$$

we deduce

$$\int_{u>4} u\,d\nu \leqslant 4k\int_{u>4} h(u)\,d\nu \leqslant 4kH(\mu|\nu).$$

Our conclusion is that, for any constant $Z \geqslant 0$,

$$\int_{u>4} \varphi\, u\, d\nu \leqslant (1+4kZ)H(\mu|\nu) + \sqrt{k}\sqrt{\int_X e^{2(\varphi-Z)}\, d\nu}\,\sqrt{H(\mu|\nu)}. \qquad (3.6)$$

We now choose $Z$ in such a way that

$$\int_X e^{2(\varphi-Z)}\, d\nu = 1;$$

in other words,

$$Z := \frac{1}{2}\log\int e^{2\varphi}\, d\nu \geqslant 0.$$

Plugging this into (3.6), we conclude that

$$\int_{u>4} \varphi\, u\, d\nu \leqslant \left(1+2k\log\int e^{2\varphi}\, d\nu\right) H(\mu|\nu) + \sqrt{k}\,\sqrt{H(\mu|\nu)}. \qquad (3.7)$$

Now inequality (i) follows from (3.4) and (3.7) upon noting that $1+\sqrt{k} < \frac{3}{2}$ and $2k < \frac{1}{2}\cdot$

We next turn to the **proof of (ii)**. Although the decomposition (3.2) and the same kind of argument would also lead to the result, we prefer to proceed as follows.

Since $h(0) = h'(0) = 0$, by Taylor's formula with integral remainder, we can write

$$h(u) = u^2\int_0^1 \frac{1-t}{1+tu}\, dt,$$

and thus

$$H(\mu|\nu) = \int_X\int_0^1 \frac{u^2(x)\,(1-t)}{1+tu(x)}\, d\nu(x)\, dt.$$

On the other hand, by Cauchy-Schwarz inequality on $(0,1) \times X$

$$\left(\int_0^1 (1-t)\,dt\right)^2 \left(\int_X \varphi|u|\,d\nu\right)^2 = \left(\int_{(0,1)\times X} (1-t)\varphi|u|\,d\nu\,dt\right)^2$$

$$\leqslant \left[\iint (1-t)\,(1+tu)\,\varphi^2\,d\nu\,dt\right] \left[\iint \frac{1-t}{1+tu}|u|^2\,d\nu\,dt\right];$$

thus

$$\left(\int \varphi\,|u|\,d\nu\right)^2 \leqslant CH(\mu|\nu)$$

where

$$C := \frac{\displaystyle\iint (1-t)\,(1+tu)\,\varphi^2\,d\nu\,dt}{\left(\displaystyle\int_0^1 (1-t)\,dt\right)^2}. \tag{3.8}$$

We decompose the numerator as follows:

$$\iint (1-t)\,(1+tu)\,\varphi^2\,d\nu\,dt = \int (1-t)t\,dt \int (1+u)\,\varphi^2\,d\nu + \int (1-t)^2\,dt \int \varphi^2\,d\nu$$

$$= \frac{1}{6}\int \varphi^2\,d\mu + \frac{1}{3}\int \varphi^2\,d\nu. \tag{3.9}$$

From the convexity inequality

$$\int \varphi^2\,d\mu \leqslant H(\mu|\nu) + \log \int e^{\varphi^2}\,d\nu, \tag{3.10}$$

(a well-known consequence of (3.5), see for instance [6, eq. (5.13)]) and Jensen's inequality, in the form

$$\int \varphi^2\,d\nu \leqslant \log \int e^{\varphi^2}\,d\nu, \tag{3.11}$$

we deduce that the right-hand side of (3.9) is bounded above by

$$\frac{1}{6}H(\mu|\nu) + \frac{1}{2}\log \int e^{\varphi^2}\,d\nu.$$

Plugging this into (3.8), we conclude that

$$\left(\int \varphi\,|u|\,d\nu\right)^2 \leqslant \left(\frac{2}{3}H + 2L\right)H, \tag{3.12}$$

where $H$ stands for $H(\mu|\nu)$ and $L$ for $\log \int e^{\varphi^2}\,d\nu$.

The preceding bound is good only for "small" values of $H$. We now complement it with another bound which is relevant for "large" values of $H$. To do so, we write

$$\begin{aligned}\left(\int \varphi|u|\,d\nu\right)^2 &\leqslant \int \varphi^2|u|\,d\nu \int |u|\,d\nu \\ &\leqslant \left(\int \varphi^2\,d\mu + \int \varphi^2\,d\nu\right)\left(\int d\mu + \int d\nu\right) \\ &\leqslant (H+2L)\,2\end{aligned}$$

where we have successively used Cauchy-Schwarz inequality, the inequality $|u| \leqslant 1+u+1$ on $[-1,+\infty)$ (which results in $|u|\,\nu \leqslant \mu+\nu$), and finally (3.10) and (3.11).

Combining this with (3.12), we obtain

$$\left(\int \varphi|u|\,d\nu\right)^2 \leqslant \min\left((2H)\left(\frac{H}{3}+L\right),\, 2(H+2L)\right).$$

From the elementary inequality

$$\min(at^2+bt,\, t+d) \leqslant Mt, \qquad M = \frac{1}{2}\left\{1+b+\sqrt{(b-1)^2+4ad}\right\}$$

we get

$$\int \varphi|u|\,d\nu \leqslant m\sqrt{H(\mu|\nu)}$$

where

$$m \leqslant \sqrt{1+L+\sqrt{(L-1)^2+\frac{8}{3}L}} \leqslant \sqrt{2}\,\sqrt{L+1}.$$

This conclues the proof. $\square$

*Remark 3.1. — If $\varphi \equiv 1$, we can replace the inequality (3.10) by just $\int d\mu = 1$; then the first part of the proof of (ii) becomes a proof of the usual CKP inequality, with the sharp constant $\sqrt{2}$.*

## 4. Application to random dynamical systems

Let now be given a Polish space $X$, an arbitrary element $x_0 \in X$ and a set of conditional Borel probability measures $(P_k(\,\cdot\,|x^{k-1}))_{x^{k-1}\in X^{k-1},\,k\geqslant 1}$, depending on $x^{k-1} = (x_1,\ldots,x_{k-1}) \in X^{k-1}$ in a measurable way. We interpret $x_0$ as the (deterministic) initial position of a random dynamical system $(X_k)_{k\in\mathbb{N}}$, with values in $X$, and $P_k(\cdot|x^{k-1})$ as the law of $X_k$, knowing that $X_0 = x_0$ and $(X_1,\ldots,X_{k-1}) = x^{k-1}$. The question is whether it is possible, knowing some nice bounds on the conditional probability measures, to get a $T_1$ inequality for the law $P^n$ of $(X_1,\ldots,X_n)$ on $X^n$, with a nice dependence on $n$.

Let us first assume that all the conditional probability measures satisfy a $T_1$ inequality, say with a uniform constant. In the context of independent random variables, it is rather easy [6, p. 122] to show that $P^n$ satisfies $T_1(\lambda)$ for $\lambda^{-1} = O(\sqrt{n})$, and that this is sharp in general. Now we want to know whether the same behavior is generic for dependent random variables. Some results in that direction have been obtained by Marton and by Rio; they are summarized and slightly improved in [5]. In those references it is shown that if each $P_k(\,\cdot\,|x^{k-1})$ satisfies $T_1(\kappa)$ for some fixed $\kappa > 0$, and the random dynamical system is weakly dependent, in the sense that the future does not depend too much on the present, then the answer is positive. See [5, Section 4] for precise assumptions. For instance, a sufficient condition is that the dynamical system is Markovian and that the map

$$x_{k-1} \longmapsto P_k(\,\cdot\,|x^{k-1})$$

is $L$-Lipschitz from $X$ to $P(X)$, equipped with the $W_1$ distance, uniformly in $k$, for some $L < 1$.

In the present section, we shall establish a variant of this result under a different set of assumptions, which seems to be easier to check in practical situations, because it is expressed in terms of exponential moments with respect to a given origin point (which we chose, arbitrarily, as the starting point of the dynamical system). What will make our argument work (in a very straightforward way) is the simple and explicit dependence of the constants in Theorem 2.1 upon $n$ when $X$ is replaced by $X^n$.

In the sequel we consider a Polish space $X$, equipped with a measurable distance $d$, $x_0$ an arbitrary element in $X$, and $(P_k(\,\cdot\,|x^{k-1}))_{x^{k-1}\in X^{k-1},\,k\geqslant 1}$ a family of Borel probability measures on $X$, depending on $x^{k-1} := (x_1,\ldots,x_{k-1}) \in X^{k-1}$ in a measurable way. For all $n \geqslant 1$, we define the probability measure $P^n$ on $X^n$ by

$$dP^n(x_1,\ldots,x_n) = dP_1(x_1)\,dP_2(x_2|x_1)\,\cdots\,dP_n(x_n|x_1,\ldots,x_{n-1}),$$

and equip $X^n$ with the distance $D$ defined by

$$D(x,y) = D_2(x,y) := \sqrt{\sum_{k=1}^{n} d(x_k, y_k)^2}.$$

There is an important difference with the above-mentioned works, namely the choice of the distance on the product space $X^n$: instead of $D_2$, they consider the distance

$$D_1(x,y) := \sum_{k=1}^{n} d(x_k, y_k).$$

While $D_2$ is often more natural than $D_1$, the latter is better adapted for arguments involving tensorization and Lipschitz functions. Of course, $D_1 \leqslant \sqrt{n} D_2$, so the distance $D_2$ is stronger than $D_1$ for each finite $n$, but does not behave similarly in the asymptotic regime $n \to +\infty$. Accordingly, if we try to deduce natural concentration estimates from our results, we typically obtain

$$P^n \left[ \left| \frac{1}{n} \sum_{k=1}^{n} \varphi(x_k) - \int \left( \frac{1}{n} \sum_{k=1}^{n} \varphi(x_k) \right) dP^n(x^n) \right| \geqslant \varepsilon \right] \leqslant 2 \exp\left( -\frac{\lambda \varepsilon^2}{2} \right)$$

for any $n \geqslant 1$ and any Lipschitz function $\varphi$ on $X$ with Lipschitz norm 1. The fact that this bound does not go to 0 as $n \to \infty$ is probably linked to Remark 2.2 (2).

THEOREM 4.1 ($T_1$ inequalities for random dynamical systems). — *With the above notation, assume the existence of $\alpha_0 > 0$, a sequence $(z_k)_{k\geqslant 1}$ in $X$ and families of nonnegative numbers $(\gamma_k)_{k\geqslant 1}$, $(\beta_j)_{j\geqslant 1}$ with*

$$\gamma := \sup_{n\geqslant 1} \left[ \frac{1}{n} \sum_{k=1}^{n} \gamma_k \right] < +\infty, \qquad \beta := \sum_{j\geqslant 1} \beta_j < \alpha_0,$$

*such that for all $k \geqslant 1$, $x^{k-1} \in X^{k-1}$,*

$$\log \int e^{\alpha_0 \, d(z_k, x_k)^2} \, dP_k(x_k | x^{k-1}) \leqslant \gamma_k + \sum_{j=1}^{k-1} \beta_j \, d(z_{k-j}, x_{k-j})^2.$$

*Then, there exists $\lambda > 0$ such that for all $n \geqslant 1$, $P^n$ satisfies $T_1(\lambda/n)$.*

*Particular case 4.2.* — Consider a homogeneous Markov chain on $X$ with transition kernel $P(dy|x)$. Assume the existence of $(x_0, y_0) \in X \times X$, $\alpha_0 > 0$, $\beta < \alpha_0$ and $C < +\infty$ such that

$$\forall x \in X, \qquad \int_X e^{\alpha_0\, d(y_0,y)^2}\, P(dy|x) \leqslant C e^{\beta\, d(x_0,x)^2}. \tag{4.1}$$

Then there exists $\lambda > 0$ such that for all $n \geqslant 1$, $P^n$ satisfies $T_1(\lambda/n)$.

*Remark 4.3.* — *If Condition (4.1) is satisfied for some choice of $(x_0, y_0, \alpha_0, \beta, C)$, then for any $\alpha_0' < \alpha_0$ and $(x_0', y_0') \in X \times X$ we can find $\beta' \in [\beta, \alpha_0')$, $C' < +\infty$ such that Condition (4.1) is satisfied for $(x_0', y_0', \alpha_0', \beta', C')$. Thus the choice of reference points $x_0$ and $y_0$ is arbitrary: for instance, if $X = \mathbb{R}^d$, we can choose 0 for both, and the condition becomes*

$$\exists \alpha > 0,\ \beta < \alpha,\ C < +\infty; \qquad \forall x \in \mathbb{R}^d, \qquad \int_{\mathbb{R}^d} e^{\alpha\, |y|^2}\, P(dy|x) \leqslant C e^{\beta\, |x|^2}. \tag{4.2}$$

*Proof of Theorem 4.1.*— Let $\alpha := \alpha_0 - \beta$. Since $\alpha \leqslant \alpha_0$, by assumption,

$$\int e^{\alpha d(x_0,x_n)^2}\, P_n(dx_n|x^{n-1}) \leqslant e^{\gamma_n} \left(1 + \sum_{j=1}^{n} \beta_j d(z_{n-j}, x_{n-j})^2\right).$$

In particular,

$$\int e^{\alpha D(z^n,x^n)^2}\, P^n(dx^n)$$

$$\leqslant e^{\gamma_n} \int \exp\left(\sum_{k=1}^{n-1} (\alpha + \beta_k) d(z_{n-k}, x_{n-k})^2\right) P^{n-1}(dx^{n-1}).$$

Here $z^n = (z_0, \ldots, z_n)$; note that $\alpha + \beta_k \leqslant \alpha_0$ for all $k$, and in particular we can repeat the argument with $n-1$ in place of $n$. Using an induction argument, one easily shows that

$$\int e^{\alpha D(x^n,z^n)^2}\, P^n(dx^n) \leqslant e^{\sum_{k=1}^{n} \gamma_k} \leqslant e^{n\gamma}.$$

In particular,

$$\log \int e^{\alpha D(x^n,z^n)^2}\, P^n(dx^n) = O(n),$$

and we conclude by applying the results presented in section 2. $\square$

As examples of application we now consider the following two particular cases:

Example 4.4. — Let $(X_i)$ be a Markovian dynamical system on a Polish space $X$, with transition kernel $P(\,\cdot\,|x)$ such that

(i) $P(\,\cdot\,|x)$ satisfies $T_1(\lambda)$ for a constant $\lambda$ independent of $x$;

(ii) the map $x \mapsto P(\,\cdot\,|x)$ is $L$-Lipschitz from $X$ to $P(X)$, equipped with the $W_1$ distance, with $L < 1$.

Then there exist $\alpha > 0$ and $\beta < \alpha$ such that for any $x_0, y_0 \in X$, there exists $\gamma < +\infty$ such that

$$\log \int_X e^{\alpha\, d(y_0,y)^2}\, P(dy|x) \leqslant \gamma + \beta\, d(x_0,x)^2$$

for all $x \in X$. In particular the hypotheses of Theorem 4.1 hold in view of the Particular case 4.2.

Example 4.5. — Let $(X_k)_{k\in\mathbb{N}}$ be a dynamical system on $\mathbb{R}^d$ such that the hypotheses of [5, Theorem 4.1] hold, that is, with the notation introduced above,

(i) there exists some constant $\lambda$ such that

$$W_1(\nu, P_k(\,\cdot\,|x^{k-1})) \leqslant \sqrt{\frac{2}{\lambda} H\big(\nu|P_k(\,\cdot\,|x^{k-1})\big)}$$

for all $k \geqslant 1$, $x^{k-1}$ in $(\mathbb{R}^d)^{k-1}$ and all probability measures $\nu$ on $\mathbb{R}^d$;

(ii) there exist some nonnegative numbers $a_j$ such that $\displaystyle\sum_{j=1}^{+\infty} a_j < 1$ and

$$W_1\big(P_k(\,\cdot\,|x^{k-1}), P_k(\,\cdot\,|\tilde{x}^{k-1})\big) < \sum_{j=1}^{k-1} a_j |x_{k-j} - \tilde{x}_{k-j}|$$

for all $k \geqslant 1$ and $x^{k-1}, \tilde{x}^{k-1}$ in $(\mathbb{R}^d)^{k-1}$.

Then the assumptions of Theorem 4.1 also hold for this system.

This last example shows that our assumptions are not less general than those in [5]. Note carefully that when we apply Theorem 4.1 to this system, we do not recover such a strong conclusion as in [5] because of the choice of distances on product spaces ($D_2$ instead of $D_1$).

Since the proofs for both Examples 4.4 and 4.5 are similar, we only study the second example.

*Proof of the assertion in Example 4.5.* — In a **first step** we prove that for any $k \geqslant 1$, $x^{k-1}, z^{k-1}$ in $(\mathbb{R}^d)^{k-1}$, $z_k$ in $\mathbb{R}^d$, $\varepsilon, \delta > 0$ and $a < \frac{\lambda}{2}$, we have

$$\log \int e^{a(1-\varepsilon)|y_k - z_k|^2} P(dy_k|z^{k-1})$$
$$\leqslant -\frac{1}{2}\log(1-\frac{2a}{\lambda}) + a\left(\frac{1}{\varepsilon}-1\right)\left(1+\frac{1}{\delta}\right)\left(\int |t_k - z_k|\, P(dt_k|z^{k-1})\right)^2$$
$$+ a\left(\frac{1}{\varepsilon}-1\right)(1+\delta)\sum_{j=1}^{k-1} a_{k-j}\sum_{j=1}^{k-1} a_{k-j}|x_j - z_j|^2.$$

Indeed, the probability measure $P_k(\,\cdot\,|x^{k-1})$ satisfies $T_1(\lambda)$ and the map $y \mapsto |y - z_k|$ is 1-Lipschitz, so by the Bobkov-Götze formulation of the $T_1$ inequality (see [3, Theorem 1.3] and [5, Section 1]) we have

$$\int e^{a\left[|y_k - z_k| - \int |t_k - z_k|\, P_k(dt_k|x^{k-1})\right]^2} P_k(dy_k|x^{k-1}) \leqslant \frac{1}{\sqrt{1-2a/\lambda}} \tag{4.3}$$

for any $a < \frac{\lambda}{2}$, $z_k \in \mathbb{R}^d$ and $x^{k-1} \in (\mathbb{R}^d)^{k-1}$.

Let then $\varepsilon$ be some positive number. Integrating the inequality

$$(1-\varepsilon)|y_k - z_k|^2 \leqslant \left| |y_k - z_k| - \int |t_k - z_k|\, P_k(dt_k|x^{k-1})\right|^2$$
$$+ \left(\frac{1}{\varepsilon}-1\right)\left(\int |t_k - z_k|\, P_k(dt_k|x^{k-1})\right)^2$$

and using (4.3) lead to

$$\log \int e^{a(1-\varepsilon)|y_k - z_k|^2} P_k(dy_k|x^{k-1}) \leqslant -\frac{1}{2}\log(1-\frac{2a}{\lambda})$$
$$+ a\left(\frac{1}{\varepsilon}-1\right)\left(\int |t_k - z_k|\, P_k(dt_k|x^{k-1})\right)^2.$$

Recall the Kantorovich-Rubinstein formulation of the $W_1$ distance [11, Theorem 1.14]:

$$W_1(\mu,\nu) = \sup_{g\ 1-\text{Lipschitz}} \left( \int g\, d\mu - \int g\, d\nu \right)$$

This and Assumption (ii), with $\tilde{x}^{n-1} = z^{n-1}$, imply

$$\begin{aligned}\int |t_k - z_k|\, P_k(dt_k|x^{k-1}) \quad &- \quad \int |t_k - z_k|\, P_k(dt_k|z^{k-1}) \\ &\leqslant \quad W_1(P_k(\,\cdot\,, x^{k-1}), P_k(\,\cdot\,, z^{k-1})) \\ &\leqslant \quad \sum_{j=1}^{k-1} a_{k-j}\, |x_j - z_j|.\end{aligned}$$

Thus for any positive number $\delta$

$$\begin{aligned}&\left( \int |t_k - z_k|\, P_k(dt_k|x^{k-1}) \right)^2 \\ &\leqslant \left(1 + \frac{1}{\delta}\right) \left( \int |t_k - z_k|\, P_k(dt_k|z^{k-1}) \right)^2 + (1+\delta) \left( \sum_{j=1}^{k-1} a_{k-j}\, |x_j - z_j| \right)^2,\end{aligned}$$

and by Cauchy-Schwarz inequality we can bound this quantity by

$$\left(1 + \frac{1}{\delta}\right) \left( \int |t_k - z_k|\, P_k(dt_k|z^{k-1}) \right)^2 + (1+\delta) \sum_{j=1}^{k-1} a_{k-j} \sum_{j=1}^{k-1} a_{k-j}\, |x_j - z_j|^2.$$

This concludes this first step.

In the **second step**, we build the sequence $(z_k)$ by the following induction process. Let $z_1$ be arbitrary in $\mathbb{R}^d$; assuming that we have defined $z^{k-1} = (z_1, \ldots, z_{k-1})$, we let

$$z_k := \int_{\mathbb{R}^d} t_k\, P_k(dt_k|z^{k-1}).$$

Then

$$\begin{aligned}\left( \int |t_k - z_k|\, P_k(dt_k|z^{k-1}) \right)^2 \quad &\leqslant \quad \log \int e^{|t_k - z_k|^2}\, P_k(dt_k|z^{k-1}) \\ &= \quad \log \int e^{\left|t_k - \int t_k\, P_k(dt_k|z^{k-1})\right|^2}\, P_k(dt_k|z^{k-1}) \\ &\leqslant \quad \log \frac{1}{\sqrt{1 - 2a/\lambda}}\end{aligned}$$

thanks to Jensen's inequality and again the Bobkov-Götze formulation of the $T_1$ inequality, which is satisfied by $P_k(\,\cdot\,|z^{k-1})$.

Now we choose $\varepsilon \in (0,1)$ and $\delta > 0$ in such a way that

$$a\left(\frac{1}{\varepsilon}-1\right)(1+\delta)\sum_{j=1}^{+\infty} a_j \; < a(1-\varepsilon)\;:$$

for instance, $\varepsilon := \left(\sum_{j=1}^{+\infty} a_j\right)^{1/2}$ and $\delta := \frac{1}{2}\left(\sum_{j=1}^{+\infty} a_j\right)^{-1/2} - 1$ will do. Then the assumptions of Theorem 4.1 can be checked to hold for

$$\alpha := a(1-\varepsilon),$$

$$\gamma_j := -\frac{1}{2}\log\left(1-\frac{2a}{\lambda}\right)\left[1+a\left(\frac{1}{\varepsilon}-1\right)\left(1+\frac{1}{\delta}\right)\right]$$

and

$$\beta_j := a\left(\frac{1}{\varepsilon}-1\right)(1+\frac{1}{\delta})\,a_j.$$

□

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