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# Real cohomology groups of the space of nonsingular curves of degree 5 in $\mathbb{CP}^2$

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ALEXEI G. GORINOV [1]

**ABSTRACT.** — We give a modification of V. A. Vassiliev's method of calculating cohomology groups of spaces of nonsingular projective complex hypersufaces. Our construction is less "canonical" than V. A. Vassiliev's one, but in some cases it allows to simplify the calculations. We apply our method to prove that the Poincaré polynomial of the space of homogeneous polynomials that define nonsingular quintics in $\mathbf{CP}^2$ is equal to $(1 + t)(1 + t^3)(1 + t^5)$.

**RÉSUMÉ.** — Nous considérons une variante de la méthode de V. A. Vassiliev du calcul des groupes de cohomologie des espaces d'hypersurfaces projectives complexes non-singulières. Bien qu'étant moins « canonique » que celle de V. A. Vassiliev, notre construction permet de simplifier les calculs dans certains cas. On l'applique ensuite pour démontrer que le polynôme de Poincaré de l'espace des polynômes homogènes qui définissent des quintiques non singulières de $\mathbf{CP}^2$ est égal à $(1 + t)(1 + t^3)(1 + t^5)$.

## 1. Main result

Let us denote by $\Pi_5$ the space of all homogeneous polynomials $\mathbf{C}^3 \to \mathbf{C}$ of degree 5 and by $P_5$ its subspace consisting of all nonsingular polynomials (i.e. the polynomials, whose gradient is non-zero outside the origin).

THEOREM 1.1. — *The Poincaré polynomial of $P_5$ is equal to*

$$(1 + t)(1 + t^3)(1 + t^5).$$

A general method of calculating the cohomology of spaces of nonsingular algebraic hypersurfaces of given degree was described in [2]. In particular, the real cohomology groups of the spaces of nonsingular plane curves of degree $\leqslant 4$ were calculated there. In the present work, we apply a modification of the same method to the case of nonsingular quintics in $\mathbf{CP}^2$.

Denote by $\Sigma_5$ the space $\Pi_5 \setminus P_5$. By the Alexander duality, the cohomology group of $P_5$ is isomorphic to the Borel-Moore homology (i.e. the homology of the complex of locally finite chains) of $\Sigma_5$:

$$H^i(P_5, \mathbf{R}) = \bar{H}_{2D-1-i}(\Sigma_5, \mathbf{R}),$$

where $D = \dim_{\mathbf{C}}(\Pi_5) = 21$ and $0 < i < 2D - 1$. This reduction was used first by V. I. Arnold in [1]. To calculate the latter group $\bar{H}_*(\Sigma_5, \mathbf{R})$ we use a version of the spectral sequence constructed in [2]. It is described in the following theorem.

THEOREM 1.2. — *The spectral sequence for the real Borel-Moore homology of the space $\Sigma_5$ is defined by the following conditions:*

*1. Any its nontrivial term $E^1_{p,q}$ belongs to the quadrilateral in the $(p,q)$-plane, defined by the conditions $[1 \leqslant p \leqslant 3, 29 \leqslant q \leqslant 39]$.*

*2. In this quadrilateral all the nontrivial terms $E^1_{p,q}$ look as is shown in (1.1).*

*3. The spectral sequence stabilizes in this term, i.e. $E^1 \equiv E^\infty$.*

| | 1 | 2 | 3 |
|---|---|---|---|
| 39 | $\mathbf{R}$ | | |
| 38 | | | |
| 37 | $\mathbf{R}$ | | |
| 36 | | | |
| 35 | $\mathbf{R}$ | $\mathbf{R}$ | |
| 34 | | | |
| 33 | | $\mathbf{R}$ | |
| 32 | | | |
| 31 | | $\mathbf{R}$ | |
| 30 | | | |
| 29 | | | $\mathbf{R}$ |

(1.1)

The proof of Theorem 1.2 will be given in Section 4.

It follows from Theorem 1.1 that the Poincaré polynomial of the space $P_5$ coincides with the Poincaré polynomial of the group $GL_3(\mathbf{C})$. Recently, C.A.M. Peters and J.H.M. Steenbrink proved the following general statement (see [4]): let $\Pi_{d,n}$ be the space of all homogeneous complex polynomials of degree $d$ in $n + 1$ variables, and let $\Sigma_{d,n}$ be the subspace of $\Pi_{d,n}$ consisting of polynomials that define singular hypersurfaces in $\mathbf{CP}^n$; in this situation, if $d > 2$, then the cohomological Leray sequence of the map $\Pi_{d,n} \setminus \Sigma_{d,n} \to (\Pi_{d,n} \setminus \Sigma_{d,n})/GL_{n+1}(\mathbf{C})$ degenerates in the term $E_2$.

I wish to express my deep gratitude to V. A. Vassiliev for proposing the problem and stimulating discussions, to A. A. Oblomkov and A. V. Inshakov, who explained me, how singular sets of plane quintics look like, and to P. Vogel for useful discussions.

## 2. The method of conical resolutions

Consider the following general situation: set $\mathbf{k}$ to be either $\mathbf{R}$ or $\mathbf{C}$, and suppose $V$ is a vector space of $\mathbf{k}$-valued functions on a manifold $\tilde{M}$ and $\Sigma \subset V$ is a closed subset formed by the functions that have singularities of a certain type. (Such a subset is often called a *discriminant*.) Suppose that $D = \dim_{\mathbf{k}} V < \infty$. We want to calculate the Borel-Moore homology of $\Sigma$. In order to do this we construct a resolution, i.e., a topological space $\sigma$ and a proper map $\pi : \sigma \to \Sigma$ such that the preimage of every point is contractible. We are going to describe a construction of $\sigma$ via configuration spaces. Our construction generalizes that from the article [2].

*Remark 2.1.* — The method described below can be extended with obvious modifications to the case when $V$ is an affine space. We assume $V$ to be a vector space, since, on the one hand, the vector case is somewhat simpler, and on the other hand it is sufficient for the application that we have in mind.

Suppose that with every function $f \in \Sigma$ a compact nonempty subset $K_f$ of some compact CW-complex $M$ is associated. In the sequel we shall be interested in the case when $\tilde{M} = \mathbf{C}^3 \setminus \{0\}, V = \Pi_5, \Sigma = \Sigma_5$. In this case it is natural to set $M$ equal to $\mathbf{CP}^2$ and $K_f$ equal to the image of the set of singular points of $f$ under the evident map $\tilde{M} \to \mathbf{CP}^2$.

In general, we suppose that the following conditions are satisfied:

- If $f, g \in \Sigma$, and $K_f \cap K_g \neq \varnothing$, then $f + g \in \Sigma$ and $K_f \cap K_g \subset K_{f+g}$,

- If $f \in \Sigma$, then for any $\lambda \neq 0$ we have $K_{\lambda f} = K_f$,

- The zero function $0 \in \Sigma$, and $K_0 = M$.

- For any $K \subset M$ set $L(K) \subset V$ to be the subset consisting of all $f$ such that $K \subset K_f$. The previous three conditions imply that $L(K)$ is a vector space. We suppose that there exists a positive integer $d$ such that for any $x \in M$ one can find a neighborhood $U \ni x$ in $M$ and continuous functions $l_1, \ldots, l_d$ from $U$ to the Grassmannian $G_{D-1}(V)$ of $(D-1)$-dimensional **k**-vector subspaces of $V$ such that we have

$$L(\{x'\}) = \bigcap_{i=1}^{d} l_i(x')$$

for any $x' \in U$.

*Remark 2.2.* — One may ask a natural question: if we are dealing with functions on some manifold $\tilde{M}$, why should we introduce some additional space $M$? The problem is that for our construction it will be convenient to associate with a singular function a compact subset of a compact $CW$-complex. In the case when the manifold $\tilde{M}$ itself is compact, we can assume, of course, that $M = \tilde{M}$, and $K_f$ is the set of points where $f$ has singularities of some given type.

By a *configuration* in a compact $CW$-complex $M$ we shall mean a compact nonempty subset of $M$. Denote by $2^M$ the space of all configurations in $M$. Suppose that the topology on $M$ is induced by a metric $\rho$. We introduce the Hausdorff metric on $2^M$ by the usual rule:

$$\tilde{\rho}(K, L) = \max_{x \in K} \rho(x, L) + \max_{x \in L} \rho(x, K).$$

It is easy to check that if $M$ is compact, then the space $2^M$ equipped with the metric $\tilde{\rho}$ is also compact. Let us denote by $B(M, k)$ the subspace of $2^M$ that consists of all configurations that contain exactly $k$ elements. For any subspace $A \subset 2^M$ we denote by $\bar{A}$ the closure of $A$ in $2^M$. We have $\overline{B(M, k)} = \bigcup_{j \leqslant k} B(M, j)$.

PROPOSITION 2.3. — *Let $(K_j)$ be a Cauchy sequence in $2^M$, and let $K$ be the set consisting of the limits of all sequences $(a_j)$ such that $a_j \in K_j$ for every $j$. Then $K$ is nonempty and compact, and $\lim_{j \to \infty} \tilde{\rho}(K_j, K) = 0$.*

PROPOSITION 2.4. — *Let $(K_i), (L_i)$ be two sequences in $2^M$. Suppose that there exist $\lim_{i \to \infty} K_i, \lim_{i \to \infty} L_i$, and denote these limits by $K$ and $L$ respectively. Suppose also that $K_i \subset L_i$ for every $i$. Then $K \subset L$.*

Suppose that $X_1, \ldots, X_N$ is a finite collection of subspaces of $2^M$ satisfying the following conditions:

1. For every $f \in \Sigma$ the set $K_f$ belongs to some $X_i$.

2. Suppose that $K \in X_i, L \in X_j, K \subsetneq L$. Then $i < j$.

3. Recall that $L(K)$ is the space of all functions $f$ such that $K \subset K_f$. If we fix $i$, then the dimension of $L(K)$ is the same for all configurations $K \in X_i$. (We denote this dimension by $d_i$.)

4. $X_i \cap X_j = \varnothing$ if $i \neq j$.

5. Any $K \in \bar{X}_i \setminus X_i$ belongs to some $X_j$ with $j < i$.

6. For every $i$ the space $\mathcal{T}_i$ consisting of pairs $(x, K), x \in K, K \in X_i$ is the total space of a locally trivial bundle over $X_i$ (the projection $pr_i : \mathcal{T}_i \to X_i$ is evident). This bundle will be called the *tautological bundle*[2] over $X_i$.

7. Note that any local trivialization of $\mathcal{T}_i$ has the following form:
$$(x, K') \mapsto (t(x, K'), K').$$
   Here $x$ is a point in some $K \in X_i$, $K'$ belongs to some neighborhood $U \ni K$ in $X_i$, and $t : K \times U \to M$ is a continuous map such that if we fix $K' \in U$, then we obtain a homeomorphism $t_{K'} : K \to K'$. We require that for every $K \in X_i$ there exist a neighborhood $U \ni K$ and a local trivialization of $\mathcal{T}_i$ over $U$ such that every map $t_{K'} : K \to K'$ establishes a bijective correspondence between the subsets of $K$ and $K'$ that belong to $\bigcup_{j \leqslant i} X_j$.

Under these assumptions we are going to construct a resolution $\sigma$ of $\Sigma$ and a filtration on it such that the $i$-th term of the filtration is the total space of a fiber bundle over $X_i$.

Note that due to condition 3, for any $i = 1 \ldots, N$ there exists an evident map $K \mapsto L(K)$ from $X_i$ to the Grassmann manifold $G_{d_i}(V)$, which is continuous due to the last condition on page 4.

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[2] For instance, if $M = \mathbf{CP}^n$ and $X_i$ consists of projective subspaces of $M$ of the same dimension, then this is just the projectivization of the usual tautological bundle over some Grassmann manifold of $\mathbf{C}^{n+1}$.

*Remark 2.5.* — The rather strange-looking condition 7 follows immediately from condition 6 in the following situation: suppose $X_i$ consists of finite configurations, and for all $K, L$, such that $K \in X_i, L \subset K$ there is an index $j < i$ such that $L \in X_j$. In this case any trivialization of $\mathcal{T}_i$ fits.

Consider the space $Y = \bigcup_{i=1}^{N} \bar{X}_i = \bigcup_{i=1}^{N} X_i$. Denote by $X$ the $N$-th self-join $Y^{*N}$ of $Y$ [3]. Note that the spaces $Y, Y^{*N}$ are compact. Call a simplex $\triangle \subset X$ *coherent* if the configurations corresponding to its vertices form an ascending sequence. Note that then its vertices belong to different $X_i$ (condition 2). Let $\triangle$ be a coherent simplex. Among the vertices of $\triangle$ there is a vertex such that the corresponding configuration contains the configurations that correspond to all other vertices of $\triangle$. Such vertex will be called the *main vertex* of $\triangle$. Denote by $\Lambda$ the union of all coherent simplices. For any $K \in X_i$ denote by $\Lambda(K)$ the union of all coherent simplices, whose main vertices coincide with $K$. Note that the space $\Lambda(K)$ is contractible.

Denote by $\Phi_i$ the union $\bigcup_{j \leqslant i} \bigcup_{K \in X_j} \Lambda(K)$. There is a filtration on $\Lambda$: $\varnothing \subset \Phi_1 \subset \cdots \subset \Phi_N = \Lambda$.

For any simplex $\triangle \subset X$ denote by $\overset{\circ}{\triangle}$ its interior, i.e. the union of its points that do not belong to the faces of lower dimension. Note that for every $x \in X$ there exists a unique simplex $\triangle$ such that $x \in \overset{\circ}{\triangle}$.

PROPOSITION 2.6. — *Let $(x_i)$ be a sequence in $X$ such that $\lim_{i \to \infty} x_i = x$. Suppose $x_i \in \overset{\circ}{\triangle_i}, x \in \overset{\circ}{\triangle}$, where $\triangle_i$ are coherent simplices, and suppose $K$ is a vertex of $\triangle$. Then there exists a sequence $(K_i)$ such that $K_i$ is a vertex of $\triangle_i$ and $\lim_{i \to \infty} K_i = K$.*

PROPOSITION 2.7. — *All spaces $\Lambda, \Lambda(K), \Phi_i$ are compact.*

*Proof.* — This follows immediately from Propositions 2.6 and 2.4.  □

For any $K \in X_i$ denote by $\partial \Lambda(K)$ the union $\bigcup_{\kappa} \Lambda(\kappa)$ over all maximal subconfigurations $\kappa \in \cup_{j<i} X_j, \kappa \subsetneq K$. The space $\Lambda(K)$ is the union of all segments that join points of $\partial \Lambda(K)$ with $K$, and hence it is homeomorphic to the cone over $\partial \Lambda(K)$.

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[3] Recall that for any finite $CW$-complex $Y$ and any $k \geqslant 1$ the $k$-th self-join of $Y$ (denoted by $Y^{*k}$) can be defined as follows: take a generic embedding $i : Y \to \mathbf{R}^{\Omega}$ for some very large (but finite) $\Omega$ and define $Y^{*k}$ to be the union of all $(k-1)$-dimensional simplices with vertices in $i(Y)$ ("generic" means that the intersection of any two such simplices is their common face if there is any).

Define the conical resolution $\sigma$ as the subspace of $V \times \Lambda$ consisting of pairs $(f, x)$ such that $f \in \Sigma$, $x \in \Lambda(K_f)$. There exist evident projections $\pi : \sigma \to \Sigma$ and $p : \sigma \to \Lambda$. We introduce a filtration on $\sigma$ setting $F_i = p^{-1}(\Phi_i)$. The space $\sigma$ is closed in $\Sigma \times \Lambda$ (this can be deduced from the last condition on page 4). The map $\pi$ is proper, since the preimage of each compact set $C \subset \Sigma$ is a closed subspace of $C \times \Lambda$, which is compact.

THEOREM 2.8. — *Suppose $X_1, \ldots, X_N$ are subspaces of $2^M$ that satisfy Conditions 1–7 of page 399. Then*

1. *$\pi$ induces an isomorphism of the Borel-Moore homology groups of the spaces $\sigma$ and $\Sigma$.*

2. *Every space $F_i \setminus F_{i-1}$ is a $\mathbf{k}$-vector bundle over $\Phi_i \setminus \Phi_{i-1}$ of dimension $\dim_{\mathbf{k}}(L(K))$, $K \in X_i$.*

3. *The space $\Phi_i \setminus \Phi_{i-1}$ is a fiber bundle over $X_i$, the fiber being homeomorphic to $\Lambda(K) \setminus \partial\Lambda(K)$.*

*Proof.* — The first statement of the theorem follows from the fact that $\pi$ is proper and $\pi^{-1}(f) = \Lambda(K_f)$, the latter space being contractible. To prove the second statement, let us study the preimage of a point $x \in \Phi_i \setminus \Phi_{i-1}$ under the map $p : F_i \setminus F_{i-1} \to \Phi_i \setminus \Phi_{i-1}$. We claim that $p^{-1}(x) = L(K)$ for some $K \in X_i$.

Recall that each point $x \in \Phi_i \setminus \Phi_{i-1}$ belongs to the interior of some coherent simplex $\triangle$, whose main vertex lies in $X_i$. Denote this vertex by $K$ and denote the map $\Phi_i \setminus \Phi_{i-1} \ni x \mapsto K \in X_i$ by $f_i$.

Now suppose $f \in L(K)$, or, which is the same, $K_f \supset K$. This implies $\Lambda(K) \subset \Lambda(K_f)$. So we have $x \in \Lambda(K) \subset \Lambda(K_f)$ and $(f, x) \in \sigma$, $p(f, x) = x$, hence $f \in p^{-1}(x)$.

Suppose now that $f \in p^{-1}(x)$. We have $(f, x) \in \sigma$, hence $x \in \Lambda(K_f)$. This means that $x$ belongs to some coherent simplex $\triangle'$, whose main vertex is $K_f$. But $x$ belongs to the interior of $\triangle$, hence $\triangle$ is a face of $\triangle'$ and $K$ is a vertex of $\triangle'$. But $K_f$ is the main vertex of $\triangle'$, hence $K \subset K_f$ and $f \in L(K)$.

So, we see that $p^{-1}(x) = L(K)$, $K \in X_i$, and the second statement of Theorem 2.8 follows immediately from the fact that the dimension of $L(K)$ is the same for all $K \in X_i$ (condition 3). In fact the bundle $p : F_i \setminus F_{i-1} \to \Phi_i \setminus \Phi_{i-1}$ is the inverse image of the tautological bundle over $G_{d_i}(V)$ under the composite map $x \overset{f_i}{\mapsto} K \mapsto L(K)$.

We shall show now that the map $f_i$ is a locally-trivial fibration with the fiber $\Lambda(K) \setminus \partial\Lambda(K)$. This will prove the third statement of Theorem 2.8.

Denote by $\mathcal{X}_i$ the space consisting of all couples $(L, K)$ such that $L \subset K, L \in \bigcup_{j \leqslant i} X_j$. Let $P : \mathcal{X}_i \to X_i$ be the evident projection. We shall construct a trivialization of $f_i : \Phi_i \setminus \Phi_{i-1} \to X_i$ from a particular trivialization of $\mathcal{T}_i$ that exists due to condition 7.

For any $K \in X_i$ let $U$ and $t$ be the neighborhood of $K$ and the trivialization of $\mathcal{T}_i$ over $U$ that exist due to condition 7. For any $K, K' \in U, x \in K$, set $t_{K'}(x) = t(x, K')$. Define a map $T : P^{-1}(K) \times U \to 2^M$ by the following rule: take any $L \subset K, L \in \bigcup_{j \leqslant i} X_j$ and set $T(L, K')$ equal to the image of $L$ under $t_{K'}$. Due to condition 7, $T(\bullet, K')$ is a bijection between the set of $L \in 2^M$ such that $L \subset K$, $L \in \bigcup_{j \leqslant i} X_j$, and the set of $L' \in 2^M$ such that $L' \subset K', L' \in \bigcup_{j \leqslant i} X_j$.

PROPOSITION 2.9. —

1.  *The map $T_1 : P^{-1}(K) \times U \to 2^M \times U$ defined by the formula $T_1(L, K') = (T(L, K'), K')$ maps $P^{-1}(K) \times U$ homeomorphically onto $P^{-1}(U)$.*

2.  *For any $K' \in U$ we have $L \subset M \subset K$ if and only if $T(L, K') \subset T(M, K') \subset T(K, K') = K'$.*

This implies that $\mathcal{X}_i$ is a locally trivial bundle over $X_i$.

*Proof.* — The second statement is obvious. We have already seen that the map $T_1$ is a bijection. The verification of the fact that $T_1$ and its inverse are continuous is straightforward but rather boring.  □

Now we can construct a trivialization of the fiber bundle $f_i : \Phi_i \setminus \Phi_{i-1} \to X_i$ over the same neighborhood $U \ni K$ as above: take any $x \in f_i^{-1}(K)$ and any $K' \in U$. $x$ can be written in the form $x = \sum_j \alpha_j L_j$, where all $\alpha_j > 0, \sum \alpha_j = 1, L_j \in \bigcup_{k \leqslant i} X_k$, and exactly one $L_j$ belongs to $X_i$. Set $F(x, K') = \sum_k \alpha_k T(L_k, K')$. Again a straightforward argument shows that $F$ is a homeomorphism $f_i^{-1}(K) \times U \to f_i^{-1}(U)$. Theorem 2.8 is proved.  □

Since every $\Lambda(K)$ is compact in the topology of $\Lambda$, we have

$$\bar{H}_*(\Lambda(K) \setminus \partial\Lambda(K)) = H_*(\Lambda(K), \partial\Lambda(K)) = H_{*-1}(\partial\Lambda(K), pt).$$

In all known examples the majority of configurations of singular points are discrete (consist of finitely many points). To make the fibers of the bundle $\Phi_i \setminus \Phi_{i-1} \to X_i$ as simple as possible, we introduce one more condition:

8. if $K$ is a finite configuration from $X_i$, then every subset $L \subset K$ is contained in some $X_j$ with $j \leqslant i$.

Note that if we are given spaces $X_1, \ldots, X_N$ that satisfy Conditions 1-7, we can construct another collection of spaces $X_1', \ldots, X_{N'}'$ that satisfy Conditions 1-7 and Condition 8. Indeed, take the union $\bigcup_{i=1}^N X_i \subset 2^M$ and add all subsets of all finite configurations $K \in \bigcup_{i=1}^N X_i$; then take an appropriate stratification of the resulting subspace of $2^M$.

We have the following two lemmas.

LEMMA 2.10. — *If Condition 8 is satisfied, then for any $i$ such that $X_i$ consists of finite configurations the fiber of the bundle $f_i : (\Phi_i \setminus \Phi_{i-1}) \to X_i$ over a point $K \in X_i$ is an open simplex, whose vertices correspond to the points of the configuration $K$.*

The proof is by induction on the number of points in $K \in X_i$. $\qquad \square$

Note that in this situation the simplicial complex $\Lambda(K)$ is (piecewise linearly) isomorphic to the first barycentric subdivision of the simplex $\triangle$ spanned by the vertices of $K$. The complex $\partial\Lambda(K)$ is isomorphic to the first barycentric subdivision of the boundary of $\triangle$.

Moreover, denote by $\Lambda^{fin}$ the union of the spaces $\Lambda(K)$ over all finite $K \in \bigcup_{i=1}^N X_i$.

LEMMA 2.11. — *If Condition 8 is satisfied, then there exists a continuous map $C : \Lambda^{fin} \to M^{*N}$ that takes every $K \in \Lambda^{fin}$ to an element of the interior of the simplex $\triangle$ spanned by the points of $K$. This map is a homeomorphism on its image, and it maps $\Lambda(K)$ (respectively, $\partial\Lambda(K)$) homeomorphically on $\triangle$ (respectively, $\partial\triangle$).*

It follows from Lemmas 2.10, 2.11 that for any $i$ such that $X_i$ consists of finite configurations, the fiber bundle $\Phi_i \setminus \Phi_{i-1}$ is isomorphic to the restriction to $X_i$ of the evident bundle $M^{*k} \setminus M^{*(k-1)} \to B(M, k)$, where $k$ is the number of points in a configuration from $X_i$. So we have

$$\bar{H}_*(\Phi_i \setminus \Phi_{i-1}) = \bar{H}_{*-(k-1)}(X_i, \pm\mathbf{R}),$$

where $\pm\mathbf{R}$ is a local system (that will be described explicitly a little later). Suppose now that the functions that form $V$ are complex-valued. Recalling that all complex vector bundles are orientable and applying statement 2 of Theorem 2.8, we get

$$\bar{H}_*(F_i \setminus F_{i-1}) = \bar{H}_{*-2d_i}(\Phi_i \setminus \Phi_{i-1}) = \bar{H}_{*-2d_i-(k-1)}(X_i, \pm\mathbf{R}),$$

where $d_i = \dim_{\mathbf{C}} L(K), K \in X_i$.

## 3. Some further preliminaries

DEFINITION 3.1. — *For any space $X$ denote by $F(X, k)$ the space of all ordered $k$-ples from $X$, i.e. the space*

$$X \times \cdots \times X \setminus \{(x_1, \ldots, x_k) | x_i = x_j \text{for some } i \neq j\}.$$

The space $B(X, k)$ defined on page 398 is the quotient of $F(X, k)$ by the evident action of the symmetric group $S_k$. We shall denote by $\check{B}(\mathbf{CP}^2, k)$ the subspace of $B(\mathbf{CP}^2, k)$ consisting of generic $k$-configurations (i.e. such configurations that contain no three points that belong to a line, no 6 points that belong to a conic etc.). The spaces $\tilde{F}(\mathbf{CP}^2, k)$ are defined in a similar way.

DEFINITION 3.2. — *For any subspace $Y \subset B(X, k)$ of some configuration space denote by $\pm\mathbf{R}$ the "alternating" system, i.e. the local system on $Y$ with the fiber $\mathbf{R}$ that changes it sign under the action of any loop defining an odd permutation in a configuration from $Y$.*

Throughout the text, $P(X, \mathcal{L})$ stands for the Poincaré polynomial of $X$ "with coefficients in $\mathcal{L}$", i.e. the polynomial $\sum_i a_i t^i$, where $a_i = \dim(H^i(X, \mathcal{L}))$. In a similar way, we denote by $\bar{P}(X, \mathcal{L})$ the polynomial $\sum_i a_i t^i$, where $a_i = \dim(\bar{H}_i(X, \mathcal{L}))$.

*We shall consider only homology and cohomology groups with real coefficients, and the fibers of all local systems will be real vector spaces of finite dimension.*

The following statement will be frequently used in the sequel:

THEOREM 3.3. — *Let $p : E \to B$ be a locally trivial fiber bundle with fiber $F$, and let $\mathcal{L}$ be a local system of groups or vector spaces on $E$. Then the Borel-Moore homology groups of $E$ with coefficients in $\mathcal{L}$ can be obtained from the spectral sequence with the term $E^2$ defined by the equality $E^2_{p,q} = \bar{H}_p(B, \bar{\mathcal{H}}_q)$, where $\bar{\mathcal{H}}_q$ is the local system with fiber $\bar{H}_q(F, \mathcal{L}|F)$ corresponding to the natural action of $\pi_1(B)$ on $\bar{H}_q(F, \mathcal{L}|F)$.*

This is a version of the Leray theorem on the spectral sequence of a locally trivial fibration. Let us describe the action of $\pi_1(B, x_0)$ on the fiber $\bar{\mathcal{H}}_q|_{x_0}$, where $x_0$ is a distinguished point in $B$. Identify $\bar{\mathcal{H}}_q|_{x_0}$ with $\bar{H}_q(F, \mathcal{L}|F)$, and set $F_{x_0} = p^{-1}(x_0)$. A loop $\gamma$ in $B$ defines a map $\tilde{f} : \mathcal{L}|F_{x_0} \to \mathcal{L}|F_{x_0}$ covering some map $f : F_{x_0} \to F_{x_0}$. Recall the construction of $f$. Consider a family of curves $\gamma_x(t), x \in F$ that cover $\gamma$. Then we can set $f(x) = \gamma_x(1), x \in F_{x_0}$. The map $\tilde{f} : \mathcal{L}|F_{x_0} \to \mathcal{L}|F_{x_0}$ consists simply of the maps $\mathcal{L}_x \to \mathcal{L}_{f(x)}$ induced by $\gamma_x$.

The map $\tilde{f}$ induces for every $q$ a map $\tilde{f}_* : \bar{H}_q(F_{x_0}, \mathcal{L}|F_{x_0}) \to \bar{H}_q(F_{x_0}, \mathcal{L}|F_{x_0})$, which is exactly the map $\bar{\mathcal{H}}_q|_{x_0} \to \bar{\mathcal{H}}_q|_{x_0}$ induced by $\gamma$.

THEOREM 3.4. — *Let $E_1 \to B_1, E_2 \to B_2$ be two bundles and $\mathcal{L}_1, \mathcal{L}_2$ be local coefficient systems of groups or vector spaces on $E_1, E_2$ respectively. Let $f : E_1 \to E_2$ be a proper map that covers some map $g : B_1 \to B_2$ (i.e., $f$ is a proper bundle map). Suppose $\tilde{f} : \mathcal{L}_1 \to \mathcal{L}_2$ is a morphism of coefficient systems that covers $f$. Then the map $\tilde{f}$ induces a homomorphism of the terms $E^2$ of spectral sequences of Theorem 3.3.*

The map of the spectral sequences induced by $\tilde{f}$ can be described explicitly in the following way. Let $F_x^{(i)}$ be the fiber of $E_i$ over $x \in B_i$, $\bar{\mathcal{H}}_q^{(i)}$ be the coefficient system on $B_i$ from Theorem 3.3, $i = 1, 2$. Since $f$ is a bundle map, it maps $F_x^{(1)}$ into $F_{g(x)}^{(2)}$, and $\tilde{f}$ maps $\mathcal{L}|F_x^{(1)}$ into $\mathcal{L}|F_{g(x)}^{(2)}$. The latter map induces for any $x \in B_1$ a map $\bar{H}_q(F_x^{(1)}, \mathcal{L}|F_x^{(1)}) \to \bar{H}_q(F_{g(x)}^{(2)}, \mathcal{L}|F_{g(x)}^{(2)})$, which can be considered as restriction to $\bar{\mathcal{H}}_q^{(1)}|_x$ of a map $f'_q : \bar{\mathcal{H}}_q^{(1)} \to \bar{\mathcal{H}}_q^{(2)}$ that covers $g$. The desired map of the terms $E^2$ of spectral sequences is just the map $\bar{H}_*(B_1, \bar{\mathcal{H}}_q^{(1)}) \to \bar{H}_*(B_2, \bar{\mathcal{H}}_q^{(2)})$ induced by $f'$.

Theorem 3.3 has the following corollary:

COROLLARY 3.5. — *Let $N, \tilde{N}$ be manifolds, let $p : \tilde{N} \to N$ be a finite-sheeted covering, and let $\mathcal{L}$ be a local system of groups on $\tilde{N}$. Then $H^*(\tilde{N}, \mathcal{L}) = H^*(N, p(\mathcal{L}))$, where $p(\mathcal{L})$ denotes the direct image of the system $\mathcal{L}$.*

If $\mathcal{L}$ is the constant local system with fiber $\mathbf{R}$, then the representation of $\pi_1(N, x), x \in N$ in the fiber of $p(\mathcal{L})$ is isomorphic to the natural action of $\pi_1(N, x)$ on the vector space spanned by the elements of $p^{-1}(x)$. In particular, if $\tilde{N}$ is simply connected, we get the regular representation of $G$ (the group acts on its group algebra by left shifts).

Recall that any irreducible real or complex representation of a finite group $G$ is included into the regular representation. If the representation is complex, this is obvious. In the real case it follows from Schur's Lemma and from the well-known fact that if $R : G \to GL(V)$ and $S : G \to GL(W)$ are real representations of $G$, and $R_{\mathbf{C}}, S_{\mathbf{C}}$ are their complexifications, then $\dim_{\mathbf{R}}(\mathrm{Hom}(R, S)) = \dim_{\mathbf{C}}(\mathrm{Hom}(R_{\mathbf{C}}, S_{\mathbf{C}}))$, where $\mathrm{Hom}(R, S)$ is the space of representation homomorphisms between $R$ and $S$ (i.e. operators $f : V \to W$ such that $f(R(g)x) = S(g)f(x)$ for any $g \in G, x \in V$).

The homological analogues of Theorems 3.3 and 3.4 can be obtained by omitting all bars over $H$-es and $\mathcal{H}$-es. The cohomological versions of these theorems can be obtained as follows: in Theorem 3.3 the action of a loop $\gamma$ is the inverse of the cohomology map induced by $\tilde{f} : \mathcal{L}_x \to \mathcal{L}_{f(x)}$, and in Theorem 3.4 we have to suppose that $\mathcal{L}_1$ is the inverse image of $\mathcal{L}_2$, i.e. that the restriction of $\tilde{f}$ over each point is bijective.

Neither in homological, nor in cohomological analogue of Theorem 3.4 we have to require that $f$ is proper.

We shall also need the following version of Poincaré duality theorem:

THEOREM 3.6. — *Let $M$ be a orbifold of dimension $n$, and let $\mathcal{L}$ be a local system on $M$, whose fiber is a real or complex vector space. Then we have*

$$H^*(M, \mathcal{L} \otimes Or(M)) \cong \bar{H}_{n-*}(M, \mathcal{L}),$$

*where $Or(M)$ is the orienting sheaf of $M$.*

The following lemma allows us to calculate real cohomology groups of the quotient of a semisimple connected Lie group by a finite subgroup with coefficients in arbitrary local systems. We shall use this lemma several times.

When this does not lead to a confusion, we shall use the symbol $\mathbf{R}$ to denote the constant sheaf with fiber $\mathbf{R}$.

LEMMA 3.7. — *Let $G$ be a connected and simply connected Lie group. Let $G_1$ be a finite subgroup of $G$, and let $\mathcal{L}$ be a local system on $G/G_1$.*

The cohomology group $H^*(G/G_1, \mathcal{L})$ is trivial if $\mathcal{L} \neq \mathbf{R}$ and the action of $\pi_1(G/G_1) \cong G_1$ on the fiber of $\mathcal{L}$ is irreducible.

*Proof.* — If we apply Corollary 3.5 to the covering $G \to G/G_1$, we obtain

$$H^*(G, \mathbf{R}) = \oplus_i H^*(G/G_1, \mathcal{L}_i),$$

where $\mathcal{L}_i$ are coefficient systems corresponding to irreducible representations of $G_1$ that are included into the regular real representation. Recall that in fact all real irreducible representations of a finite group are included into the regular real representation.

But $H^*(G, \mathbf{R}) = H^*(G/G_1, \mathbf{R})$, since every cohomology class of $G$ is $G_1$-invariant; therefore all the groups $H^*(G/G_1, \mathcal{L}_i)$ for $\mathcal{L}_i \neq \mathbf{R}$ are zero. $\square$

After Theorem 3.3 on page 405 we gave an explicit construction of local systems that appear in the Leray spectral sequence of a locally-trivial fibration. However, in many interesting cases we have a map that is "almost" a fibration, say the quotient of a smooth manifold by an almost free action of a compact group etc., and we would like to know, what the Leray sequence (which is defined for any continuous map) looks like in this case. It turns out that in the case of a quotient map the sheaves that occur in the Leray sequence can be explicitly described (at least for some actions of some groups).

Let us fix a smooth action of a Lie group $G$ on a manifold $M$. A submanifold $S \subset M$ is called a *slice* at $x \in S$ for the action of $G$ iff $GS$ is open in $M$, and there is a $G$-equivariant map $GS \to G/Stab(x)$ such that the preimage of $Stab(x)$ under this map is $S$. Here $GS$ is the union of the orbits of all points of $S$, and $Stab(x)$ is the stabilizer of $x$.

If $G$ is compact, a slice exists for any action at every point $x \in M$: provide $M$ with a $G$-invariant Riemannian metric and set $S$ equal to the exponential of an $\varepsilon$-neighborhood of zero of the orthogonal complement of $T_x(Gx)$ for any sufficiently small $\varepsilon$. (Here $Gx$ is the orbit of $x$.)

THEOREM 3.8. — *Suppose that $G$ is connected and simply connected, and for any $x \in M$ the group $Stab(x)$ is finite. Let $\mathcal{L}$ be a local system on $M$. If there exists a slice for the action of $G$ at every point $x \in M$, then for any $i$ the sheaf $\mathcal{H}^i_{\mathcal{L}}$ on $M/G$ generated by the presheaf $U \mapsto H^i(p^{-1}(U), \mathcal{L})$ is isomorphic to $p(\mathcal{L}) \otimes H^i(G, \mathbf{R})$, where $p : M \to M/G$ is the natural*

*projection, $p(\mathcal{L})$ is the direct image of $\mathcal{L}$, and $H^i(G, \mathbf{R})$ is the constant sheaf with fiber $H^i(G, \mathbf{R})$. Moreover, if $\mathcal{L} = \mathbf{R}$, then $p(\mathcal{L}) = \mathbf{R}$.*

Note that the sheaf $\mathcal{H}^0_{\mathcal{L}}$ is canonically isomorphic to $p(\mathcal{L})$ for any $\mathcal{L}$.

*Proof.* — Let us first consider the case $\mathcal{L} = \mathbf{R}$. We have to show that for any $i$ the sheaf $\mathcal{H}^i_{\mathbf{R}}$ is constant with fiber isomorphic to $H^i(G, \mathbf{R})$. Let $S$ be a slice for the action of $G$ at $x \in M$. It is easy to show that $S$ is invariant with respect to $Stab(x)$ and $GS$ is homeomorphic to $G \times_{Stab(x)} S$, which is the quotient of $G \times S$ by the following action of $Stab(x)$ : $g(g_1, x_1) = (g_1 g^{-1}, g x_1)$ for any $g \in Stab(x), g_1 \in G, x_1 \in S$. This implies easily that for any $x' \in M/G$ a local basis at $x'$ is formed by open sets $U \ni x'$ such that $p^{-1}(U)$ contracts to $p^{-1}(x')$. Hence the canonical map $\rho_{x'} : \mathcal{H}^i_{\mathbf{R}}(x') \to H^i(p^{-1}(x'), \mathbf{R})$, where $\mathcal{H}^i_{\mathbf{R}}(x)$ is the fiber of $\mathcal{H}^i_{\mathbf{R}}$ over $x$, is an isomorphism.

Note that for any $x \in M$ the action map $\tau_x : G \to Gx, \tau_x(g) = gx$ induces an isomorphism of real cohomology groups (the existence of a slice at $x$ implies that $Gx$ is homeomorphic to $G/Stab(x)$, and, since $G$ is connected, and $Stab(x)$ is finite, the real cohomology map induced by $G \to G/Stab(x)$ is an isomorphism). Let $\sigma : M/G \to M$ be any map, such that $p \circ \sigma = Id_{M/G}$. Now for any $i, y \in H^i(G, \mathbf{R})$ define the section $s_y$ of $\mathcal{H}^i_{\mathbf{R}}$ as follows: $s_y(x') = \rho_{x'}^{-1} \circ (\tau^*_{\sigma(x')})^{-1}(y)$.

Let $U$ be a neighborhood of $x'$ such that $p^{-1}(U)$ contracts to $p^{-1}(x')$. It is easy to check that for any $y \in H^i(G, \mathbf{R})$ there is a $y' \in H^i(p^{-1}(U))$ such that $s_y$ coincides on $U$ with the canonical section of $\mathcal{H}^i_{\mathbf{R}}$ over $U$ defined by $y'$, so all maps $x' \mapsto s_y(x')$ are indeed sections. It follows immediately from the definition that the section $s_y$ is nowhere zero if $y \neq 0$. Setting $y$ equal to the elements of some basis of $H^i(G, \mathbf{R})$, we obtain $\dim(H^i(G, \mathbf{R}))$ everywhere linearly independent sections of $\mathcal{H}^i_{\mathbf{R}}$, whose values span $\mathcal{H}^i_{\mathbf{R}}(x')$ for any $x'$. Hence the map $(x', y) \mapsto f_{x'}(y)$ establishes an isomorphism between $\mathcal{H}^i_{\mathbf{R}}$ and $H^i(G, \mathbf{R})$. The theorem is proved in the case, when $\mathcal{L} = \mathbf{R}$.

Now suppose that the system $\mathcal{L}$ is arbitrary. Note that for any open subset $U$ of $M/G$ there is a natural map (the $\smile$-product)

$$H^0(p^{-1}(U), \mathcal{L}) \otimes H^i(p^{-1}(U), \mathbf{R}) \to H^i(p^{-1}(U), \mathcal{L} \otimes \mathbf{R}) \cong H^i(p^{-1}(U), \mathcal{L})$$

This gives us a map

$$\mathcal{H}^0_{\mathcal{L}} \otimes \mathcal{H}^i_{\mathbf{R}} \to \mathcal{H}^i_{\mathcal{L}}. \tag{3.1}$$

Due to the existence of a slice at each point of $M$, the groups $\mathcal{H}^0_{\mathcal{L}}(x')$, $\mathcal{H}^i_{\mathbf{R}}(x'), \mathcal{H}^i_{\mathcal{L}}(x')$ are canonically isomorphic to $H^0(p^{-1}(x'), \mathcal{L}), H^i(p^{-1}(x'),$

$\mathbf{R}$), $H^i(p^{-1}(x'), \mathcal{L})$ respectively (for any $x' \in M/G$). Under this identification the restriction of the map (3.1) to the fiber over a point $x' \in M/G$ is the cup product map

$$H^0(p^{-1}(x'), \mathcal{L}) \otimes H^i(p^{-1}(x'), \mathbf{R}) \to H^i(p^{-1}(x'), \mathcal{L}). \qquad (3.2)$$

Now take some $x \in p^{-1}(x')$. The action of $\pi_1(p^{-1}(x')) \cong Stab(x)$ in the fiber of $\mathcal{L}|p^{-1}(x')$ splits into a sum of irreducible representations, hence $\mathcal{L}|p^{-1}(x')$ can be decomposed into a sum $\mathcal{L} = \oplus \mathcal{L}_j$ such that the action of $\pi_1(p^{-1}(x'))$ on a fiber of each $\mathcal{L}_j$ is irreducible. Note that the map (3.2) respects the decomposition of $\mathcal{L}$ into a direct sum. Applying Lemma 3.7 to each $\mathcal{L}_j$, we see that (3.1) is an isomorphism. We have already shown that $\mathcal{H}^i_{\mathbf{R}}$ is constant with fiber isomorphic to $H^i(G, \mathbf{R})$. The theorem is proved. $\qquad \square$

*Remark 3.9.* — There exist evident versions of Theorem 3.8 and Lemma 3.7 for non necessarily simply connected Lie groups, but we shall not need them in the sequel.

The following three lemmas will be frequently used in our calculations (see [2], [3] for a proof):

LEMMA 3.10. — *The group* $\bar{H}_*(B(\mathbf{C}^n, k), \pm\mathbf{R})$ *is trivial for any* $k \geqslant 2, n \geqslant 1$.

LEMMA 3.11. — *The group* $\bar{H}_*(B(\mathbf{CP}^n, k), \pm\mathbf{R})$ *for* $n \geqslant 1$ *is isomorphic to* $H_{*-k(k-1)}(G_k(\mathbf{C}^{n+1}), \mathbf{R})$, *where* $G_k(\mathbf{C}^{n+1})$ *is the Grassmann manifold of* $k$-*dimensional subspaces in* $\mathbf{C}^{n+1}$.

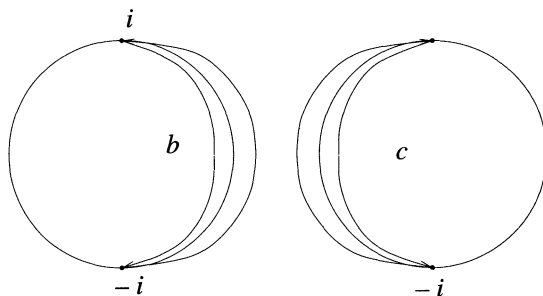In particular, the group $\bar{H}_*(B(\mathbf{CP}^n, k), \pm\mathbf{R})$ is trivial if $k > n + 1$.



Figure 1. —

LEMMA 3.12. — *If $k \geqslant 2$, then the group $H_*((S^2)^{*k}, \mathbf{R})$ is trivial in all positive dimensions, where $(S^2)^{*k}$ is the $k$-th self-join of $S^2$.*

Consider the space $\mathbf{C} \setminus \{1, -1\}$ and the coefficient system $\mathcal{L}$ on it that changes its sign under any loop based at 0 that passes once around 1 or $-1$. Let $f$ be the map $z \mapsto -z$ and let $\tilde{f} : \mathcal{L} \to \mathcal{L}$ be the map that covers $f$ and is identical over 0.

PROPOSITION 3.13. — *The map $\tilde{f}$ acts on the groups $H_1(\mathbf{C} \setminus \{1, -1\}, \mathcal{L})$, $H^1(\mathbf{C} \setminus \{1, -1\}, \mathcal{L})$, and $\bar{H}_1(\mathbf{C} \setminus \{1, -1\}, \mathcal{L})$ as multiplication by $-1$.*

Consider $B(\mathbf{C}^*, 2)$, i.e. the space of pairs of points in $\mathbf{C} \setminus \{0\}$. It is a fiber bundle over $\mathbf{C}^*$, the projection $p : B(\mathbf{C}^*, 2) \to \mathbf{C}^*$ being the multiplication of complex numbers. The fiber is homeomorphic to $\mathbf{C} \setminus \{1, -1\}$, and the action of the generator of $\pi_1(\mathbf{C}^*)$ is $z \mapsto -z$. The fiber $p^{-1}(1)$ contracts to the character "8". Denote by $b, c$ the loops in $p^{-1}(1)$ based at $\{\pm i\}$ and represented schematically on Figure 1; note that they correspond to the circles of the "8". Denote by $a$ the loop $t \mapsto \{ie^{\pi it}, -ie^{\pi it}\}$.

Consider the following three local systems on $B(\mathbf{C}^*, 2)$ (the fiber of each of them is $\mathbf{R}$):

1. $\mathcal{A}_1$ changes its sign under $a$ and does not change its sign under $b$ and $c$.

2. $\mathcal{A}_2$ changes its sign under $b$ and $c$ and does not change its sign under $a$.

3. $\mathcal{A}_3$ changes its sign under all loops $a, b, c$.

Let $f$ be the map $B(\mathbf{C}^*, 2) \to B(\mathbf{C}^*, 2)$ induced by the map $z \mapsto 1/z$ and let $f^i : \mathcal{A}_i \to \mathcal{A}_i, i = 1, 2, 3$ be the maps that cover $f$ and are identical over the fiber $p^{-1}(1)$. Note that we have $\mathcal{A}_3 = \pm\mathbf{R}$.

PROPOSITION 3.14. —

1. $\bar{P}(B(\mathbf{C}^*, 2), \mathcal{A}_1) = \bar{P}(B(\mathbf{C}^*, 2), \mathcal{A}_3) = t^2(1+t)$, $\bar{P}(B(\mathbf{C}^*, 2), \mathcal{A}_2) = 0$.

2. For $i = 1, 3$ the map $f^i_*$ acts identically $\bar{H}_3(B(\mathbf{C}^*, 2), \mathcal{A}_i)$ and acts on $\bar{H}_2(B(\mathbf{C}^*, 2), \mathcal{A}_i)$ as multiplication by $-1$.

Note that a lifting of the generator of $\pi_1(\mathbf{C}^*)$ is given by: $\gamma_{\{a,b\}}(t) = \{ae^{\pi it}, be^{\pi it}\}$, where $\{a,b\} \in p^{-1}(1)$.

This statement follows immediately from Theorems 3.3 and 3.4.

## 4. The spectral sequence for plane quintics

In this section we give a proof of Theorem 1.2. Set $V = \Pi_5, \Sigma = \Sigma_5$. In the sequel, by a *conic* we shall mean any curve of degree 2 in $\mathbf{CP}^2$. We shall say that points $x_1, \ldots, x_k \in \mathbf{CP}^2$ are *in general position*, if among these points there are no 3 points that are on the same line, no 6 points on a conic, no 10 points on a cubic, etc. A line $l \subset \mathbf{CP}^2$ is said to be nontangential to an algebraic curve $C$, if $l \cap Q$ consists of $\deg C$ points. By definition, lines $l_1, \ldots, l_k \subset \mathbf{CP}^2$ are *in general position*, if the elements of $\mathbf{CP}^{2\vee}$ corresponding to $l_1, \ldots, l_k$ are in general position.

### 4.1. Configuration spaces

PROPOSITION 4.1. — *The configuration spaces $X_1, \ldots, X_{42}$, that consist of the following configurations satisfy Conditions 1–7 and Condition 8 (see pages 399 and 403). The number indicated in brackets equals the dimension of $L(K)$ for $K$ lying in the corresponding $X_i$.*

1. *One point in $\mathbf{CP}^2$ (18).*

2. *2 points in $\mathbf{CP}^2$ (15).*

3. *3 points in $\mathbf{CP}^2$ (12).*

4. *4 points on a line (11).*

5. *5 points on a line (10).*

6. *6 points on a line (10).*

7. *7 points on a line (10).*

8. *8 points on a line (10).*

9. *9 points on a line (10).*

10. *10 points on a line (10).*

11. *A line in $\mathbf{CP}^2$ (10).*

12. *4 points in* $\mathbf{CP}^2$ *not on a line (9). (Any three of them may belong to a line though.)*

13. *4 points on a line + a point not belonging to the line (8).*

14. *5 points on a line + one point not belonging to the line (7).*

15. *6 points on a line + one point not belonging to the line (7).*

16. *7 points on a line + one point not belonging to the line (7).*

17. *A line in* $\mathbf{CP}^2+$ *a point not belonging to the line (7).*

18. *5 points in* $\mathbf{CP}^2$ *such that there is no line containing 4 of them (6).*

19. *4 points on a line + 2 points not belonging to the line (5).*

20. *5 points on a line + 2 points not belonging to the line (4).*

21. *6 points on a line + 2 points not belonging to the line (4).*

22. *A line in* $\mathbf{CP}^2+$ *2 points not belonging to it (4).*

23. *3 points on each of two intersecting lines such that none of the points coincides with the point of intersection (4).*

24. *6 points on a nondegenerate conic (4).*

25. *A configuration of type 23 + the point of intersection of the lines (4).*

26. *6 points not belonging to a (possibly degenerate) conic such that there is no line containing 4 of those points (3).*

27. *4 points on a line + 3 points on another line such that none of the 7 points coincides with the point of intersection of the lines (3).*

28. *5 points on a line* $l_1$ *+ 3 points on* $l_2 \setminus l_1$*, where* $l_2$ *is a line* $\neq l_1$ *(3).*

29. *A line + 3 points of some other line, none of which coincides with the point of intersection of the lines (3).*

30. *4 points on a line + 4 points on some other line such that none of the 8 points coincides with the point of intersection of the lines (3).*

31. *The union of two lines in* $\mathbf{CP}^2$ *(3).*

32. *7 points on a nondegenerate conic (3).*

33. *A nondegenerate conic (3).*

34. *4 points on a line + 3 points in general position not belonging to the line (2).*

35. *A configuration of type 23 + a point not belonging to the union of the lines (1).*

36. *6 points on a nondegenerate conic + a point not belonging to the conic (1).*

37. *A configuration of type 35 + the point of intersection of the lines (1).*

38. *4 points $A, B, C, D \in \mathbf{CP}^2$ in general position + 4 points of intersection of a line $l$ not passing through $A, B, C, D$ and two (possibly degenerate) conics passing through $A, B, C, D$ and not tangential to $l$ (1).*

39. *3 points $A, B, C \in \mathbf{CP}^2$ in general position + 6 points of intersection of 3 lines $AB, BC, AC$ and a (possibly degenerate) conic not passing through $A, B, C$, and not tangential to the lines $AB, BC, AC$ (1).*

40. *10 points of intersection of 5 lines in $\mathbf{CP}^2$ in general position (1).*

41. *A line in $\mathbf{CP}^2 +$ 3 points in general position not belonging to the line (1).*

42. *The whole $\mathbf{CP}^2$ (0).*

*Proof.* — Condition 1 follows from the following observations: 1. the singular locus of a curve defined by the product of two polynomials is the union of the singular loci of the curves defined by those polynomials and the intersection points of the curves; 2. the singular locus an irreducible curve of degree 5 consists of 1 to 6 points in general position; 3. all possible singular sets of curves of degree $\leqslant 4$ are described in [2]. Note that some spaces $X_i$ contain (or consist of) configurations that are not singular loci of curves of degree 5. We introduce them to make sure that Conditions 5 and 8 are satisfied.

The verification of Conditions 2, 4 and 8 is straightforward.

Condition 3 will be deduced below from the following lemma.

LEMMA 4.2. —

1. Let $x_1, \ldots, x_k, k \leqslant 6$, be several points in general position in $\mathbf{CP}^2$. Then the complex dimension of the space $L(\{x_1, \ldots, x_k\})$ (which consists of polynomials of degree 5 that have singularities at all points $x_1, \ldots, x_k$ and maybe elsewhere) is equal to $21 - 3k$.

2. Let $l_1, l_2$ be two distinct lines in $\mathbf{CP}^2$. Suppose that
   a) $x_1, x_2, x_3 \in l_1 \setminus l_2$,
   b) $y_1, y_2, y_3 \in l_2 \setminus l_1$,
   c) $A \notin l_1 \cup l_2$.
   Then there exists exactly one cubic passing through all the points $x_i, y_j, i, j = 1, 2, 3$ and having a singularity at $A$.

3. Let $Q$ be a nondegenerate conic in $\mathbf{CP}^2$, and suppose that $x_1, \ldots, x_6 \in Q, A \notin Q$. Then there exists exactly one cubic passing through $x_1, \ldots, x_6$ and having a singularity at $A$.

4. If a curve of degree 5 has three singular points on a line, then it contains the line. If a curve of degree 5 has six singular points on a nondegenerate conic, then it contains the conic.

5. Consider a point $A \in \mathbf{CP}^2$. For any $d$ define $L_d^x(A)$ (respectively, $L_d^y(A), L_d^z(A), M_d(A)$) as the linear space of homogeneous polynomials $f$ of degree $d$ such that $\frac{\partial f}{\partial x} = 0$ (respectively, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$, $f = 0$) at every point of the preimage of $A$ under the natural map $\mathbf{C}^3 \setminus \{0\} \to \mathbf{CP}^2$. Suppose that $l$ is a line in $\mathbf{CP}^2, x_1, x_2, x_3, x_4 \in l, y_1, y_2, y_3 \notin l$, and suppose that $y_1, y_2, y_3$ are not on a line. Then

$$dim_{\mathbf{C}}\left((\cap_{i=1}^4 M_4(x_i)) \cap (\cap_{i=1}^3 L_4^x(y_i)) \cap (\cap_{i=1}^3 L_4^y(y_i)) \cap (\cap_{i=1}^3 L_4^z(y_i))\right) = 2.$$

Remark 4.3. — Statement 5 of Lemma 4.2 implies that the thirteen hyperplanes $M_4(x_i), i = 1, \ldots, 4, L_4^x(y_i), L_4^y(y_i), L_4^z(y_i), i = 1, 2, 3$ intersect transversally.

Let us prove the first statement of the lemma. Suppose that $x_1, \ldots, x_6$ are points of $\mathbf{CP}^2$ in general position. It suffices to prove that 18 linear conditions on the space $\Pi_5$ that define the space $L(\{x_1, \ldots, x_6\})$ are independent. Suppose they are not, then $dim_{\mathbf{C}} L(\{x_1, \ldots, x_6\}) \geqslant 4$. Choose a point $x'$ such that $x_1, \ldots, x_6, x'$ are in general position. The space $L(\{x_1, \ldots, x_6, x'\})$ would be then of dimension $\geqslant 1$, which is impossible, because no curve of

degree 5 can have 7 singular points in general position (if the curve is irreducible, this follows from [5, exercise 3, §2 of Chapter III], otherwise this is trivial).

The statements 2, 3., 5 can be proved in an analogous way. The statements 4 follows from Bezout's theorem. $\Diamond$

Let us prove now that the spaces $X_i$ introduced in Proposition 4.1 satisfy Condition 3. The case of $X_1$ and $X_2$ is evident. If we have a configuration $K$ that consists of three points in $\mathbf{CP}^2$ that do not belong to any line, then using the first statement of Lemma 4.2, we get $\dim_{\mathbf{C}} L(K) = 21 - 9 = 12$. If $K \subset \mathbf{CP}^2$ consists of three points on a line $l \subset \mathbf{CP}^2$, then due to statement 4 of Lemma 4.2, any function $f \in L(K)$ has the form $f = gh$, where $g$ is a fixed linear homogeneous function, and $h$ is a polynomial that defines a curve that intersects $l$ in every point of $K$ and maybe elsewhere. Using statement 5 of Lemma 4.2, we obtain $\dim_{\mathbf{C}} L(K) = 15 - 3 = 12$. We have thus proved that $\dim_{\mathbf{C}} L(K) = 12$ for any $K \in X_3$.

The case of $X_4$ can be considered in an analogous way.

If a curve of degree 5 contains five singular points on a line, then this curve is defined by a polynomial of the form $f^2 g$, where $f$ is a polynomial that defines the line, and $g$ is a polynomial of degree 3. This gives the dimensions of all spaces $L(K), K \in X_5., \ldots, X_{11}, X_{14}, \ldots, X_{17}, X_{20}, \ldots, X_{22}$.

Consider a configuration $K \in X_{12}$. If no three of the points of $K$ are on a line, we have $\dim_{\mathbf{C}}(L(K)) = 21 - 12 = 9$ by statement 1 of Lemma 4.2. If $K$ contains 3 points on a line $l$, then, due to statement 4 of Lemma 4.2, every $f \in L(K)$ has the form $f = gh$, where $g$ is a fixed polynomial of degree 1, and $h$ is an arbitrary polynomial of degree 4 that defines a curve that has 3 fixed intersection points with $l$ and a fixed singular point outside $l$. Using statement 5 of the same lemma (the transversality of intersection), we see that the dimension of $L(K)$ is equal to $15 - 3 - 3 = 9$. The same argument gives the dimensions of $L(K), K \in X_{13}, X_{19}, X_{34}, X_{18}, X_{26}$.

Note that if $l_1, l_2$ are two distinct lines in $\mathbf{CP}^2$, $x_1, x_2, x_3 \in l_1 \backslash l_2, y_1, y_2, y_3 \in l_2 \backslash l_1, A \notin l_1 \cup l_2$, then statement 2 of Lemma 4.2 implies that

$$\dim_{\mathbf{C}} \left( \bigcap_{i=1}^{3} M_3(x_i)) \cap (\bigcap_{i=1}^{3} M_3(y_i)) \cap L_3^x(A) \cap L_3^y(A) \cap L_3^z(A) \right) = 1,$$

which means that nine hyperplanes $M_3(x_i), M_3(y_i), i = 1, 2, 3, L_3^x(A), L_3^y(A)$, and $L_3^z(A)$ intersect transversally. This gives the dimensions of $L(K), K \in X_{23}, X_{25.}, X_{35}, X_{37}$. The same argument works for $X_{24}, X_{36}$, except that we apply statement 3. of Lemma 4.2 (instead of statement 2).

It is easy to see that for any $K \in X_{27}, \ldots, X_{31}$ the vector space $L(K)$ consists of polynomials of the form $f^2 g^2 h$, where $f, g$ are some fixed polynomials of degree 1 that define two distinct lines, and $h$ is an arbitrary polynomial of degree 1. Analogously, for any $K \in X_{32}, X_{33}$ the space $L(K)$ consists of polynomials of the form $f^2 g$, where $f$ is a fixed polynomial of degree 2 that defines a nondegenerate conic, and $h$ is an arbitrary polynomial of degree 1.

Consider a configuration $K \in X_{38}$ and $f \in L(K)$. It follows from statement 4 of Lemma 4.2 that $f = gh$, where $g$ is a polynomial of degree 1, and $h$ is a polynomial of degree 4 that has singularities at four points $A, B, C, D$ in general position outside the line $l$ defined by $g$. This implies that $h = h_1 h_2$, where $h_1, h_2$ are polynomials of degree 2 that define two conics $Q_1, Q_2$ passing through $A, B, C, D$. $f$ must also have singularities at four points on $l$, hence each of these four points belongs to exactly one of the conics $Q_1, Q_2$. It follows that $f$ is defined by $K$ up to multiplication by a nonzero constant.

Analogously it can be proved that for any $K \in X_{39}, X_{40}$ and any $f \in L(K)$, $f$ is defined by $K$ up to nonzero constant. The cases $X_{41}, X_{42}$. are trivial. Thus, we have proved that the spaces $X_i$ satisfy Condition 3.

Let us prove now that these spaces satisfy Conditions 6 and 7. Recall that the spaces $X_i$ satisfy Condition 8. This implies (see p. 6) that for the spaces $X_i$ consisting of finite configurations Condition 7 follows from Condition 6. Consider some $X_i$ that consists of finite configurations. It is immediate to check that the number of elements in all configurations from $X_i$ is the same. Denote this number by $k$. Set $\mathcal{M}_k = \{(x, K) | x \in \mathbf{CP}^2, K \subset \mathbf{CP}^2, \#(K) = k, x \in K\}$ (here and below, for a finite set $K$ we denote by $\#(K)$ the cardinality of $K$). It is easy to see that $\mathcal{M}_k$ is the total space of a fiber bundle over $B(\mathbf{CP}^2, k)$ (with projection $(x, K) \mapsto K$). The triple $(\mathcal{T}_i, X_i, pr_i)$ is the restriction of this fiber bundle to $X_i$.

Thus, all spaces $X_i$ that consist of finite configurations satisfy Conditions 6 and 7.

Now consider, for instance, the space $X_{31}$. Note that if $G$ is a Lie group that acts smoothly on a smooth manifold $M$ and $a \in M$, there exist submanifolds $S \subset M, S' \subset G$ such that

1. $a \in S, e \in S'$ ($e$ is the unit element of $G$),

2. $S'$ is transversal to $Stab(a)$ at $e$,

3. $S'a \subset M$ is a submanifold that intersects $S$ transversally at $a$ (here $S'x = \{gx | g \in S'\}$),

4. the map $S \times S' \to M$, defined by $(a', g) \mapsto ga', g \in S', a' \in S$ is a diffeomorphism onto an open neighborhood of $a$ in $M$.

Set $M = X_{31}, G = PGL(\mathbf{CP}^2)$. The action of $G$ on $M$ is transitive, so for any $K \in X_{31}$ the above remark gives us a neighborhood $U \ni K$ and a diffeomorphism $r : U \to S', S' \subset PGL(\mathbf{CP}^2)$ such that for any $K' \in U$ we have $r(K')K = K'$. Now for any $x \in K$ and $K' \in U$, set $t(x, K') = r(K')x$. It is clear that the map $(x, K') \mapsto (t(x, K'), K')$ is a local trivialization of $\mathcal{T}_{31}$ over $U$. This trivialization satisfies Condition 7, since all spaces $X_i$ are invariant under $PGL(\mathbf{CP}^2)$.

Local trivializations of the tautological bundles over the spaces $X_{11}, X_{17},$ $X_{22}, X_{41},$ and $X_{33}$ can be constructed in the same way.

However, this method does not work for $X_{29}$, because the action of $PGL(\mathbf{CP}^2)$ on this space is no longer transitive. But we can proceed as follows. Consider $K \in X_{29}$. We have $K = K_1 \sqcup K_2$, where $K_1$ is a line, and $K_2$ consists of three points of another line. Denote by $B'$ the space of all configurations in $\mathbf{CP}^2$ consisting of three points on a line. Let $U_1$ (respectively, $U_2$) be neighborhoods of $K_1$ in $X_{11}$ (respectively, of $K_2$ in $B'$) such that the bundle $(\mathcal{T}_{11}, X_{11}, pr_{11})$ is trivial over $U_1$, $(\mathcal{T}_3, X_3, pr_3)$ is trivial over $U_2$, and for every $K_1' \in U_1', K_2' \in U_2'$ we have $K_1' \cap K_2' = \varnothing$. For $j = 1, 2$ let $t_j : K_j \times U_j \to \mathbf{CP}^2$ be a map such that the map $(x, K_j') \mapsto (t_j(x, K_j'), K_j'), x \in K_j, K_j' \in U_j$ is a trivialization of the corresponding tautological bundle over $U_j$. Set $U = \{K_1' \sqcup K_2' | K_1' \in U_1', K_2' \in U_2'\}$, and for any $K' = K_1' \sqcup K_2' \in U$ set $t(x, K')$ equal to $t_j(x, K_j')$, if $x \in K_j, j = 1, 2$. It is clear that $U$ is an open neighborhood of $K$ in $X_{29}$ and that the map $(x, K') \mapsto (t(x, K'), K')$ is a trivialization of $(\mathcal{T}_{29}, X_{29}, pr_{29})$ over $U$. It follows from the construction of $t$ that for any fixed $K' \in U$, the map $x \mapsto t(x, K')$ establishes a bijective correspondence between the subsets of $K$ and $K'$ that belong to $\bigcup_{j<29} X_j$. (Due to Condition 8, this needs to be checked only for maximal finite subconfigurations of $K$ (which belong to $X_{28}, X_{21}, X_{16}, X_{10}$) and for nondiscrete subconfigurations (which belong to $X_{11}, X_{17}, X_{22}$).)

We have shown that the spaces $X_i$ satisfy Conditions 6 and 7. It remains to verify Condition 5.

Let us begin with the following three lemmas.

LEMMA 4.4. — *Denote by $\Pi_d$ the vector space of all homogeneous polynomials $\mathbf{C}^3 \to \mathbf{C}$ of degree $d$. The map $\Pi_d \setminus \{0\} \to 2^{\mathbf{CP}^2}$ that takes a polynomial to the projectivization of the set of its zeroes is continuous.*

COROLLARY 4.5. — *The subspace of* $2^{\mathbf{CP}^2}$ *consisting of all zero sets of homogeneous polynomials of some fixed degree is closed.*

*Remark 4.6.* — In the case of real polynomials, the analogous map to the real projective plane is neither everywhere defined nor continuous on its domain of definition.

For any $f \in \Pi_d \setminus \{0\}$ denote by $[f]$ the image of $f$ under the natural map $\Pi_d \setminus \{0\} \to (\Pi_d \setminus \{0\})/\mathbf{C}^*$.

LEMMA 4.7. — *Suppose we have a sequence* $(K_i), K_i \in 2^{\mathbf{CP}^2}$, *and a sequence* $(f_i), f_i \in \Pi_d \setminus \{0\}$, *and suppose that* $f_i$ *has a singularity at every point of* $K_i$. *If* $K \in 2^{\mathbf{CP}^2}, f \in \Pi_d \setminus \{0\}$ *are such that* $\lim_{i \to \infty} K_i = K, \lim_{i \to \infty} [f_i] = [f]$, *then* $f$ *has a singularity at every point of* $K$.

LEMMA 4.8. — *Suppose we have sequences* $(L_i)$ *and* $(M_i)$ *in* $2^{\mathbf{CP}^2}$ *and suppose that* $K \in 2^{\mathbf{CP}^2}, K = \lim_{i \to \infty}(L_i \cup M_i)$. *Then there exist a sequence of indices* $(i_j)$ *such that* $K = (\lim_{j \to \infty} L_{i_j}) \cup (\lim_{j \to \infty} M_{i_j})$.

*Proof of Lemma 4.8.* — Choose a sequence $(i_j)$ such that there exist $\lim_{i \to \infty} L_i, \lim_{i \to \infty} M_i$, and denote these limits by $L, M$ respectively. Let $\rho$ be a metric that induces the usual topology on $\mathbf{CP}^2$, and let $\tilde{\rho}$ be the corresponding Hausdorff metric on $2^{\mathbf{CP}^2}$. If $A, B, C, D \in 2^{\mathbf{CP}^2}$, then $\tilde{\rho}(A \cup B, C \cup D) \leqslant \tilde{\rho}(A, C) + \tilde{\rho}(B, D)$. This implies that $\tilde{\rho}(M_{i_j} \cup L_{i_j}, M \cup L) \leqslant \tilde{\rho}(M_{i_j}, M) + \tilde{\rho}(L_{i_j}, L)$. Hence $M \cup L = \lim_{j \to \infty}(M_{i_j} \cup L_{i_j}) = \lim_{i \to \infty}(M_i \cup L_i) = K$. □

Now the verification of Condition 5 becomes straightforward in all cases except $X_{38}, X_{39}, X_{40}$. Consider, for instance, $K \in \bar{X}_{30}$. We have $K = \lim_{i \to \infty} K_i$, since all $K_i \in X_{30}$, they can be represented as $K_i = (K_i \cap l_1^i) \cup (K_i \cap l_2^i)$, where $l_1^i, l_2^i$ are lines. Due to Lemma 4.8, we can suppose that $K = (\lim_{i \to \infty}(K_i \cap l_1^i)) \cup (\lim_{i \to \infty}(K_i \cap l_2^i))$. Using Corollary 4.5, we can suppose that the sequences $(l_1^i), (l_2^i)$ converge. Applying Proposition 2.4, we see that $K$ is a configuration of the form ($\leqslant 4$ points on a line $l_1$)$\cup$($\leqslant 4$ points on another line). All such configurations belong to $\bigcup_{i=1}^{30} X_i$.

However, this argument does not work for $X_{38}, X_{39}, X_{40}$. Let us see, what happens in these cases. Consider, for instance, $K \in X_{38}$. Using Lemma 4.7, we see that $K$ is included into the singular set of some polynomial $f$ of degree 5. If this singular set is discrete, there is nothing to prove: due to

Conditions 1 and 8, if a subset of a discrete singular set consists of $\leqslant 8$ elements, then this subset belongs to $\bigcup_{i=1}^{38} X_i$. Otherwise we can do the following.

We have $K = \lim_{i \to \infty} K_i$, where all $K_i$ are of the form $(Q_1^i \cap Q_2^i) \cup (Q_1^i \cap l^i) \cup (Q_2^i \cap l^i)$, $Q_1^i$ and $Q_2^i$ are conics, $l^i$ are lines. Applying Lemma 4.8 and Corollary 4.5, we can assume that

$$K = \big( \lim_{i \to \infty} (Q_1^i \cap Q_2^i) \big) \cup \big( \lim_{i \to \infty} (Q_1^i \cap l^i) \big) \cup \big( \lim_{i \to \infty} (Q_2^i \cap l^i) \big)$$

and that the sequences $(Q_1^i), (Q_2^i)$ and $(l^i)$ converge. Denote the limits of these sequences by $Q_1, Q_2$ and $l$ respectively.

$f$ can have the following nondiscrete singular sets: a line of multiplicity $\geqslant 2$, two double lines, a double line + a triple line, a double nondegenerate conic. Let us consider all these cases.

*A double line.* In this case we have the following possibilities:

1. $Q_1 = m_1 \cup m_2, Q_2 = m_1 \cup m_3$, where $m_1, m_2, m_3, l$ are 4 pairwise distinct lines. $\lim_{i \to \infty} (Q_1^i \cap Q_2^i)$ is included into a configuration of the form (3 points on $m_1$){the point $m_2 \cap m_3$}. Thus, $K$ is included into a configuration of the form (the points $l \cap m_1, l \cap m_2, l \cap m_3$ and $m_2 \cap m_3$)$\cup$(3 points on $m_1$). Such a configuration is a subset of a configuration from $X_{34}$ or $X_{20}$.

2. $Q_1 = l \cup m$, where $m \neq l$, and $Q_2$ contains neither $l$ nor $m$. $\lim_{i \to \infty} (Q_1^i \cap l^i)$ consists of one or two points on $l$. Hence $K$ is included into a configuration of the form $(Q_2 \cap l) \cup (Q_2 \cap m) \cup$(2 points on $l$), which contains $\leqslant 6$ points.

3. $Q_1 = m \neq l, Q_2$ contains neither $l$ nor $m$. $K = (m \cap l) \cup (Q_2 \cap l) \cup (Q_2 \cap m)$, hence $K$ contains $\leqslant 5$ points.

*A triple line.* In this case $K$ is included into a configuration of the form (a line)+(a point). Hence $K$ is a subset of a configuration from $X_8$ or $X_{16}$.

*A line of multiplicity $\geqslant 4$.* $K$ is a subset of a configuration from $X_8$.

*Two double lines or a double line + a triple line.* $K$ is a subset of the union of 2 lines and $\#(K) \leqslant 8$. All such configurations belong to $\bigcup_{i=1}^{30} X_i$.

*A double nondegenerate conic.* We have that $Q_1 = Q_2$ is nondegenerate. $\lim_{i \to \infty} (Q_1^i \cap Q_2^i)$ is included into a subconfiguration of the form (4 points on $Q_1$), $\lim_{i \to \infty} (Q_1^i \cap l^i) = \lim_{i \to \infty} (Q_2^i \cap l^i)$ contains 1 or 2 points on $Q_1$. Thus, $K$ is included into a configuration from $X_{24}$.

We have checked Condition 5 for $X_{38}$. The spaces $X_{39}, X_{40}$ can be considered in a similar way.

Proposition 4.1 is proved.    □

Now we apply Theorem 2.8 and Lemmas 2.10,2.11 to construct a conical resolution $\sigma$ and a filtration $\varnothing \subset F_1 \subset \cdots \subset F_{42.} = \sigma$. The spectral sequence (1.1) is the sequence corresponding to this filtration.

Most of the columns of the sequence (1.1) can be investigated in essentially the same way as in the case of nonsingular quartics considered in [2]. We shall only discuss the columns that need a somewhat different argument.

## 4.2. Column 38

Let $X_{38}$ be the space of all configurations of type 38 (see Proposition 4.1). From Lemma 2.10 we get

$$E^1_{38,i} = \bar{H}_{38+i-2-7}(X_{38}, \pm\mathbf{R}),  \tag{4.1}$$

where the local system $\pm\mathbf{R}$ is described in Definition 3.2.

$X_{38}$ is naturally fibered over the space $\tilde{B}(\mathbf{CP}^2, 4)$ of generic quadruples $\{A, B, C, D\} \subset \mathbf{CP}^2$. Let us denote by $Y$ the fiber of this bundle, i.e. the space of all configurations from $X_{38}$ such that the points of intersection of the conics are fixed.

LEMMA 4.9. —  *The term $E^2$ of the spectral sequence of the bundle $X_{38} \to \tilde{B}(\mathbf{CP}^2, 4)$ is zero.*

The proof will take the rest of the subsection.

Denote by $L$ the space of all lines not passing through any of the four points $A, B, C, D$ in general position in $\mathbf{CP}^2$. For any such line $l$ denote by $Z$ the space of conics passing through $A, B, C, D$ and not tangential to $l$. The space $Z$ is homeomorphic to ($S^2$ minus 2 points)$= \mathbf{C}^*$.

$Y$ is fibered over $L$ with fiber $B(Z, 2) = B(\mathbf{C}^*, 2)$.

LEMMA 4.10. — *The Borel-Moore homology group of the fiber $Y$ of the bundle $X_{38} \to \tilde{B}(\mathbf{CP}^2, 4)$ can be obtained from the spectral sequence of the bundle $Y \to L$, whose term $E^2$ is as follows:*

$$
\begin{array}{c|ccc}
3 & \mathbf{R}^3 & \mathbf{R}^3 & \mathbf{R} \\
2 & \mathbf{R} & & \\
\hline
 & 2 & 3 & 4
\end{array}
\qquad (4.2)
$$

*Proof.* — Recall that $B(\mathbf{C}^*, 2)$ is a fiber bundle with base $\mathbf{C}^*$ and fiber $\mathbf{C} \setminus \{1, -1\}$. Let us study the restriction of the coefficient system $\pm\mathbf{R}$ to the fiber $B(\mathbf{C}^*, 2)$ of the bundle $Y \to L$. This system changes its sign, when one of the points passes around zero (and the other stands still). This corresponds to the fact that if we fix all points in the configuration except the points of intersection of the line and one of the conics, we can transpose those points. On the contrary, a loop that transposes two conics, transposes two pairs of points and does not change the sign of the coefficient system. We see that the loops of the fiber do not change the sign of the coefficient system, and some loop that projects onto the generator of $\pi_1(\mathbf{C}^*)$ (and hence any other such loop) does. So $\pm\mathbf{R}|B(\mathbf{C}^*, 2)$ is the system $\mathcal{A}_1$ of Proposition 3.14. We have $\bar{H}_2(B(\mathbf{C}^*, 2), \mathcal{A}_1) = \bar{H}_3(B(\mathbf{C}^*, 2), \mathcal{A}_1) = \mathbf{R}$.

The space $L$ is homeomorphic to $\mathbf{C}^2$ minus the union of three complex lines in general position. We have $\bar{H}_i(L) = \mathbf{R}^3$ if $i = 2, 3$, $\bar{H}_i(L) = \mathbf{R}$ if $i = 4$ and $\bar{H}_i(L) = 0$ otherwise. We shall complete the proof of Lemma 4.10 in the following two lemmas.

LEMMA 4.11. — *Let $l(t)$ be a loop in $L$ that moves a line $l = l(0)$ around one of the points $A, B, C, D$. Let $Z$ be the space of conics passing through $A, B, C, D$ and not tangential to $l$. We can identify $Z$ with $\mathbf{C}^*$ (choosing an appropriate coordinate map $z : Z \to \mathbf{C}^*$) in such a way that the map $Z \to Z$ induced by $l(t)$ can be written as $z \mapsto 1/z$. If moreover $A = (1, 0)$, $B = (-1, 0), C = (0, 1), D = (0, -1), l(t) = \{x = \alpha(t)\}, \alpha(t) = 1 + \varepsilon e^{2\pi i t}$, where $\varepsilon = \frac{2}{\sqrt{3}} - 1$, then the conics $q_1 = \{xy = 0\}$ and $q_2 = \{x^2 + y^2 = 1\}$ are preserved, and the points of intersection of $q_1$ and $l$ are fixed, while the points of intersection of $q_2$ and $l$ are transposed.*

*Proof.* — Denote by $Q$ the space of conics passing through $A, B, C, D$. These conics can be written as follows:

$$
ax^2 + ay^2 + bxy - a = 0.
$$

Such a conic is tangential to $l(t)$ if and only if

$$
(b\alpha(t))^2 - 4a^2\alpha(t)^2 + 4a^2 = 0. \qquad (4.3)
$$

Note that if $t = 0$, then $\alpha(0) = \frac{2}{\sqrt{3}}$, and the condition (4.3) becomes simply $a^2 = b^2$. The map $f_t : Q \to Q$ that carries the conics tangential to $l$ to the conics tangential to $l(t)$ can be written as

$$(a, b) \mapsto (\frac{1}{2} \cdot a\sqrt{\alpha^2/(\alpha^2 - 1)}, b).$$

If $\alpha(t)$ is as above, $\sqrt{\alpha^2/(\alpha^2 - 1)}$ changes its sign, so the map $f_1(a, b) = (-a, b)$. The desired coordinate $z \in \mathbf{C}^*$ is $z = (b + a)/(b - a)$. Note that $z(q_1) = 1, z(q_2) = -1, z(-a, b) = 1/z(a, b)$. The conics $q_1$ and $q_2$ are preserved under any map $f_t$. The points of intersection of $q_1$ and $l$ are clearly fixed. The statement concerning the points of intersection of $q_2$ and $l$ can be verified immediately. Lemma 4.11 is proved.     □

Now we can describe the action of $\pi_1(L)$ on the Borel-Moore homology of the fiber $B(Z, 2) = B(\mathbf{C}^*, 2)$ of the bundle $Y \to L$.

The covering map of coefficient systems $\pm\mathbf{R}|B(\mathbf{C}^*, 2) \to \pm\mathbf{R}|B(\mathbf{C}^*, 2)$ induced by the loop considered in Lemma 4.11 is minus identity over the configuration $\{1, -1\} \in B(\mathbf{C}^*, 2)$. This implies that the fiber of the coefficient system over the pair, say $\{i, -i\}$, is mapped identically.

Applying Proposition 3.14, we obtain immediately that the corresponding map $\bar{H}_i(B(Z, 2), \pm\mathbf{R}) \to \bar{H}_i(B(Z, 2), \pm\mathbf{R})$ is the identity for $i = 3$ and minus identity for $i = 2$.

Thus, the 3-d line of the sequence (4.2) contains the Borel-Moore homology of $L$ with constant coefficients. In order to obtain the 2-nd line we must calculate the Borel-Moore homology of $L$ with coefficients in the system $\mathcal{L}$ that changes its sign under the action of any loop in $\mathbf{CP}^{2\vee}$ that embraces exactly one of the lines corresponding to the points $A, B, C, D$.

LEMMA 4.12. — *Let $L$ be the complement in $\mathbf{CP}^2$ of four complex lines in general position. Let $f : L \to L$ be the restriction to $L$ of the projective linear map that transposes two of these lines and preserves the other two, and let $\tilde{f} : \mathcal{L} \to \mathcal{L}$ be the map that covers $f$ and is identical over some point of $L$ that is fixed under $f$. Then*

a) *the Poincaré polynomial of $L$ with coefficients in $\mathcal{L}$ equals $t^2$, and*

b) *the map $\tilde{f}$ multiplies by $-1$ the groups $H^2(L, \mathcal{L})$ and $\bar{H}_2(L, \mathcal{L})$.*

Note that if $\tilde{f}$ is the identity over some fixed point of $f$, then it is the identity over any other fixed point; this follows from the fact that the set of fixed points of $f$ is connected.

*Proof of Lemma 4.12.* — Identify $L$ with the space $\mathbf{C}^2 \setminus (\{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 + z_2 = 1\})$. Consider the map $p : L \to \mathbf{C} \setminus \{1\}, p(z_1, z_2) = z_1 + z_2$. Set $A_1 = p^{-1}(U_1(0)), A_2 = p^{-1}(\mathbf{C} \setminus \{0, 1\})$, where $U_1(0)$ is the open unit disc.

The space $A_1$ is homotopically equivalent to the torus $\{(z_1, z_2) | |z_1| = \frac{1}{3}, |z_2| = \frac{1}{3}\}$. The loops in $A_1$ defined by the formulas $t \mapsto \frac{1}{3}(e^{2\pi i}, 1)$ and $t \mapsto \frac{1}{3}(1, e^{2\pi i})$ act non-trivially on the fiber of $\mathcal{L}$, which implies that the cohomology groups $H^*(A_1, \mathcal{L})$ are zero.

The restriction $p|A_2$ is a fibration. The restriction of $\mathcal{L}$ to the fiber $p^{-1}(\frac{1}{2})$ in nontrivial, hence $H^i(p^{-1}(\frac{1}{2}), \mathcal{L})$ is isomorphic to $\mathbf{R}$ if $i = 1$ and is zero otherwise. Define the loops $\alpha$ and $\beta$ in the space $\mathbf{C} \setminus \{0, 1\}$ as follows: $\alpha : t \mapsto 1 - \frac{1}{2}e^{2\pi i t}, \beta : t \mapsto \frac{1}{2}e^{2\pi i t}$. It is easy to check that both of them induce the identical mapping of $p^{-1}(\frac{1}{2})$ (hence the space $A_2$ is in fact homeomorphic to the direct product $\mathbf{C} \setminus \{0, 1\} \times p^{-1}(\frac{1}{2})$). Note, moreover, that a lifting of $\alpha$ into $A_2$ changes the sign of $\mathcal{L}$, while a lifting of $\beta$ does not. Now it is clear that the Poincaré polynomial $P(A_2, \mathcal{L})$ is equal to $t^2$ and that the inclusion $A_1 \cap A_2 = p^{-1}(U_1(0) \setminus \{0\}) \subset A_2$ induces an isomorphism of 2-dimensional cohomology groups with coefficients in $\mathcal{L}$.

Now consider the cohomological Mayer-Vietoris sequence corresponding to $L = A_1 \cup A_2$. Its only nontrivial terms will be

$$H^1(A_1 \cap A_2, \mathcal{L}) \to H^2(L, \mathcal{L}) \to H^2(A_1, \mathcal{L}) \oplus H^2(A_2, \mathcal{L}) \to H^2(A_1 \cap A_2, \mathcal{L})$$

The map on the right is an isomorphism, hence so is the map on the left. So we have $P(L, \mathcal{L}) = t^2$.

The map $f$ preserves each fiber of $p$. Moreover, using the Künneth formula and Proposition 3.13, we obtain that $\tilde{f}$ acts on the groups $H^*(A_1 \cap A_2, \mathcal{L})$ as multiplication by $-1$. Since the boundary operator commutes with $\tilde{f}$, we obtain the statement of the lemma concerning the group $H^2(L, \mathcal{L})$. The statement about the Borel-Moore homology group follows from the Poincaré duality and the fact that $f$ preserves the orientation. □

Lemma 4.10 follows immediately from Lemma 4.12. □

To complete the proof of Lemma 4.9 we must calculate the action of $\pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ on the Borel-Moore homology groups of $Y$ obtained from the spectral sequence (4.2). This will be done in the following three lemmas.

LEMMA 4.13. — *A loop $\gamma(t)$ in $\tilde{B}(\mathbf{CP}^2, 4)$ that belongs to the image of $\pi_1(\tilde{F}(\mathbf{CP}^2, 4))$ under the evident map $\tilde{F}(\mathbf{CP}^2, 4) \to \tilde{B}(\mathbf{CP}^2, 4)$ induces the identical map of the fiber $Y$ and of the coefficient system $\pm\mathbf{R}|Y$ over it.*

Note that $\pi_1(\tilde{F}(\mathbf{CP}^2, 4)) \cong \mathbf{Z}_3$ (because $\tilde{F}(\mathbf{CP}^2, 4)$ is diffeomorphic to $PGL(\mathbf{CP}^2)$, which is the quotient of $SL_3(\mathbf{C})$ by its center).

*Proof of Lemma 4.13.* — We can represent every $\gamma \in \pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ as follows: $\gamma(t) = \{A(t), B(t), C(t), D(t)\}$, where $A(t), \dots, D(t)$ are some paths in $\mathbf{CP}^2$ such that for any $t$ the points $A(t), B(t), C(t)$, and $D(t)$ are in general position. If we have a $\gamma$ that comes from $\pi_1(\tilde{F}(\mathbf{CP}^2, 4))$, we have $A(0) = A(1), \dots, D(0) = D(1)$. Denote by $Y_t$ the fiber of the bundle $X_{38} \to \tilde{B}(\mathbf{CP}^2, 4)$ over $\gamma(t)$. Note that for any $t$ there exists a unique projective linear map $M(t)$ that carries $A(0)$ to $A(t)$, $B(0)$ to $B(t)$, $C(0)$ to $C(t)$ and $D(0)$ to $D(t)$. This map induces maps $f_t : Y_0 \to Y_t$. The map $f_1$ is clearly identical. Moreover, if we have a configuration $K \in Y_0$, then the curve in $X$ starting at $K$ and covering $\gamma$ is $t \mapsto M(t)K$. Since $M(1) = \mathrm{Id}_{\mathbf{CP}^2}$, $\gamma$ does not transpose any pair of points from $K$. The lemma is proved. $\qquad\square$

LEMMA 4.14. —

1. *A loop $\gamma \in \pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ transposing the points $A$ and $B$ induces a bundle map $f_1 : Y \to Y$. This map is covered by a map $\tilde{f}_1 : \pm\mathbf{R}|Y \to \pm\mathbf{R}|Y$.*

2. *The corresponding map $\overline{f}_1$ of $L$ into itself is obtained from the projective linear map of $\mathbf{CP}^2$ that transposes the points $A$ and $B$ and fixes $C$ and $D$.*

3. *Let $l$ be a line in $L$ that is preserved under $\overline{f}_1$. The restriction of $f_1$ to the fiber over $l$ is the map $B(\mathbf{C}^*, 2) \to B(\mathbf{C}^*, 2)$ induced by $z \mapsto 1/z$. The restriction of $\tilde{f}_1$ to this fiber is minus identity over the pair $\{i, -i\}$.*

4. *The map $\tilde{f}_1|B(Z, 2)$ acts on the group $\bar{H}_3(B(Z, 2), \pm\mathbf{R}|B(Z, 2))$ as multiplication by $-1$ and acts on the group $\bar{H}_2(B(Z, 2), \pm\mathbf{R}|B(Z, 2))$ as the identity; here $B(Z, 2)$ is the fiber of the bundle $Y \to L$ over some line in $L$ that is preserved under $\overline{f}_1$.*

*Proof.* — Proceeding as in the proof of Lemma 4.13 we obtain bundle maps $f_t : Y_0 \to Y_t$. The map $f_1 : Y_0 \to Y_1 = Y_0$ is induced by the projective linear map that fixes $C$ and $D$ and transposes $A$ and $B$. Note that the map

$\tilde{f}_1 : \pm\mathbf{R}|x \to \pm\mathbf{R}|x$, where $x \in Y, f_1(x) = x$ is just the map induced by the loop $t \mapsto f_t(x)$. Now let $A, B, C, D$ be the following points of the affine plane $\mathbf{C}^2 \subset \mathbf{CP}^2$: $A = (1,0), B = (-1,0), C = (0,1), D = (0,-1)$. Set $l = \infty$. Then the map $f_1$ is induced by the linear map with matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This map preserves the line $l = \infty$ and transposes the conics tangential to $l$. Identify $Z$ with $\mathbf{C}^*$, and choose a coordinate $z \in \mathbf{C}^*$ such that the induced map can be written as $z \to 1/z$. Note that the conics in the pair corresponding to $\{i, -i\}$ are transposed, and the pair itself is preserved. Thus, the loop $\gamma$ transposes 3 pairs of points over this pair, and hence the fiber of the coefficient system $\pm\mathbf{R}|B(Z, 2)$ is multiplied by $-1$.

We have proved the first three statements of Lemma 4.14. The fourth statement follows immediately from Proposition 3.14.   □

Now we can easily obtain the map of the sequence (4.2) induced by $\tilde{f}_1$. The third line of the sequence (4.2) contains the groups $\bar{H}_i(L, \bar{\mathcal{H}}_3)$, where $\bar{\mathcal{H}}_3$ is the constant local system on $L$ with the fiber $\bar{H}_3(B(Z, 2), \pm\mathbf{R}|B(Z, 2))$. Since the map $\tilde{f}_1$ multiplies the fiber of $\bar{\mathcal{H}}_3$ by $-1$, it multiplies $\bar{H}_4(L, \bar{\mathcal{H}}_3) = E^1_{4,3}$ by $-1$. The second line of the sequence (4.2) contains the groups $\bar{H}_i(L, \bar{\mathcal{H}}_2)$, where $\bar{\mathcal{H}}_2$ is the local nonconstant system on $L$ considered in Lemma 4.12. Due to Lemma 4.14, $\tilde{f}_1$ acts identically on the fiber of the system $\bar{\mathcal{H}}_2$ over some point of $L$. We obtain from Lemma 4.12 that $\gamma$ multiplies $\bar{H}_2(L, \bar{\mathcal{H}}_2) = E^1_{2,2}$ by $-1$.

It is easy to see that the action of $\pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ on $E^1_{2,3}$ and on $E^1_{3,3}$ of the sequence (4.2) is nontrivial and irreducible.

Hence the action of $\pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ on $\bar{H}_*(Y, \pm\mathbf{R}|Y)$ is nontrivial and irreducible in any dimension. Recall that the universal covering space of $\tilde{B}(\mathbf{CP}^2, 4)$ is $SL_3(\mathbf{C})$, and the group $\pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ contains a normal subgroup isomorphic to $\mathbf{Z}_3 = \pi_1(\tilde{F}(\mathbf{CP}^2, 4))$, the quotient being isomorphic to $S_4$. Lemma 4.9 follows immediately from Lemma 3.7. In fact, setting $G = SL_3(\mathbf{C}), G_1 =$(the subgroup of $SL_3(\mathbf{C})$ generated by $e^{\frac{2}{3}\pi i}I$ ($I$ is the identity matrix) and the (complexification of the) motions of a regular tetrahedron) in Lemma 3.7 we obtain that the group $H^*(\tilde{B}(\mathbf{CP}^2, 4), \mathcal{L}) = 0$ if the action of $\pi_1(\tilde{B}(\mathbf{CP}^2, 4))$ on the fiber of $\mathcal{L}$ is nontrivial and irreducible. By Poincaré duality $\bar{H}_*(\tilde{B}(\mathbf{CP}^2, 4), \mathcal{L})$ is also zero for any such $\mathcal{L}$.

## 4.3. Column 39

Recall that we denote by $X_{39}$ the space of configurations of type 39. We have

$$E^1_{39,i} = \bar{H}_{39+i-2-8}(X_{39}, \pm\mathbf{R}).$$

$X_{39}$ is fibered over the space $\tilde{B}(\mathbf{CP}^2, 3)$ of all generic triples of points $\{A, B, C\}$, $A, B, C \in \mathbf{CP}^2$.

LEMMA 4.15. — *The term $E^2$ of the spectral sequence of the bundle $X_{39} \to \tilde{B}(\mathbf{CP}^2, 3)$ looks as follows:*

$$
\begin{array}{c|ccccccc}
7 & \mathbf{R} & & & & & & \\
6 & & & \mathbf{R} & & \mathbf{R} & & \\
5 & & & & & & & \mathbf{R} \\
\hline
 & 6 & 7 & 8 & 9 & 10 & 11 & 12
\end{array}
\tag{4.4}
$$

*and the differentials $E^2_{8,6} \to E^2_{6,7}$ and $E^2_{12,5} \to E^2_{10,6}$ are non zero.*

The proof will take the rest of the subsection.

If we fix three lines $AB, BC, AC$, then the intersection points of $AB$ and $BC$ with the conic can be chosen arbitrarily. The space of conics passing through these 4 points and not tangential to $AC$ is homeomorphic to ($S^2$ minus three points): we have to exclude 2 tangential conics and the conic consisting of the lines $AB$ and $BC$.

Thus, the fiber $Y$ of the bundle $X_{39} \to \tilde{B}(\mathbf{CP}^2, 3)$ is itself a fiber bundle over $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ with fiber ($S^2$ minus 3 points). Denote the latter fiber by $Z$. The space $Z$ can be identified with $\mathbf{C}^* \setminus \{1\}$.

LEMMA 4.16. — *The term $E^2$ of the spectral sequence for the Borel-Moore homology of the bundle $Y \to B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ looks as follows:*

$$
\begin{array}{c|ccc}
1 & \mathbf{R} & \mathbf{R}^2 & \mathbf{R} \\
\hline
 & 4 & 5 & 6
\end{array}
\tag{4.5}
$$

*Proof.* — If we fix all points in a configuration from $X_{39}$ except the points of intersection of the conic and the line $AC$, then we can transpose these two points, hence the restriction of $\pm\mathbf{R}$ to ($S^2$ minus 3 points) is nontrivial. We have $\bar{H}_i(Z, \pm\mathbf{R}) = \mathbf{R}$ if $i = 1$, and 0 otherwise.

That is why the only nontrivial line in the spectral sequence of the bundle $Y \to B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ is the first one; it contains the groups $\bar{H}_*(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2), \bar{\mathcal{H}}_1)$, where $\bar{\mathcal{H}}_1$ is the system with the fiber $\bar{H}_1(Z, \pm\mathbf{R}|Z)$ corresponding to the action of $\pi_1(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2))$. The fiber of $\bar{\mathcal{H}}_1$ is $\mathbf{R}$, and, as we shall see, every element of $\pi_1(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2))$ multiplies the fiber of $\bar{\mathcal{H}}_1$ by $\pm 1$. So we can apply the Künneth formula, and we get

$$\bar{H}_*(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2), \bar{\mathcal{H}}_1) = \bar{H}_*(B(\mathbf{C}^*, 2), \mathcal{B}_1) \otimes \bar{H}_*(B(\mathbf{C}^*, 2), \mathcal{B}_2), \quad (4.6)$$

where $\mathcal{B}_1, \mathcal{B}_2$ are the restrictions of $\bar{\mathcal{H}}_1$ on the first and the second factors of $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$. To calculate $\mathcal{B}_1$ we fix an element in the second factor of the product $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ and study the action of the loops in the first factor on the group $\bar{H}_1(Z, \pm\mathbf{R}|Z)$.

We set $A = (0 : 1 : 0), B = (0 : 0 : 1), C = (1 : 0 : 0)$. Set $AC = \infty$, so that the points of the type $(z : w : 1)$ belong to the affine plane $\mathbf{C}^2 \subset \mathbf{CP}^2$, and the spaces $B(\mathbf{C}^*, 2)$ consist of pairs of nonzero points on the coordinate axes. Now fix the pair of points $\{(0, i), (0, -i)\}$ on the $y$-axis. Denote by $Q$ the space of conics passing through $(i, 0), (-i, 0), (0, i)$, and $(0, -i)$. Note that the fiber $Z$ over this quadruple is the subspace of $Q$ that consists of the conics that are not tangential to $AC = \infty$ and are not equal to the union $AB \cup AC$. Note also that the conics from $Q$ have the form

$$ax^2 + bxy + ay^2 + a.$$

Set $z = (2a - b)/2a + b$. This identifies the space $Z \subset Q$ with $\mathbf{C} \setminus \{0, -1\}$.

In the following two lemmas we identify the space $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ with the space of configurations in $\mathbf{C}^2$ that consist of two nonzero points on the $x$-axis and two nonzero points on the $y$-axis.

LEMMA 4.17. — *Consider the following loop in the space $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$: $\gamma(t) = \{(\alpha(t), 0), (1/\alpha(t), 0), (0, i), (0, -i)\}$, where $\alpha(t)$ is a simple curve in $\mathbf{C} \setminus \{0\}$ such that $\alpha(0) = i, \alpha(1) = -i$. Then $\gamma$ induces the identical map of $Z$, and for any $q \in Z, \gamma$ fixes both points of $\gamma \cap \infty$.*

LEMMA 4.18. — *Consider the following loop in the space $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$: $\gamma(t) = \{(ie^{\pi it}, 0), (-ie^{\pi it}, 0), (0, i), (0, -i)\}$. The map $Z \to Z$ induced by $\gamma$ can be written as $z \mapsto 1/z$. This map preserves the conics $q_1 = xy$ and $q_2 = x^2 + y^2 + 1$. Both points of $q_1 \cap \infty$ are fixed, and the points of $q_2 \cap \infty$ are transposed.*

The proof of these lemmas is an exercise in analytic geometry.     □

Now we shall use Lemmas 4.17 and 4.18 to calculate the action of $\pi_1(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2))$ on the Borel-Moore homology of the fiber of the bundle $Y \to B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$.

Let us note that the loop $\gamma$ considered in Lemma 4.17 can be viewed as the loop in the fiber of the bundle $B(\mathbf{C}^*, 2)$ over 1 (recall that on page 410 we defined a structure of a fiber bundle over $\mathbf{C}^*$ on the space $B(\mathbf{C}^*, 2)$, the projection being the multiplication of complex numbers). We obtain from Lemma 4.17 that such $\gamma$ induces the identical map of the space of $Z$, and for any $q \in Z, \gamma$ fixes both points of $q \cap \infty$. So $\gamma$ transposes two points in a configuration from $Y$. Thus, the system $\pm\mathbf{R}|Z$ is multiplied by $-1$, and $\gamma$ acts on the group $\bar{H}_1(Z, \pm\mathbf{R}|Z)$ as multiplication by $-1$.

Now let $\gamma$ be the loop from Lemma 4.18. Note that it can be identified with the loop $a$ from Proposition 3.14. Applying Lemma 4.18, we obtain that this loop transposes the tangential conics. Identify the space $Z$ of non-tangential conics with $\mathbf{C}^*$ taking the coordinate $z$ as in Lemma 4.18. The map $g : Z \to Z$ induced by $\gamma$ is $z \mapsto 1/z$. Due to Lemma 4.18, the loop $\gamma$ transposes the points of intersection of $q_2$ and $AC$. This implies that the map $\tilde{g} : \pm\mathbf{R}|Z \to \pm\mathbf{R}|Z$ induced by $\gamma$ is identity over $1 = z(q_2)$ (it transposes two pairs of points).

Now introduce another coordinate $w$ on $Z$, $w = (z - 1)/(z + 1)$. This identifies $Z$ with $\mathbf{C} \setminus \{1, -1\}$. Since we have $w(1/z) = -w(z)$, the map $g : Z \to Z$ can be written as $w \mapsto -w$. The map $\tilde{g}$ is identity over $0 = w(1)$. Applying Proposition 3.13, we obtain immediately that the map $\tilde{g}_* : \bar{H}_1(Z, \pm\mathbf{R}|Z) \to \bar{H}_1(Z, \pm\mathbf{R}|Z)$ is minus identity.

So we see that the restriction of the local system $\bar{\mathcal{H}}_1$ to the first factor of $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ is in fact the system $\mathcal{A}_3$ of Proposition 3.14 (it changes its sign both under the action of the loops of the fiber of $B(\mathbf{C}^*, 2) \to \mathbf{C}^*$ and under the "middle line"). Due to Proposition 3.13, we have $\bar{P}(B(\mathbf{C}^*, 2), \mathcal{A}_3) = t^2(t+1)$. Lemma 4.16 follows now from formula (4.6).

Now we shall study the action of $\pi_1(\tilde{B}(\mathbf{CP}^2, 3)$ on the group $\bar{H}_*(Y, \pm\mathbf{R}|Y)$. The fundamental group of $\tilde{B}(\mathbf{CP}^2, 3)$ equals $S_3$ (since $\tilde{F}(\mathbf{CP}^2, 3)$ is simply-connected). We shall describe the map of the sequence (4.5) induced by the transposition of the points $A$ and $C$.

LEMMA 4.19. —

1. Let $\gamma$ be a loop in $\tilde{B}(\mathbf{CP}^2, 3)$ that transposes the points $A$ and $C$. Above we represented $Y$ as a fiber bundle over $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$, the projection being defined as follows: $Q \mapsto (Q \cap AB, Q \cap BC)$, where $Q$ is the conic that corresponds to an element of $Y^4$. The map induced by $\gamma$ preserves this structure of a fiber bundle on $Y$.

2. The corresponding map $h : B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2) \to B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ is the transposition of factors. (Recall that the first (respectively, the second) factor in this product is identified with the space of pairs of nonzero points on the $x$- (respectively, the $y$-)axis.)

3. Identify the space $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ with the space of configurations in $\mathbf{C}^2$ that consist of two nonzero points on the $x$-axis and two nonzero points on the $y$-axis. Let $Z$ be the fiber of $Y \to B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ over the point $\{(i, 0), (-i, 0), (0, i), (0, -i)\}$ (this point is clearly fixed under $h$). The map $Z \to Z$ induced by $\gamma$ is identical, and the points of intersection of each conic $q \in Z$ with the line $AC$ are transposed by $\gamma$. Hence a loop corresponding to the movement of any $q \in Z$ from this fiber transposes four pairs of points and acts identically on the coefficient system $\pm \mathbf{R}$ over this fiber.

Proof. — Since $\pi_1(\tilde{B}(\mathbf{CP}^2, 3)) = S_3$, any two loops that transpose $A$ and $C$ define the same map $\bar{H}_*(Y, \pm\mathbf{R}) \to \bar{H}_*(Y, \pm\mathbf{R})$. Recall that $A = (1 : 0 : 0), B = (0 : 0 : 1), C = (0 : 1 : 0) \in \mathbf{CP}^2$. Set $A(t) = (\frac{1}{2}(1 + e^{i\pi t}) : \frac{1}{2}(1 - e^{i\pi t}) : 0), C(t) = (\frac{1}{2}(1 - e^{i\pi t}) : \frac{1}{2}(1 + e^{i\pi t}) : 0)$, $B(t) = B$. Set $\gamma(t) = \{A(t), B(t), C(t)\}$. Denote by $Y_t$ the fiber of the bundle $X_{39} \to \tilde{B}(\mathbf{CP}^2, 4)$ over $\gamma(t)$. There exists a projective linear map that carries $A$ into $A(t)$, $C$ into $C(t)$ and fixes $B$. It can be chosen so that its restriction to the affine plane $\mathbf{C}^2 = \mathbf{CP}^2 \setminus AC$ will be the linear map with matrix

$$\begin{pmatrix} \frac{1}{2}(1 + e^{i\pi t}) & \frac{1}{2}(1 - e^{i\pi t}) \\ \frac{1}{2}(1 - e^{i\pi t}) & \frac{1}{2}(1 + e^{i\pi t}) \end{pmatrix}$$

(in appropriate coordinates).

This map induces a map $f_t : Y_0 \to Y_t$. In particular, the map $f_1$ is induced by the transposition of the axes in $\mathbf{C}^2$. The first and the second statements of the lemma follow immediately. To prove the third statement

---

[4] Note that there are three ways to represent $Y$ as a fiber bundle over $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$; instead of the lines $AB$ and $BC$ we could have taken any other two of the lines $AB, BC$ and $AC$.

note that the fiber $Z$ over the point $\{(i,0), (-i,0), (0,i), (0,-i)\}$ consists of conics of the type $ax^2 + bxy + ay^2 + a$. Such conics are preserved, if we change $x$ and $y$, and their points of intersection with $AC = \infty$ are clearly transposed. □

The map $f_1$ induces the identical map of the fiber $Z$ over some "diagonal" point of $B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2)$ and acts identically on the restriction of the coefficient system $\pm\mathbf{R}$ to that fiber.

In general, suppose we have a connected space $A$ and a local system $\mathcal{L}$ on $A \times A$ with fiber $\mathbf{R}$, and suppose that every element of $\pi_1(A \times A)$ multiplies the fiber of $\mathcal{L}$ by 1 or $-1$. Let $\mathcal{L}_1, \mathcal{L}_2$ be the restrictions of $\mathcal{L}$ to the first and the second factor, and let $f : A \times A \to A \times A$ be the transposition of factors. Suppose that $\tilde{f} : \mathcal{L} \to \mathcal{L}$ is the map that covers $f$ and is identical over some point of the type $(x,x), x \in A$. Then, the map of $\bar{H}_*(A \times A, \mathcal{L}) = \bar{H}_*(A, \mathcal{L}_1) \otimes \bar{H}_*(A, \mathcal{L}_2)$ into itself can be written as $a \otimes b \mapsto (-1)^{deg(a)deg(b)} b \otimes a$.

Applying this to our situation, we obtain that $\gamma$ acts identically on the group $\bar{H}_4(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2), \bar{\mathcal{H}}_1) = E^2_{4,1}$, and multiplies the group $\bar{H}_6(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2), \bar{\mathcal{H}}_1) = E^2_{6,1}$ by $-1$.

We have $\bar{P}(\tilde{B}(\mathbf{CP}^2, 3), \mathbf{R}) = t^{12}$, $\bar{P}(\tilde{B}(\mathbf{CP}^2, 3), \pm\mathbf{R}) = t^6$. This gives us the 5-th and the 7-th lines of (4.4).

Let us calculate $\bar{P}(\tilde{B}(\mathbf{CP}^2, 3), \mathcal{S})$, where $\mathcal{S}$ is the local system corresponding to the 2-dimensional irreducible representation of $S_3$. It is easy to show that $P(\tilde{F}(\mathbf{CP}^2, 3)) = (1 + t + t^2)(1 + t)$. Applying Corollary 3.5 (see page 405) to the covering $\tilde{F}(\mathbf{CP}^2, 3) \to \tilde{B}(\mathbf{CP}^2, 3)$, we obtain

$$P(\tilde{F}(\mathbf{CP}^2, 3)) = P(\tilde{B}(\mathbf{CP}^2, 3), \mathbf{R}) + P(\tilde{B}(\mathbf{CP}^2, 3), \pm\mathbf{R}) + 2P(\tilde{B}(\mathbf{CP}^2, 3), \mathcal{S}),$$

since the regular representation of $S_3$ contains one trivial, one alternating representation and two copies of the 2-dimensional irreducible representation. Thus, $P(\tilde{B}(\mathbf{CP}^2, 3), \mathcal{S}) = t^2(1 + t^2)$, and by the Poincaré duality we obtain $\bar{P}(\tilde{B}(\mathbf{CP}^2, 3), \mathcal{S}) = t^8(1 + t^2)$.

It remains to prove that the 6-th line of the Leray sequence corresponding to the fibration $X_{39} \to \tilde{B}(\mathbf{CP}^2, 3)$ is as in (4.4) (i.e., the action of $S_3$ in $\bar{H}_5(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2), \bar{\mathcal{H}}_1) = \mathbf{R}^2$ is irreducible), and that the differentials $E^2_{8,6} \to E^2_{6,7}$ and $E^2_{12,5} \to E^2_{10,6}$ are non zero.

To this end note that the group $SU_3$ acts almost freely on $X_{39}$ (via $SU_3 \to SU_3/\langle e^{\frac{2}{3}\pi i} I\rangle \hookrightarrow PGL(\mathbf{CP}^2)$, where $I$ is the identity matrix). Apply Theorem 3.8 setting $M = X_{39}, G = SU_3, \mathcal{L} = \pm\mathbf{R}|X_{39}$. From the Leray

sequence of the map $M \to M/G$ (and from the fact that the cohomological dimension of $M/G$ is clearly finite) we obtain that either $H^*(M, \mathcal{L}) = 0$ or $d_{max} - d_{min} \geqslant 8$ (here $d_{max}$, (respectively, $d_{min}$) is the greatest (respectively the smallest) $i$ such that $H^i(M, \mathcal{L}) \neq 0$).

By the Poincaré duality, we have either $\bar{H}_*(M, \mathcal{L}) = 0$ or $d'_{max} - d'_{min} \geqslant 8$ (here $d'_{max}$ (respectively, $d'_{min}$) is the greatest (respectively, the smallest) $i$ such that $\bar{H}_i(M, \mathcal{L}) \neq 0$. Obviously, neither one of these statements holds if the action of $S_3$ in $\bar{H}_5(B(\mathbf{C}^*, 2) \times B(\mathbf{C}^*, 2), \bar{\mathcal{H}}_1) = \mathbf{R}^2$ is reducible or any of the differentials $E^2_{8,6} \to E^2_{6,7}, E^2_{12,5} \to E^2_{10,6}$ is trivial. Lemma 4.15 is proved. $\quad\square$

## 4.4. Nondiscrete singular sets

We are going to show that the columns of the spectral sequence 1.1 corresponding to all nondiscrete singular sets are zero. The columns 11 and 33 are considered in exactly the same way as in [2].

PROPOSITION 4.20. — *Let $l$ be a line in $\mathbf{CP}^2$, $A_1, \ldots, A_k, k \geqslant 0$ be points not on $l$, $m$ be an integer $> 1$. Denote the union of all simplices in $(\mathbf{CP}^2)^{*(m+k)}$ with the vertices in $A_1, \ldots, A_k$ and $m$ vertices on $l$ by $\Lambda(l, m, A_1, \ldots, A_k)$. The space $\Lambda(l, m, A_1, \ldots, A_k)$ has zero real homology groups modulo a point. If $k > 0$, this space is contractible.*

*Proof of Proposition 4.20.* — If $k = 0$, the statement of the proposition follows from Lemma 3.12. If $k > 0$, the space $\Lambda(l, m, A_1, \ldots, A_k)$ is contractible, since this space is a union of simplices that all contain the vertex $A_1$. $\quad\square$

Using this proposition, we can easily prove that $\partial\Lambda(K)$ has zero real homology groups modulo a point if $K \in X_i, i = 17, 22, 29, 41$. Let us consider, for instance, the case $i = 41$. Consider a configuration $K$ consisting of a line $l$ and 3 points $A, B, C$ outside $l$ such that the points $A, B, C$ do not belong to any line.

Note that $\partial\Lambda(K) = L_1 \cup L_2 \cup L_3 \cup L_4$, where $L_1 = \Lambda(l + A + B)$, $L_2 = \Lambda(l + B + C), L_3 = \Lambda(l + A + C), L_4 = \bigcup_\kappa \Lambda(\kappa)$, where $\kappa$ runs through the set of all configurations of type "$A, B, C$+ 4 points on $l$". Using Lemma 2.11 (see page 403), we conclude that $L_4$ is homeomorphic to the space $\Lambda(l, 4, A, B, C)$, which is contractible due to Proposition 4.20.

The intersections $L_1 \cap L_2, L_2 \cap L_3, L_1 \cap L_3$ are all spaces of the form $\Lambda(l + \text{ a point not on } l)$. The intersection $L_1 \cap L_2 \cap L_3$ is just $\Lambda(l)$. All these spaces are contractible. Now, the intersections $L_i \cap L_4, i = 1, 2, 3$ are the unions of all $\Lambda(\kappa)$, $\kappa$ running through the set of all configurations of type "the points $(A, B)$ (resp., $(B, C), (A, C)$) outside $l + 4$ points on $l$". These spaces are homeomorphic to the space $\Lambda(l, 4, 2 \text{ points outside } l)$ and are contractible due to Proposition 4.20. Analogously, the intersections $L_1 \cap L_2 \cap L_4, L_2 \cap L_3 \cap L_4, L_1 \cap L_3 \cap L_4$ are all homeomorphic to spaces of type $\Lambda(l, 4, \text{a point outside } l)$ and are also contractible. Finally, the quadruple intersection $L_1 \cap L_2 \cap L_3 \cap L_4$ is the union of all $\Lambda(\kappa)$, for all $\kappa = $"4 points in $l$", which is homeomorphic to $l^{*4} = \Lambda(l, 4, \varnothing)$.

So we see that the spaces $L_i, i = 1, \ldots, 4$ have zero real homology groups modulo a point, and so do all their intersections. This implies that real homology groups of their union $\partial \Lambda(K)$ modulo a point are also zero.

Let us now consider the case $K = l_1 \cup l_2$, where $l_1, l_2$ are two lines (column 31). Here $\partial \Lambda(K)$ is the union $L_1 \cup L_2 \cup L_3$, where $L_i$ for $i = 1, 2$ is the union of the spaces $\Lambda(\kappa)$, where $\kappa$ runs through the set of configurations of the type "$l_i + 3$ points on the other line", and $L_3$ is the union of $\Lambda(K')$, for $K'$ running through the set $\{K' \subset l_1 \cup l_2 | \#(K') = 8, \#(K' \cap l_i) \geqslant 4, i = 1, 2\}$.

First note that the intersection $L_1 \cap L_2$ is the union of the spaces $\Lambda(\kappa)$, where $\kappa$ runs through the space of configurations of the type "3 points on $l_1 \setminus l_2$, 3 points on $l_2 \setminus l_1$, the point of intersection". It follows from Lemma 2.11 that $L_1 \cap L_2$ is contractible.

The space $L_1$ admits the following filtration $\varnothing \subset \Lambda(l_1) \subset M_1 \subset M_2 \subset M_3 \subset M_4 \subset M_5 \subset M_6 = L_1$. Here $M_i, i = 1, \ldots, 5$, is the union of all $\Lambda(\kappa)$, where $\kappa \subset K$ is a configuration of type 16, 17, 21, 22, 28 respectively.

The space $M_1 \setminus \Lambda(l_1)$ is fibered over $l_2 \setminus l_1$, the fiber over a point $A$ being homeomorphic to $\Lambda(l_1, 7, A) \setminus l_1^{*7}$. This fiber has trivial real Borel-Moore homology. The space $M_2 \setminus M_1$ is fibered over $l_2 \setminus l_1$, the fiber over a point $A$ being homeomorphic to $\Lambda(l_1 + A) \setminus \partial \Lambda(l_1 + A)$, whose Borel-Moore homology is also zero.

The space $M_3 \setminus M_2$ is fibered over the space $B(l_2 \setminus l_1, 2)$, the fiber over a pair $\{A, B\}$ being homeomorphic to $\Lambda(l_1, 6, A, B) \setminus (\Lambda(l_1, 6, A) \cup \Lambda(l_1, 6, B))$. This space also has zero Borel-Moore homology.

The spaces $M_4 \setminus M_3$ and $M_6 \setminus M_5$ are considered in the same way as $M_2 \setminus M_1$. The space $M_5 \setminus M_4$ is fibered over $B(l_2 \setminus l_1, 3)$. The fiber over $\{A, B, C\}$ is homeomorphic to the space $\Lambda(l_1, 5, A, B, C) \setminus (\Lambda(l_1, 5, A, B) \cup$

$\Lambda(l_1, 5, B, C) \cup \Lambda(l_1, 5, A, C))$, whose Borel-Moore homology groups are zero. This implies that $L_1$ (and also $L_2$) have zero real homology groups modulo a point.

Now consider the space $L_3$. Let $L_3'$ (respectively, $L_3''$) be the union of all $\Lambda(K')$ for $K'$ running through the set $\{K' \subset l_1 \cup l_2 | \#(K') = 8, \#(K' \cap l_2) = 5, \#(K' \cap l_1) = 4\}$ (respectively, $\{K' \subset l_1 \cup l_2 | \#(K') = 8, \#(K' \cap l_1) = 5, \#(K' \cap l_2) = 4\}$). It is easy to see that all spaces $L_3', L_3'', L_3' \cap L_3'', L_3' \cup L_3''$ are contractible. The space $L_3 \setminus (L_3' \cup L_3'')$ is the union of $\Lambda(K') \setminus \partial\Lambda(K')$ for $K'$ running through the set of configurations that consist of 4 points on $l_2 \setminus l_1$ and 4 points on $l_1 \setminus l_2$. Using Lemma 2.11, we get $\bar{H}_*(L_3 \setminus (L_3' \cup L_3''), \mathbf{R}) = \bar{H}_{*-7}(B(\mathbf{C}, 4) \times B(\mathbf{C}, 4), \pm\mathbf{R} \times \pm\mathbf{R}) = 0$. Hence the real homology groups of $L_3$ modulo a point are zero.

We have also $L_1 \cap L_3 = L_3'', L_2 \cap L_3 = L_3', L_1 \cap L_2 \cap L_3 = L_3' \cap L_3''$. These spaces are all contractible. This completes the proof that real homology groups of $\partial\Lambda(K)$ modulo a point are zero, when $K$ is the union of two lines.

The fact that the last column of the sequence (1.1) is zero is proved exactly in the same way as in the case of plane cubics in $\mathbf{CP}^2$, see [2, Section 4].

## 4.5. End of the proof of Theorem 1.2

We have now proved the first two statements of Theorem 1.2. In order to complete the proof of the theorem, it remains to show that the differential $E_{2,35}^1 \to E_{1,35}^1$ of the spectral sequence (1.1) is zero. This can be done as follows (cf. [2, Lemma 6]).

Let $S$ be the image of $\Sigma_5 \setminus \{0\}$ under the evident map $\Pi_5 \setminus \{0\} \to \mathbf{CP}^{20}$, and let $c_1 \in H^2(\mathbf{CP}^{20}, \mathbf{R})$ be the first Chern class of the tautological bundle over $\mathbf{CP}^{20}$. Since the fundamental class of any irreducible algebraic hypersurface is dual to a nonzero multiple of $c_1$, the restriction of $c_1$ to $\mathbf{CP}^{20} \setminus S$ is zero, which implies that $H^*(\Pi_5 \setminus \Sigma_5, \mathbf{R}) = H^*(\mathbf{C}^*, \mathbf{R}) \otimes H^*(\mathbf{CP}^{20} \setminus S, \mathbf{R})$. Thus, the Poincaré polynomial of the space $\Pi_5 \setminus \Sigma_5$ is divisible by $1 + t$, which implies easily that the differential $E_{2,35}^1 \to E_{1,35}^1$ is zero. Theorem 1.2 is proved. $\square$

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