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Higher order Poincaré-Pontryagin functions and iterated path integrals[(*)]

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ABSTRACT. — We prove that the Higher order Pointcaré-Pontryagin functions associated to the perturbed polynomial foliation

$$df - \varepsilon(P\,dx + Q\,dy) = 0$$

satisfy a differential eqation of Fuchs type.

RÉSUMÉ. — Nous montrons que toute fonction de Poincaré-Pontryagin d'ordre supérieur, associée au feuilletage polynomial perturbé défini par

$$df \;-\; \varepsilon\;(Pdx + Qdy) \;=\; 0,$$

vérifie une équation fuchsienne.

1. Statement of the result

Let $f, P, Q \in \mathbb{R}[x, y]$ be real polynomials in two variables. How many limit cycles the perturbed foliation

$$df - \varepsilon(Pdx + Qdy) = 0 \tag{1.1}$$

can have ? This problem is usually referred to as the weakened 16th Hilbert problem (see Hilbert [15], Arnold [1, p.313]).

Suppose that the foliation defined on the real plane by $\{df = 0\}$ possesses a family of periodic orbits $\gamma(t) \subset f^{-1}(t)$, continuously depending on a parameter $t \in (a, b) \subset \mathbb{R}$. Take a segment σ, transversal to each orbit $\gamma(t)$ and suppose that it can be parameterized by $t = f|_\sigma$ (this identifies

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σ to (a,b)). The first return map $\mathcal{P}(t,\varepsilon)$ associated to the period annulus $\mathcal{A} = \cup_{t\in\sigma}\gamma(t)$ and to (1.1) is analytic in t,ε and can be expressed as

$$\mathcal{P}(t,\varepsilon) = t + \varepsilon^k M_k(t) + \varepsilon^{k+1} M_{k+1}(t) + \cdots \tag{1.2}$$

where $M_k(t) \not\equiv 0$ is the kth order Poincaré-Pontryagin function. The maximal number of the zeros of M_k on σ provides an upper bound for the number of the limit cycles bifurcating from the annulus $\mathcal{A}$. For this reason $M_k(t)$ was called in [11] *generating function of limit cycles.* The above construction can be carried out in the complex domain. In this case the polynomials f, P, Q are complex and $\gamma(t)$ is a continuous family of closed loops contained in the fibers $f^{-1}(t)$, parameterized by a transversal open disc $\sigma^{\mathbb{C}}$. The maximal number of the complex zeros of the generating function M_k on $\sigma^{\mathbb{C}}$ provides an upper bound for the complex limit cycles bifurcating from the family $\{\gamma(t)\}_t$, see [16].

The main result of the paper is the following

THEOREM 1.1. — *The generating function of limit cycles M_k satisfies a linear differential equation of Fuchs type.*

We show also that the monodromy group of M_k is contained in $SL(n,\mathbb{Z})$ where n is the order of the equation. In this sense the differential equation satisfied by M_k is of "Picard-Fuchs" type too. As a by-product we prove that $n \leqslant r^k$ where $r = \dim H_1(f^{-1}(t_0),\mathbb{Z})$ and t_0 is a typical value of f. It is not clear, however, whether there exists an uniform bound in k for the order n. In the explicit examples known to the author $n \leqslant r$.

In the case $k = 1$ the generating function M_k is an Abelian integral depending on a parameter

$$M_1(t) = \int_{\gamma(t)} P dx + Q dy \tag{1.3}$$

and hence it satisfies a Fuchs equation of order at most r (this bound is exact). The identity (1.3) goes back at least to Pontryagin [21] and has been probably known to Poincaré. In the case $k > 1$ the (higher order) Poincaré-Pontryagin function M_k is not necessarily of the form (1.3) with P, Q rational functions. This fact is discussed in Appendix B. We show in section 2 that $M_k(t)$ is a linear combination of iterated path integrals of length k along $\gamma(t)$ whose entries are essentially rational one-forms. This observation is crucial for the proof of Theorem 1.1. It implies that the monodromy representation of M_k is finite-dimensional, as well that M_k is a function of moderate growth. The universal monodromy representation of *all* generating functions M_k was recently described in [11]. It is not known, however, whether this representation is finite-dimensional.

We note that iterated path integrals appeared recently in a similar context in the study of the polynomial Abel equation [3, 4, 8]. Some of their basic properties used in the paper are summarized in the Appendix.

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2. The integral representation of $M_k(t)$

From now on we consider (1.1) as a perturbed complex foliation in $\mathbb{C}^2$. Let $l(t) \in f^{-1}(t) \subset \mathbb{C}^2$ be a continuous family of closed loops, defined for all t which belong to some complex neighborhood of the typical value t_0 of f. There exists a constant $c > 0$ such that the holonomy (or monodromy) map $P(t, \varepsilon)$ of the foliation ((1.1) associated to the family $l(t)$ is well defined and analytic in $\{(t, \varepsilon) : |t - t_0| < c, |\varepsilon| < c\}$. Therefore it has there the representation (1.2). Of course the continuous deformation of a given closed loop $l(t_0) \subset f^{-1}(t_0)$ is not unique. The free homotopy class of the loop is however unique and the first non-zero Poincaré-Pontryagin function $M_k(t)$, defined by (1.2) depends only on the free homotopy class $\gamma(t)$ of $l(t)$ [11]. The main result of this section is the following

THEOREM 2.1. — *Let $\gamma(t) : [0, 1] \to f^{-1}(t)$ be a continuous family of closed loops. For every regular value t_0 of f there exists a neighborhood U_0 of t_0 in which the first non-vanishing Poincaré-Pontryagin function $M_k(t)$, associated to $\gamma(t)$ and (1.1) is a finite linear combination of iterated integrals of length at most k, whose entries are differential one-forms analytic in $f^{-1}(U_0)$.*

The function $M_k(t)$ is computed according to the Françoise's recursion formula [7]

$$M_k(t) = \int_{\gamma(t)} \Omega_k$$

where

$$\Omega_1 = Pdx + Qdy, \Omega_m = r_{m-1}(Pdx + Qdy), 2 \leqslant m \leqslant k \tag{2.1}$$

and the functions r_i are determined successively from the (non-unique) representation $\Omega_i = dR_i + r_i df$. We intend to derive explicit expressions for the functions r_i. For this purpose consider a trivial smooth fibration

$$f : V_0 \to U_0 = \{t \in C : |t - t_0| < c\}$$

where V_0 is a connected analytic two-dimensional manifold, and f is an analytic surjection. The fibers $f^{-1}(t)$ are mutually diffeomorphic Riemann surfaces. Suppose that there exists an analytic curve

$$\tau : t \to P_0(t) \in f^{-1}(t) \subset V_0, t \in U_0 \tag{2.2}$$

transversal to the fibers $f^{-1}(t)$. For an analytic one-form in V_0 define the function

$$F(P) = \int_{P_0(t)}^{P} \omega$$

where $t = f(P)$ and the integration is along some path contained in $f^{-1}(t)$ and connecting the points $P_0(t), P \in f^{-1}(t)$. Finally we shall suppose that when varying $P \in f^{-1}(U_0)$ the path connecting $P_0(t)$ and P varies continuously in P. The function $F(P)$ is multivalued but locally analytic in $V_0 = f^{-1}(U_0)$.

LEMMA 2.2. — *Under the above conditions the following identity holds*

$$d \int_{P_0(t)}^{P} \omega = \left(\int_{P_0(t)}^{P} \frac{d\omega}{df} \right) df + \omega - (\tau \circ f)^* \omega \tag{2.3}$$

where $\frac{d\omega}{df}$ is the Gelfand-Lerray form of $d\omega$ and $(\tau \circ f)^*\omega$ is the pull back of ω under the map

$$\tau \circ f : V_0 \xrightarrow{f} U_0 \xrightarrow{\tau} V_0 \, .$$

Remark. — If $\tilde{\tau} : t \to \tilde{P}_0(t) \in f^{-1}(t)$ is another transversal curve (as in (2.2)) then (2.3) implies

$$d \int_{P_0(t)}^{\tilde{P}_0(t)} \omega = \left(\int_{P_0(t)}^{\tilde{P}_0(t)} \frac{d\omega}{df} \right) df + (\tilde{\tau} \circ f)^* \omega - (\tau \circ f)^* \omega .$$

If, in particular $P \equiv P_0(t)$ (so the path of integration τ is closed) we get the well known identity [2]

$$d \int_{\tau(t)} \omega = \left(\int_{\tau(t)} \frac{d\omega}{df} \right) dt.$$

Proof of Lemma 2.2. — Suppose that when t is sufficiently close to t_0 the path of integration connecting $P_0(t)$ to P is contained in some open polydisc $D_\varepsilon \subset V_0$, in which we may choose local coordinates x, y. We claim that for such a family of paths and for t sufficiently close to t_0 the identity (2.3) holds true. As (2.3) is linear in ω we may suppose without loss of generality that $\omega|_{D_\varepsilon} = Q(x,y)dx$ where Q is analytic in D_ε. Suppose further that the path of integration from $P_0(t) = (x_0(t), y_0(t))$ to $P = (x,y)$ is projected under the map $(x,y) \to x$ into an analytic path, avoiding the ramification points of this projection, and connecting $x_0(t)$ and x in the complex x-plane. Along such a path we may express $y = y(x,t)$ from the identity $f(x,y) = t$ and hence

$$\begin{aligned} d\int_{P_0(t)}^{P} \omega &= d\int_{x_0(t)}^{x} Q dx \\ &= Qdx - Qx_0'(t)df + \left(\int_{x_0(t)}^{x} Q_y \frac{\partial y}{\partial t} dx\right) df \\ &= Qdx - Qx_0'(t)df + \left(\int_{x_0(t)}^{x} \frac{Q_y}{f_y} dx\right) df \\ &= \omega - (\tau \circ f)^* \omega + \left(\int_{P_0(t)}^{P} \frac{d\omega}{df}\right) df. \end{aligned}$$

Therefore (2.3) holds true in a neighborhood of $P_0(t_0)$. By analytic continuation it holds true for arbitrary P and arbitrary continuous family of paths connecting $P_0(t)$ to P. The Lemma is proved. $\square$

Let $f \in \mathbb{C}[x,y]$, $\gamma(t) \subset f^{-1}(t)$ be a continuous family of closed loops such that $\int_{\gamma(t)} \omega \equiv 0$. If $\gamma(t)$ generates the fundamental group of $f^{-1}(t)$ then (2.3) implies that $\omega = dA + Bdf$ where A, B are analytic functions in $f^{-1}(U_0)$. In the case when the fundamental group of $f^{-1}(t)$ is not infinite cyclic we consider a covering

$$\tilde{V}_0 \xrightarrow{p} V_0 \tag{2.4}$$

such that the fundamental group of $\tilde{V}_0$ is infinite cyclic with a generator represented by a closed loop $\tilde{\gamma}(t_0)$ projected to $\gamma(t_0)$ under p. Such a covering exists and is unique up to an isomorphism [9]. Moreover $\tilde{V}_0$ has a canonical structure of analytic two-manifold induced by p. If we define $\tilde{f} = f \circ p$, then the fibration

$$\tilde{f} : \tilde{V}_0 \to U_0 \tag{2.5}$$

is locally trivial and the fibers are homotopy equivalent to circles. An analytic function (or differential form) on $\tilde{V}_0$ is a locally analytic function (differential form) on $V_0 = f^{-1}(U_0)$ such that *(i)* it has an analytic continuation along any arc in $f^{-1}(U_0)$

(ii) its determination does not change as (x,y) varies along any closed loop homotopic to $\gamma(t_0)$.

We shall denote the space of such functions (differential forms) by $\tilde{\mathcal{O}}(f^{-1}(U_0))$ $(\tilde{\Omega}^k(f^{-1}(U_0)))$. Lemma 2.2 implies the following

COROLLARY 2.3. — *If $\tilde{\omega} \in \tilde{\Omega}^1(f^{-1}(U_0))$ is such that $\int_{\gamma(t)} \tilde{\omega} \equiv 0$, then*

$$\tilde{\omega} = d\tilde{A} + \tilde{B}df$$

where $\tilde{A}, \tilde{B} \in \tilde{\mathcal{O}}(f^{-1}(U_0))$,

$$\tilde{A} = \int_{P_0(t)}^{P} \tilde{\omega}, \tilde{B} = -\int_{P_0(t)}^{P} \frac{d\tilde{\omega}}{df} + R(f)$$

and $R(.)$ is analytic in U_0.

In the proof of Theorem 2.1 we shall use the following well known

PROPOSITION 2.4. — *Let $f \in \mathbb{C}[x,y]$ be a non-constant polynomial. Then there exists a polynomial $m \in \mathbb{C}[f]$ such that*

1. *$m(f)$ belongs to the gradient ideal of f*
2. *$m(c) = 0$ if and only if c is a critical value of f.*

The identity

$$(\alpha f_x + \beta f_y)dx \wedge dy = df \wedge (\alpha dy - \beta dx) \tag{2.6}$$

combined with Proposition 2.4 shows that when ω is a polynomial (analytic) one-form, then the Gelfand-Leray form

$$m(f)\frac{d\omega}{df}$$

can be chosen polynomial (analytic).

Proof of Proposition 2.4. — Consider the reduced gradient ideal $J_{red} \subset \mathbb{C}[x,y]$ generated by $f_x/D, f_y/D$ where D is the greatest common divisor of f_x, f_y. The variety $V(J_{red}) = \{c_i\}_i$ is a finite union of points which may be supposed non-empty. Therefore $\mathbb{C}[x,y]/J_{red}$ is a vector space of finite dimension[9] and the multiplication by f defines an endomorphism. Therefore $m(f) \in J_{red}$ where $m(.)$ is the minimal polynomial of the endomorphism defined by f. We note that $m(c) = 0$ if and only if $c = c_i$ for some i. Taking into consideration that $\prod_i (f - c_i)/D$ is a polynomial we conclude that $\prod_i (f - c_i) m(f)$ belongs to the gradient ideal of f.

Proof of Theorem 2.1. — Suppose that $M_1 = \cdots = M_{k-1} = 0$ but $M_k \neq 0$, $k \geqslant 3$. The recursion formula (2.1) implies that $M_i(t) = \int_{\gamma(t)} \tilde{\omega}_i$, where $\tilde{\omega}_i \in \tilde{\Omega}^1(f^{-1}(U_0))$, $i \leqslant k$. Indeed, $\tilde{\omega}_1 = \omega \in \Omega^1(f^{-1}(U_0))$ and if $\tilde{\omega}_i \in \tilde{\Omega}^1(f^{-1}(U_0))$, then by Corollary 2.3

$$\tilde{\omega}_{i+1} = -\omega \int_{P_0(t)}^{P} \frac{d\tilde{\omega}_i}{df}.$$

The Gelfand-Leray form $\frac{d\tilde{\omega}_i}{df}$ may be supposed analytic (according to Proposition 2.4 and (2.6)). $M_i'(t) = \int_{\gamma(t)} \frac{d\tilde{\omega}_i}{df} = 0$ implies that $\int_{P_0(t)}^{P} \frac{d\tilde{\omega}_i}{df} \in \tilde{\mathcal{O}}(f^{-1}(U_0))$ and hence $\tilde{\omega}_{i+1} \in \tilde{\Omega}^1(f^{-1}(U_0))$. We obtain in particular that

$$M_k(t) = -\int_{\gamma(t)} \omega \int_{P_0(t)}^{P} \frac{d\tilde{\omega}}{df}$$

where

$$M_{k-1}(t) = \int_{\gamma(t)} \tilde{\omega} \equiv 0.$$

We shall prove the Theorem by induction on k. Suppose that that $M_{k-1}(t)$, is a finite linear combination of iterated integrals of length at most $k-1$, whose entries are differential one-forms analytic in $f^{-1}(U_0)$. We need to show that the same holds true for

$$\int_{P_0(t)}^{P} \frac{d\tilde{\omega}}{df}. \tag{2.7}$$

Let $\omega_1, \omega_2, \ldots, \omega_{k-1}$ be analytic one-forms in $f^{-1}(U_0)$. Lemma 2.2 implies

$$\frac{d}{df}\left(\omega_1 \int_{P_0(t)}^{P_1} \omega_2 \int_{P_0(t)}^{P_2} \omega_3 \ldots \int_{P_0(t)}^{P_{k-1}} \omega_{k-1}\right) = \frac{d\omega_1}{df} \int_{P_0(t)}^{P_1} \omega_2 \ldots \int_{P_0(t)}^{P_{k-1}} \omega_{k-1}$$

$$+\omega_1 \int_{P_0(t)}^{P_1} \frac{d\omega_2}{df} \ldots \int_{P_0(t)}^{P_{k-1}} \omega_{k-1} + \cdots + \omega_1 \int_{P_0(t)}^{P_1} \omega_2 \ldots \int_{P_0(t)}^{P_{k-1}} \frac{d\omega_{k-1}}{df}$$

$$-\frac{\omega_1 \wedge \omega_2}{df} \int_{P_0(t)}^{P_2} \omega_3 \ldots \int_{P_0(t)}^{P_{k-1}} \omega_{k-1}$$

$$-\omega_1 \int_{P_0(t)}^{P_1} \frac{\omega_2 \wedge \omega_3}{df} \int_{P_0(t)}^{P_3} \omega_4 \dots \int_{P_0(t)}^{P_{k-1}} \omega_{k-1}$$

$$\dots$$

$$-\omega_1 \int_{P_0(t)}^{P_1} \omega_2 \dots \int_{P_0(t)}^{P_{k-4}} \omega_{k-3} \int_{P_0(t)}^{P_{k-3}} \frac{\omega_{k-2} \wedge \omega_{k-1}}{df}$$

$$+\omega_1 \int_{P_0(t)}^{P_1} \omega_2 \int_{P_0(t)}^{P_2} \omega_3 \dots \int_{P_0(t)}^{P_{k-2}} \frac{\omega_{k-2} \wedge (\tau \circ f)^* \omega_{k-1}}{df}.$$

The differential form $\frac{\omega_{k-2} \wedge (\tau \circ f)^* \omega_{k-1}}{df}$ can be written in the form $\omega_{k-2} R(f)$ where $(\tau \circ f)^* \omega_{k-1} = -R(f) df$. This shows that (2.7) is a linear combination of iterated integrals of length at most $k-1$. As the Gelfand-Leray forms

$$\frac{\omega_i \wedge \omega_{i+1}}{df}, \frac{d\omega i}{df}$$

may always be chosen analytic in $f^{-1}(U_0)$ (see (2.6)) then Theorem 2.1 is proved. $\square$

The proof of the above theorem provides also an algorithm for computing the higher-order Poincaré-Pontryagin functions M_k in terms of iterated integrals. To illustrate this we consider few examples. To simplify the notations, for every given one-form ω on $\mathbb{C}^2$, we denote by ω' some *fixed* one-form, such that $df \wedge w' = d\omega$ (that is to say ω' is a Gelfand-Leray form of $d\omega$).

Examples. —

1. It is well known that
$$M_1(t) = \int_{\gamma(t)} \omega.$$

2. If $M_1 = 0$ then Lemma 2.2 and Corollary 2.3 imply
$$\omega = dA(x,y) + B(x,y) df + dR(f)$$
where
$$A(x,y) = \int_{P_0(t)}^{P} \omega, B(x,y) = -\int_{P_0(t)}^{P} \frac{d\omega}{df} + R(f), P = (x,y).$$
We have
$$M_2(t) = \int_{\gamma(t)} B\omega = -\int_{\gamma(t)} \omega \int_{P_0(t)}^{P} \omega'$$

and hence

$$M_2(t) = -\int_{\gamma(t)} \omega\omega'. \tag{2.8}$$

As $\int_{\gamma(t)} \omega' = \int_{\gamma(t)} \omega \equiv 0$ then the iterated integral (2.8) depends on the free homotopy class of $\gamma(t)$ (and not on the initial point $P_0(t)$).

3. If $M_2 = 0$, then

$$M_3(t) = \int_{\gamma(t)} \omega \int_{P_0(t)}^{P} \frac{d}{df}(\omega \int_{P_0(t)}^{Q} \omega')$$

where

$$\frac{d}{df}\left(\omega \int_{P_0(t)}^{P} \omega'\right) = \omega' \int_{P_0(t)}^{P} \omega' + \omega \int_{P_0(t)}^{P} \omega'' - \int_{\gamma(t)} \frac{\omega \wedge \omega'}{df} + R(t) \int_{P_0(t)}^{P} \omega$$

and $R(t)$ is an analytic function computed from the identity $(f \circ \tau)^* \omega' = R(f) df$. As $\int_{\gamma(t)} \omega\omega = 0$ then

$$M_3(t) = \int_{\gamma(t)} \omega(\omega')^2 + \int_{\gamma(t)} \omega^2 \omega'' - \omega \frac{\omega \wedge \omega'}{df}. \tag{2.9}$$

Both of the iterated integrals in (2.9) depend on the free homotopy class of $\gamma(t)$ only and do not depend on the particular choice of the Gelfand-Leray form ω'.

4. If $M_3 = 0$, then

$$\begin{aligned} M_4(t) \quad = \quad & -\int_{\gamma(t)} \omega \int_{P_0(t)}^{P} \frac{d}{df}(\omega \int_{P_0(t)}^{Q} (\omega')^2) \\ & -\int_{\gamma(t)} \omega \int_{P_0(t)}^{P} \frac{d}{df}(\omega \int_{P_0(t)}^{Q} \omega\omega'') \\ & -\int_{\gamma(t)} \omega \int_{P_0(t)}^{P} \frac{d}{df}(\omega \int_{P_0(t)}^{Q} \frac{\omega \wedge \omega'}{df}) \end{aligned}$$

If we make the particular choice $\omega' = -dB$, $B = \int_{P_0(t)}^{P} \frac{d\omega}{df}$ for the Gelfand-Leray form of $d\omega$, as well $w'' = 0$ then the formula for M_4 becomes

$$M_4(t) \quad = \quad \int_{\gamma(t)} \omega(\omega')^3 + \omega\omega' \frac{\omega \wedge \omega'}{df} + \omega^2 \frac{d \frac{\omega \wedge \omega'}{df}}{df}$$

$$
\begin{aligned}
&+\int_{\gamma(t)} \omega\frac{\omega\wedge\omega'}{df}\omega' - \omega\frac{\omega\wedge\frac{\omega\wedge\omega'}{df}}{df}\\
&-(\int_{\gamma(t)} \omega^2\omega')(f\circ\tau)^*\omega' + (\int_{\gamma(t)} \omega^2)(f\circ\tau)^*\frac{\omega\wedge\omega'}{df}\\
= \quad & \int_{\gamma(t)} \omega(\omega')^3 + \omega\omega'\frac{\omega\wedge\omega'}{df} + \omega^2\frac{d\frac{\omega\wedge\omega'}{df}}{df}\\
&+\int_{\gamma(t)} \omega\frac{\omega\wedge\omega'}{df}\omega' - \omega\frac{\omega\wedge\frac{\omega\wedge\omega'}{df}}{df}.
\end{aligned}
$$

Note that the last expression depends on the choice of ω' and hence on the initial point $P_0(t)$. It is an open question to find a general closed formula for M_k, $k \geqslant 4$, in terms of iterated integrals with rational entries, depending on the free homotopy class of $\gamma(t)$ only.

3. Proof of Theorem 1.1

The proof is split in two parts. First we show that $M_k(t)$ satisfies a linear differential equation of finite order (possibly with irregular singularities). For this we need to study the monodromy group of $M_k(t)$. Second, we shall show that the generating function $M_k(t)$ is of moderate growth on the projective plane $\mathbb{CP}^1$, and hence the equation is Fuchsian.

3.1. The monodromy representation of M_k

Recall first that the universal monodromy representation for $M_k(t)$ (for arbitrary k) can be constructed as follows (see [11] for proofs). To the non-constant polynomial $f \in \mathbb{C}[x,y]$ we associate the locally trivial fibration

$$f^{-1}(\mathbb{C}\backslash\Delta) \xrightarrow{f} \mathbb{C}\backslash\Delta$$

where $\Delta \subset \mathbb{C}$ is the finite set of atypical values. Let $t_0 \notin \Delta$ and put $S = f^{-1}(t_0)$. The canonical group homomorphism

$$\pi_1(\mathbb{C}\setminus\Delta, t_0) \to \mathrm{Diff}\,(S)/\mathrm{Diff}_0(S). \tag{3.1}$$

where $\mathrm{Diff}\,(S)/\mathrm{Diff}_0(S)$ is the mapping class group of S, induces a homomorphism (group action on $\pi_1(S)$)

$$\pi_1(\mathbb{C}\setminus D, t_0) \to \mathit{Perm}\,(\pi_1(S)) \tag{3.2}$$

where $\mathit{Perm}\,(\pi_1(S))$ is the group of permutations of $\pi_1(S)$, and $\pi_1(S)$ is the set of free homotopy classes of closed loops on S.

Let $\gamma(t_0) \in \pi_1(S)$ be a free homotopy class of closed loops on S and consider the orbit $\mathcal{O}_{\gamma(t_0)}$ of $\gamma(t_0)$ under the group action (3.2). For a given point $P_0 \in S$ we denote $F = \pi_1(S, P_0)$ and let $G \subset \pi_1(S, P_0)$ be the normal subgroup generated by the pre-image of the orbit $\mathcal{O}_{\gamma(t_0)}$ under the canonical projection

$$\pi_1(S, P_0) \to \pi_1(S) .$$

Let (G, F) be the normal sub-group of G generated by commutators

$$g^{-1}f^{-1}gf, g \in G, f \in F$$

and denote

$$H_1^\gamma(S, \mathbb{Z}) = \mathbb{G}/(\mathbb{G}, \mathbb{F}).$$

From the definition of G it follows that the Abelian group $H_1^\gamma(S, \mathbb{Z})$ is invariant under the action of $\pi_1(\mathbb{C} \setminus \Delta)$ and hence we obtain a homomorphism (the universal representation)

$$\pi_1(\mathbb{C} \setminus \Delta, t_0) \to Aut(H_1^\gamma(S, \mathbb{Z})). \tag{3.3}$$

On the other hand, the monodromy representation of the generating function $M_k(t) = M_k(\gamma, \mathcal{F}_\varepsilon, t)$ is defined as follows. The function $M_k(t)$ is multivalued on $\mathbb{C} \setminus \Delta$. Let us consider all its possible determinations in a sufficiently small neighborhood of $t = t_0$. All integer linear combinations of such functions form a module over $\mathbb{Z}$ which we denote by $\mathcal{M}_k(\gamma, \mathcal{F}_\varepsilon)$. The fundamental group $\pi_1(\mathbb{C} \setminus \Delta, t_0)$ acts on $\mathcal{M}_k = \mathcal{M}_k(\gamma, \mathcal{F}_\varepsilon)$ in an obvious way. We obtain thus a homomorphism

$$\pi_1(\mathbb{C} \setminus \Delta, t_0) \to Aut(\mathcal{M}_k) \tag{3.4}$$

called the monodromy representation of the generating function $M_k(\gamma, \mathcal{F}_\varepsilon, t)$.

It is proved in [11, Theorem 1] that the map

$$H_1^\gamma(f^{-1}(t_0), \mathbb{Z}) \xrightarrow{\varphi} \mathcal{M}_k(\gamma, \mathcal{F}_\varepsilon) : \gamma \to M_k(\gamma, \mathcal{F}_\varepsilon, t) \tag{3.5}$$

is a canonical surjective homomorphism compatible with the action of the fundamental group $\pi_1(\mathbb{C} \setminus \Delta, t_0)$. Equivalently, (3.4) is a sub-representation of the representation dual to the universal representation (3.3).

If the rank of the Abelian group $H_1^\gamma(f^{-1}(t_0), \mathbb{Z})$ were finitely generated, then this would imply that each $M_k(t)$ satisfies a linear differential equation whose order is bounded by the dimension of $H_1^\gamma(f^{-1}(t_0), \mathbb{Z})$. We shall prove here a weaker statement: $M_k(t)$ satisfies a linear equation of finite order (depending on k). Our argument is based on the integral representation for

$M_k(t)$ obtained in the previous section. Namely, consider the lower central series

$$F_1 \supseteq F_2 \supseteq \cdots F_k \supseteq \cdots \tag{3.6}$$

where $F_1 = F = \pi_1(S, P_0)$, and $F_{k+1} = (F_k, F)$ is the subgroup of F_k generated by commutators $(f_k, f) = f_k^{-1} f^{-1} f_k f$, $f_k \in F_k$, $f \in F$. An iterated integral of length k along a closed loop which belongs to F_{k+1} vanishes. Therefore Theorem 2.1 implies that in (3.3) we can further truncate by F_{k+1}. Namely, for every subgroup $H \subset F$ we denote $\tilde{H} = (H \cup F_{k+1})/F_{k+1}$. As before the group action (3.2) induces a homomorphism

$$\pi_1(\mathbb{C} \setminus D, t_0) \to Aut(\tilde{G}/(\tilde{G}, \tilde{F})) \tag{3.7}$$

and there is a canonical surjective homomorphism

$$\tilde{G}/(\tilde{G}, \tilde{F}) \to \mathcal{M}_k(\gamma, \mathcal{F}_\varepsilon).$$

The lower central series of $\tilde{F} = \tilde{F}_1$ is

$$\tilde{F}_1 \supseteq \tilde{F}_2 \supseteq \cdots \tilde{F}_k \supseteq \{id\}.$$

It is easy to see that in this case $(\tilde{G}, \tilde{F}) \subset \tilde{F}$ is finitely generated (e.g. [20, Lemma 4.2, p.93]), and hence $\tilde{G}/(\tilde{G}, \tilde{F})$ is finitely generated too. This implies on its hand that $\mathcal{M}_k(\gamma, \mathcal{F}_\varepsilon)$ is finite-dimensional, and hence the generating function satisfies a linear differential equation. Its order is bounded by the dimension of $\tilde{G}/(\tilde{G}, \tilde{F})$. The latter is easily estimated to be less or equal to

$$\sum_{i=1}^{k} \dim F_i/F_{i+1} \leqslant r^k$$

(for the last inequality see [14, section 11]). To resume, we proved that *the generating function of limit cycles $M_k(t)$, $t \in \mathbb{C} \setminus \Delta$, satisfies an analytic linear differential equation of order at most r^k.*

3.2. The moderate growth of M_k

We shall show that the possible singular points (contained in $\Delta \cup \infty$) are of Fuchs type. A necessary and sufficient condition for this is the moderate growth of $M_k(t)$ in any sector centered at a singular point. For this we shall use once again the integral representation for $M_k(t)$. Let t_0 be an atypical value for f and suppose that the analytic curve

$$\tau : t \to P_0(t) \in f^{-1}(t)$$

is defined for $t \sim t_0$ and is transversal to the fibers $f^{-1}(t)$. It follows from the proof of Theorem 2.1 that $M_k(t)$ has an integral representation in a punctured neighborhood of t_0 too. More precisely $M_k(t)$ is a finite linear combination

$$\sum \frac{\alpha_i(t)}{m_i(t)} \int_{\gamma(t)} \omega_1^i \ldots \omega_{i_j}^i$$

where ω_q^p are polynomial one-forms, $i_j \leqslant k$, $\alpha_i(t)$ are analytic functions and $m_i(t)$ are polynomials. Let

$$t \to P_0(t) \in f^{-1}(t), t \to \tilde{P}_0(t) \in f^{-1}(t)$$

be two analytic curves defined in a neighborhood of the atypical value t_0 and transversal to the fibers $f^{-1}(t)$ (including $f^{-1}(t_0)$) and consider the iterated integral

$$F(t) = \int_{l(t)} \omega_1 \omega_2 \ldots \omega_k$$

where $l(t)$ is a path on $f^{-1}(t)$ connecting $P_0(t)$ and $\tilde{P}_0(t)$, and $\omega_1, \omega_2, \ldots, \omega_k$ are polynomial one-forms in $\mathbb{C}^2$.

PROPOSITION 3.1. — *Let* $S(t_0) = \{t \in \mathbb{C} : arg(t - t_0) < \varphi_0, 0 < |t - t_0| < r_0\}$ *be a sector centered at* t_0. *There exists* $r_0, N_0 > 0$ *such that* $|F(t)| < |t - t_0|^{-N_0}$. *Let* $S(\infty) = \{t \in \mathbb{C} : arg(t) < \varphi_0, |t| > r_0\}$ *be a sector centered at* ∞. *There exist* $r_0, N_0 > 0$ *such that* $|F(t)| < |t|^{N_0}$ *in* $S(\infty)$.

Remark. — Recall that an analytic function F defined on the universal covering of $\mathbb{C} \setminus \Delta$ and satisfying the claim of the above Proposition is said to be of *moderate growth.*

Proof of Proposition 3.1. — Let $(x_i(t), y_i(t)) \in f^{-1}(t)$ be the ramification points of the projection $f^{-1}(t) \to \mathbb{C}$ induced by $\pi : (x, y) \to x$. Each ramification point $x_i(t)$ has a Puiseux expansion in a neighborhood of t_0. Therefore when t tends to t_0 in a sector centered at t_0, each ramification point tends to a definite point $P \in \mathbb{CP}^1$.

Assume further that the projection of $l(t)$ on the x-plane is represented by a piece-wise straight line

$$\pi(l(t)) = \cup_{i=1}^n [x_i, x_{i+1}] \tag{3.8}$$

connecting $x_0(t), x_1(t), \cdots, x_{n+1}(t)$ where x_i, $i = 1, 2, \ldots, n$ are some ramification points, and $P_0(t) = (x_0(t), y_0(t))$, $\tilde{P}_0(t) = (x_{n+1}(t), y_{n+1}(t))$. The iterated integral $F(t)$ along $l(t)$ is expressed as an iterated integral along $\pi(l(t))$ whose entries are one-forms with algebraic coefficients. It is clear that such an iterated integral along $[x_i(t), x_{i+1}(t)]$ is of moderate growth (because $x_i(t)$ are of moderate growth). Thus, if the number n in (3.8) were bounded when t varies in $S(t_0)$, then $F(t)$ would be of moderate growth of $F(t)$. It remains to show that the number $n = n(t)$ is uniformly bounded in $S(t_0)$. Note that when $t \in S(t_0)$ and r_0 is sufficiently small the ramification points $x_i(t)$ are all distinct. Denote by $\mathbf{B}$ the subset of $S(t_0)$ of points t having the property

"there exist ramification points $x_i(t), x_j(t), x_k(t)$ such that $x_i(t) - x_j(t)$ and $x_i(t) - x_k(t)$ are collinear but not identically collinear ."

Using the fact that $x_i(t) - x_j(t)$, $x_i(t) - x_k(t)$ have Puiseux expansions, we conclude that $\mathbf{B}$ is a real analytic subset of $\mathbb{R}^2 \simeq \mathbb{C}$ of co-dimension one. The set $S(t_0) \setminus \mathbf{B}$ has a finite number of connected components and on each connected component the function $t \to n(t)$ is constant. It follows that $n(t)$ is uniformly bounded. Proposition 3.1, and hence Theorem 1.1 is proved.

A. Iterated Path Integrals

Let S be a Riemann surface and $\omega_1, \omega_2, \ldots, \omega_k$ be holomorphic one-forms. For every smooth path $l : [0,1] \to S$ we define the *iterated path integral*

$$\int_l \omega_1 \omega_2 \ldots \omega_k = \int_{0 \leqslant t_k \leqslant \ldots \leqslant t_1 \leqslant 1} f_k(t_k) \ldots f_1(t_1) dt_k \ldots dt_1 \tag{A.1}$$

where $l^* \omega_i = f_i(t) dt$. We have for instance

$$\int_l \omega_1 \omega_2 = \int_l \omega_1 \int_{l(0)}^{l(t)} \omega_2 .$$

The basic properties of the iterated path integrals (A.1) were established by Parsin [19]. The general theory of iterated integrals has been developed by Chen, e.g. [5, 6]. In the Chen's theory the iterated integrals generate the De Rham complex of the path space PX associated to an arbitrary manifold X. In this context the iterated integrals of the form (A.1) provide the 0-cochains of the path space $P_{a,b}S$, where a, b are the two ends of l. Indeed, a connected component of $P_{a,b}S$ consists of those paths (with fixed ends) which are homotopy equivalent, and (A.1) is constant on such paths. Some basic properties of iterated path integrals are summarized below. The missing proofs (and much more) may be found in R. Hain [12, 13].

LEMMA A.1. —

(i) *The value of* $\int_l \omega_1\omega_2\ldots\omega_k$ *depends only on the homotopy class of* l *in the set of loops with ends fixed at* $l(0), l(1)$.

(ii) *If* $l_1, l_2 : [0,1] \to S$ *are composable paths (i.e.* $l_1(1) = l_2(0)$ *) then*

$$\int_{l_1 l_2} \omega_1\omega_2\ldots\omega_k = \sum_{i=0}^{k} \int_{l_2} \omega_1\omega_2\ldots\omega_i \int_{l_1} \omega_{i+1}\omega_2\ldots\omega_k \tag{A.2}$$

where we set $\int_l \omega_1\omega_2\ldots\omega_i = 1$ *if* $i = 0$.

(iii)

$$\int_l \omega_1\omega_2\ldots\omega_k = (-1)^k \int_{l^{-1}} \omega_k\omega_{k-1}\ldots\omega_1.$$

From now on we suppose that $l(0) = l(1) = P_0$ and put $F = \pi_1(S, P_0)$. For $\alpha, \beta \in F$ we denote the commutator $\alpha^{-1}\beta^{-1}\alpha\beta$ by (α, β). If $A, B \subset F$ are subgroups, we denote by (A, B) the subgroup of F generated by all commutators (α, β), such that $\alpha \in A$ and $\beta \in B$. Consider the lower central series $F = F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ where $F_k = (F_{k-1}, F)$ and $F_1 = F$.

LEMMA A.2. —

(i)

$$-\int_{(\alpha,\beta)} \omega_1\omega_2 = \det\begin{pmatrix} \int_\alpha \omega_1 & \int_\beta \omega_1 \\ \int_\alpha \omega_2 & \int_\beta \omega_2 \end{pmatrix}, \forall \alpha, \beta \in F_1.$$

(ii) *Let* $\gamma \in F_k$ *and* $\omega_1, \omega_2, \ldots, \omega_k$ *be holomorphic one-forms on* S. *Then the iterated path integral* $\int_\gamma \omega_1\ \omega_2\ \ldots\ \omega_{k-1}$ *vanishes, and the value of the integral* $\int_\gamma \omega_1\ \omega_2\ \ldots\ \omega_k$ *does not depend on the initial point* P_0.

(iii) *If* $\alpha, \beta \in F_k$ *then*

$$\int_{\alpha\beta} \omega_1\ \omega_2\ \ldots\ \omega_k = \int_\alpha \omega_1\ \omega_2\ \ldots\ \omega_k + \int_\beta \omega_1\ \omega_2\ \ldots\ \omega_k.$$

(iv) *If* $\alpha \in F_p$, $\beta \in F_q$, *then*

$$\begin{aligned} -\int_{(\alpha,\beta)} \omega_1\ \omega_2\ \ldots\ \omega_{p+q} \quad &= \quad \int_\alpha \omega_1\ \omega_2\ \ldots\ \omega_p \int_\beta \omega_{p+1}\ \omega_{p+2}\ \ldots\ \omega_{p+q} \\ &- \quad \int_\beta \omega_1\ \omega_2\ \ldots\ \omega_q \int_\alpha \omega_{q+1}\ \omega_{q+2}\ \ldots\ \omega_{p+q} \end{aligned}$$

Proof. — The identity (i) follows from (A.2). It implies in particular that $\int_\gamma \omega_1\omega_2$ does not depend on the initial point P_0 provided that $\gamma \in F_2$, and vanishes provided that $\gamma \in F_3$. Suppose that the claim (ii) is proved up to order $k-1$ and let $\gamma \in F_k$. If $\gamma = (\alpha, \beta)$ where $\alpha \in F_{k-1}$ then (A.2) implies

$$\begin{aligned}\int_{(\alpha,\beta)} \omega_1\, \omega_2\, \dots\, \omega_{k-1} &= \int_\alpha \omega_1\, \omega_2\, \dots\, \omega_{k-1} + \int_{\alpha^{-1}} \omega_1\, \omega_2\, \dots\, \omega_{k-1} \\ &\quad + \sum_{i=0}^{k-1} \int_\beta \omega_1\, \omega_2\, \dots\, \omega_i \int_{\beta^{-1}} \omega_i\, \dots\, \omega_{k-1} \\ &= \int_{\alpha\alpha^{-1}} \omega_1\, \omega_2\, \dots\, \omega_{k-1} + \int_{\beta^{-1}\beta} \omega_1\, \omega_2\, \dots\, \omega_{k-1} \\ &= 0.\end{aligned}$$

If, more generally, $\gamma \in F_k$ then $\gamma = \prod_i \gamma_i$ where each γ is a commutator (α, β), such that either $\alpha \in F_{k-1}$, or $\beta \in F_{k-1}$. Therefore

$$\int_\gamma \omega_1\, \omega_2\, \dots\, \omega_{k-1} = \sum_i \int_{\gamma_i} \omega_1\, \omega_2\, \dots\, \omega_{k-1} = 0.$$

The claim that $\int_\gamma \omega_1\, \omega_2\, \dots\, \omega_k$, $\gamma \in F_k$ does not depend on the initial point P_0 follows from (iv) (by induction). The claims (iii) and (iv) follow from (ii) and (A.2). $\square$

We proved in section 2 that the generating function of limit cycles $M_k(t)$ is a linear combination of iterated path integrals of length k along a loop $\gamma(t)$. As $M(t)$ depends on the *free* homotopy class of $\gamma(t)$ then these iterated integrals are of special nature. The iterated integrals appearing in Lemma A.2 have the same property: they do not depend on the initial point P_0. Therefore they must satisfy (by analogy to M_k) a Fuchsian differential equation. The proof of this fact can be seen as a simplified version of the proof of Theorem 1.1 and for this reason it will be given below.

Let $\gamma(t) \subset f^{-1}(t)$ be a family of closed loops depending continuously on a parameter t in a neighborhood of the typical value t_0 of the non-constant polynomial $f \in \mathbb{C}[x, y]$. We put $S = f^{-1}(t_0)$ and suppose, using the notations of Lemma A.2, that $\gamma(t_0) \in F_k$. Consider the iterated integral

$$I(t) = \int_{\gamma(t)} \omega_1\, \omega_2\, \dots\, \omega_k$$

where ω_i are polynomial one-forms in $\mathbb{C}^2$. In the case $k = 1$ this is an Abelian integral depending on a parameter t and hence it satisfies a (Picard-) Fuchs equation of order at most r.

PROPOSITION A.3. — *The iterated integral I satisfies a Fuchs equation of order at most $M_r(k)$, where $M_r(k) = dim F_k/F_{k+1}$ is given by the Witt formula*

$$M_r(k) = \frac{1}{k}\sum_{d|k} \mu(d) r^{k/d} \tag{A.3}$$

and $\mu(d)$ is the Möbius function (it equals to $0, \pm 1$, see Hall [14]). For small values of k, r the numbers $M_r(k)$ are shown on the table below.

r \ k	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	2	1	2	3	6	9	18	30
3	3	3	8	18	32	116	312	810
4	4	6	20	60	204	4020	4095	8160

Proof. — Let Δ be the finite set of atypical values of f. The function $I(.)$ is locally analytic on $\mathbb{C} \setminus \Delta$ and has a moderate growth there (see section 3). A finite-dimensional representation of its monodromy group is constructed as follows. As F_k is a normal subgroup of F we may consider the Abelian factor groups F_k/F_{k+1}. Recall that F_k/F_{k+1} is free, torsion free, and finitely generated. Thus it is homomorphic to $\mathbb{Z}^{M_r(k)}$ where r is the number of generators of F and $M_r(k)$ is given by the Witt formula (A.3), e.g. Hall[14]. As the Abelian group F_k/F_{k+1} is canonically identified to a subset of $\pi_1(S)$ invariant under the action (3.2) of the fundamental group $\pi_1(\mathbb{C} \setminus \Delta, t_0)$, then we obtain a homomorphism

$$\pi_1(\mathbb{C} \setminus \Delta, t_0) \to Aut(F_k/F_{k+1}) \; . \tag{A.4}$$

Finally, Lemma 3(ii) implies that the iterated integral $I(t)$ depends on the equivalence class of $\gamma(t)$ in F_k/F_{k+1}. Therefore the monodromy representation of I is a sub-representation of (A.4). The Proposition is proved.

B. Is the generating function M_k an Abelian integral?

Equivalently, is the Fuchs equation satisfied by M_k of Picard-Fuchs type? This is an open difficult problem. The results of [11] and Theorem 1.1 provide an answer to the following related question. Let $f \in \mathbb{C}[x, y]$ be a non constant polynomial, $\gamma(t) \in f^{-1}(t)$ a family of closed loops depending continuously on t. Is there a rational one-form on $\mathbb{C}^2$, such that

$$M_k(t) = \int_{\gamma(t)} \omega \; ? \tag{B.1}$$

Consider the canonical homomorphism

$$\pi_1 : H_1^\gamma(f^{-1}(t),\mathbb{Z}) \to H_1(f^{-1}(t),\mathbb{Z}) \tag{B.2}$$

(which is neither injective, nor surjective in general) as well the surjective homomorphism

$$\pi_2 : H_1^\gamma(f^{-1}(t),\mathbb{Z}) \to \mathcal{M}_k(\gamma,\mathcal{F}_\varepsilon). \tag{B.3}$$

The homomorphism π_1 is defined in an obvious way, and π_2 was defined in section 3.1. Recall that both of them are compatible with the action of $\pi_1(\mathbb{C}\setminus\Delta, t_0)$.

THEOREM B.1.— *The generating function M_k can be written in the form (B.1) if and only if*

$$Ker(\pi_1) \subset Ker(\pi_2).$$

The theorem says, roughly speaking, that M_k is an Abelian integral in the sense (B.1) if and only if M_k "depends on the homology class of $\gamma(t)$ only". Indeed, when $M_k(t)$ is an Abelian integral, this holds true. Conversely, if $Ker(\pi_1) \subset Ker(\pi_2)$, then the injective homomorphism

$$H_1^\gamma(f^{-1}(t),\mathbb{Z})/Ker(\pi_1) \to H_1(f^{-1}(t),\mathbb{Z})$$

and the surjective homomorphism

$$H_1^\gamma(f^{-1}(t),\mathbb{Z})/Ker(\pi_1) \to \mathcal{M}_k(\gamma,\mathcal{F}_\varepsilon)$$

are both compatible with the action of $\pi_1(\mathbb{C}\setminus\Delta, t_0)$. The proof that M_k is an Abelian integral in the sense (B.1) repeats the arguments from the proof of [11, Theorem 2] and will be not reproduced here.

Theorem B.1 can be illustrated by the following two basic examples, taken from [11].

Example B.2.— The generating function M_3 associated to the perturbed foliation

$$df + \varepsilon(2 - x + \frac{1}{2}x^2)dy = 0, f = x(y^2 - (x-3)^2)$$

and to the family of ovals $\gamma(t)$ around the center of the unperturbed system can not be written in the form (B.1). Indeed, an appropriate computation shows that there is a loop $l(t)$ contained in the orbit of $\gamma(t)$ under the action of $\pi_1(\mathbb{C}\setminus\Delta, t_0)$, such that

- the homology class of $l(t)$ is trivial

- the free homotopy class of $l(t)$ is non-trivial
- the corresponding generating function $M_3(t) = M_3(l, \mathcal{F}_\varepsilon, t)$ is not identically zero.

It follows that $Ker(\pi_1) \not\subset Ker(\pi_2)$ and $M_k(t)$ is not an Abelian integral.

Example B.3. — Let ω be an *arbitrary* polynomial one-form on $\mathbb{C}^2$. The generating function M_k associated to

$$df + \varepsilon\omega = 0, f = y^2 + (x^2 - 1)^2$$

and to the exterior family of ovals $\{f = t\}, t > 1$ can be written in the form (B.1). Indeed, it can be shown that the homomorphism π_1 (B.2) is injective [11]. Thus M_k is always an Abelian integral in the sense (B.1), see also [17, 18].

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